A distributive lattice structure on noncrossing partitions*

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Abstract

In [FP2] a natural order on Dyck paths of any fixed length inducing a distributive lattice structure is defined. We transfer this order on noncrossing partitions along a well-known bijection [S], thus showing that noncrossing partitions can be endowed with a distributive lattice structure having some combinatorial relevance. Finally we prove that our lattices are isomorphic to the posets of 312-avoiding permutations with the order induced by the strong Bruhat order of the symmetric group.

1 Introduction

Every paper dealing with Catalan numbers contains a sentence somehow like the following: "In [S2] Stanley gives 66 different combinatorial interpretations of Catalan numbers". Indeed, exercise 6.19 is maybe the best source of information on the Catalan family, at least from a purely enumerative point of view. A further step should be to consider some interesting order structures on the objects of the Catalan family and try to compare them. What we would like to do in the present paper is a first instance of this program.

We start by considering noncrossing partitions. They can be endowed with the refinement order, so to obtain the well-known noncrossing partition lattices, first studied by Kreweras [Kre], which have been proved very useful in several, different contexts. These lattices possess many interesting properties, however they are not distributive (actually not even modular). Is there the possibility of defining some *interesting* distributive lattice structure on noncrossing partitions? We claim that the answer is affirmative by explicitly finding an order on noncrossing partitions which is isomorphic to at least two combinatorially meaningful distributive lattices.

We first consider Dyck paths and define an order on them as follows: given two Dyck paths P, Q of the same length, we say that $P \leq Q$ when P entirely lies below Q (possibly coinciding with Q in some points). It is possible to show [FP2] that the set of Dyck paths of any given length endowed with this order is a distributive lattice. These *Dyck lattices* are not so well known; they have been studied first in [FP2] (following some general ideas of Narayana [N]), and in [CJ] the authors show their importance in the study of some matters related with Temperley-Lieb algebras. Our idea is to transport such a structure on noncrossing partitions along a famous bijection (see [S]). We have called *Bruhat noncrossing partition lattices* the distributive lattices of noncrossing partitions arising in this way; section 3 is devoted to the study of some properties of these lattices. Moreover,

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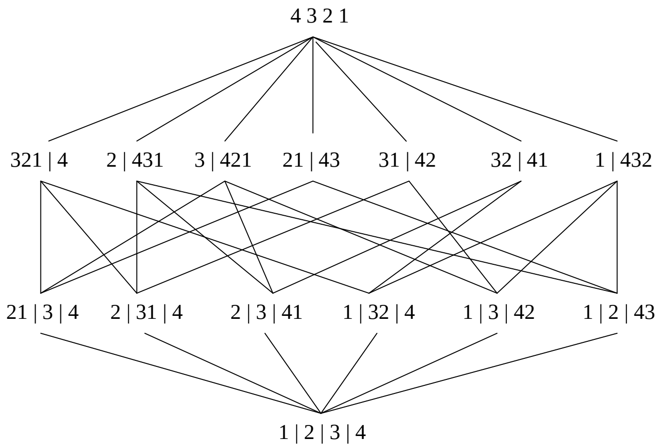


Figure 1: $\Pi(4)$.

Bruhat noncrossing partition lattices turn out to be isomorphic to an even more interesting class of lattices. It is not difficult to explicitly find a trivial bijection between noncrossing partitions and 312-avoiding permutations. More precisely, we show that such a bijection is an order-isomorphism between the Bruhat lattice of noncrossing partitions of an n set and the class $S_n(312)$ of 312-avoiding permutations of an n set endowed with the (strong) Bruhat order. As a byproduct, we have that $S_n(312)$ is a distributive sublattice of the symmetric group of order n with the Bruhat order. These results are contained in section 4, where we also find a criterion to determine the meet and the join of two 312-avoiding permutations in $S_n(312)$. To the best of our knowledge, the only paper dealing with this kind of matters is [P], where the author determines the Bruhat posets (arising from Weyl groups) which are lattices. However, the language and the aims of [P] are totally different from the ones of our approach. It would be interesting to compare our results with those of Proctor. However, it seems to us that our result is the first one concerning the order structure induced by the Bruhat order on a class of pattern-avoiding permutations.

The final part of this introduction is devoted to the explanation of the main notations we use through the paper and to the presentation of the basics of some general theories we refer to in the next pages.

The set (and the lattice) of partitions of $[n] = \{1, 2, \dots, n\}$ will be denoted by $\Pi(n)$. If $\pi \in \Pi(n)$, we will always use the notation $\pi = B_1|B_2|\dots|B_k$, where the B_i's are the blocks of π, the elements inside each block are in decreasing order and $\max B_i < \max B_j$, for $i < j$. Given $\pi, \rho \in \Pi(n)$, define $\pi \leq \rho$ when every block of π is contained into some block of ρ. The many properties of this classical order can be found in several textbooks, such as [S1, A]. Here we only mention that $\Pi(n)$ endowed with this *refinement order* is a lattice which is neither distributive nor modular. Nevertheless, it possesses a rank function: the rank of $\pi = B_1|B_2|\dots|B_k$ is $n - k$. The Whitney numbers of the partition lattices are the well-known Stirling numbers of the second kind. The Hasse diagram of $\Pi(4)$ is shown in Figure 1.

We will often deal with Dyck paths and, depending on the context, we will find convenient to describe them in several different ways. Therefore a Dyck path will be alternatively described as a particular lattice path in the discrete plane $\mathbf{N} \times \mathbf{N}$ (and denoted by capital letters like $P, Q, R, \dots$)

or as a function $f : \mathbf{N} \longrightarrow \mathbf{N}$ satisfying certain properties (and denoted by lowercase letters like $f, g, h, \ldots$) or else as a particular word of the two-letter alphabet $\{U, D\}$ (and denoted by Greek letters such as $\omega(U, D), \psi(U, D), \ldots$). We leave to the reader the details of the descriptions of Dyck paths we have sketched in the previous sentence.

In section 4 we make use of the concept of (generalized) pattern-avoiding permutation. Let $\pi = \pi_1\pi_2 \cdots \pi_n$ and $\sigma = \sigma_1\sigma_2 \cdots \sigma_k$ be two permutations of $[n]$ and $[k]$, respectively, with $k \leq n$. The permutation π *avoids the pattern* σ if there exist no indexes $i_1 < i_2 < \cdots < i_k$ such that $\pi_{i_1}\pi_{i_2} \ldots \pi_{i_k}$ is in the same relative order as $\sigma_1\sigma_2 \ldots \sigma_k$. The permutation π is called σ*-avoiding* and the subset of σ-avoiding permutations of S_n is denoted $S_n(\sigma)$. A huge amount of papers can be found dealing with pattern avoidance, see for instance [SS, F, Kra]. In [BS] the authors introduced *generalized patterns* for the study of Mahonian statistics on permutations. A generalized pattern is a permutation $\sigma \in S_k$ equipped with a dash between two of its elements (e.g. $1-32$ and $23-1$ are generalized patterns of length 3) and a permutation π contains a generalized pattern when adjacent elements in the generalized pattern correspond to adjacent elements in π. Classes of generalized pattern avoiding permutations has been widely studied in recent years (see [BS, BFP, C, CM], to cite a very few).

2 Noncrossing partitions and Dyck paths

A partition of $1, 2, \ldots, n$ is *noncrossing* when, given four elements, $1 \leq a < b < c < d \leq n$, such that a, c are in the same block and b, d are in the same block, then the two blocks coincide. The set of all noncrossing partitions of an n-set will be denoted $NC(n)$. We refer the reader to the fairly complete survey [S] and to the references therein for the plentiful applications of this notion.

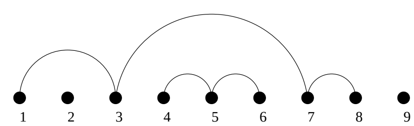


Figure 2: The noncrossing partition $2|654|8731|9 \in NC(9)$.

The refinement order can be restricted to noncrossing partitions: what we obtain is again a lattice, which is usually referred to as the *noncrossing partition lattice*. Among the main features of these lattices we recall here that they are not distributive and the lattice operations are different from those of the partition lattices (the join of two noncrossing partitions needs not be noncrossing within the full partition lattice).

Noncrossing partitions are enumerated by Catalan numbers, so, as it often happens, it is possible to find a bijection with Dyck paths. The nice bijection we are going to describe can also be found, for instance, in [D, S]. Fix a Dyck path and label its up steps by enumerating them from left to right (so that the k-th up step is labelled k). Next assign to each down step the same label of the up step it is matched with. Now consider the partition whose blocks are constituted by the labels of each sequence of consecutive down steps. Such a partition is easily seen to be noncrossing. In Figure 3 we have illustrated this bijection on a concrete example; the bold labels next to the down steps are the elements of the corresponding noncrossing partition, whereas the up steps are simply labelled in increasing order.

Now denote with $\mathcal{D}_n$ the set of Dyck paths of length $2n$. It is possible to define a natural order on $\mathcal{D}_n$ by setting $f \leq g$ whenever $f(n) \leq g(n)$, for every $n \in \mathbf{N}$. This means that $f \leq g$ when f "lies

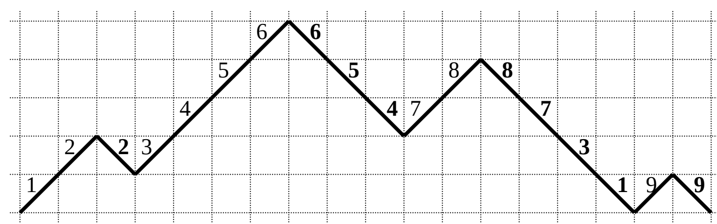


Figure 3: The Dyck path associated with 2|654|8731|9.

weakly” below g. The set $\mathcal{D}_n$, endowed with such an order, turns out to be a distributive lattice, which has been studied in some detail in [FP2] under the name of *Dyck lattice (of order n)*. We point out that Dyck lattices have also been considered in [CJ], where the authors speak of *geometric inclusion* of paths.

Our idea is to transfer the order structure of Dyck lattices along the above described bijection. In this way we define a new order on noncrossing partitions. The distributive lattices so obtained will be called *Bruhat noncrossing partition lattices*. The reason of this name, which is at present rather obscure, will become clear in the last section. Our main goal is to give a satisfactory description of such lattices.

3 The Bruhat noncrossing partition lattice

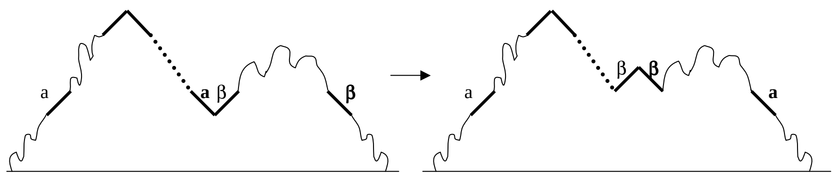
In the rest of the paper it is tacitly assumed that noncrossing partitions are endowed with the Bruhat order.

Given two noncrossing partitions π, ρ we look for some condition to recognize if $\pi \prec \rho$ or not. The following theorem gives a precise answer to this problem.

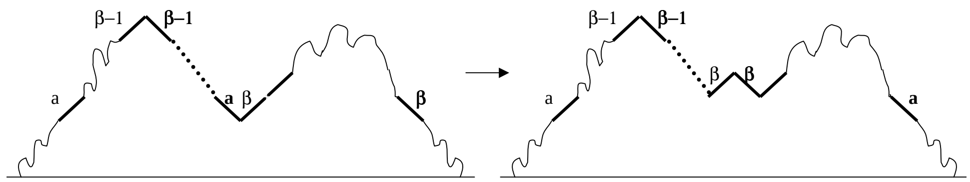
Theorem 3.1 *(Characterization of coverings) Given two noncrossing partitions $\pi, \rho \in NC(n)$, we have $\pi \prec \rho$ if and only if ρ is obtained from π by moving the minimum of some block B of π into the block $\tilde{B}$ containing the element $\beta = \max B + 1$ and either*

1. *keeping β inside $\tilde{B}$, if $\beta = \max \tilde{B}$, or*
2. *adding a new block $\overline{B} = \{\beta\}$, if $\beta \neq \max \tilde{B}$.*

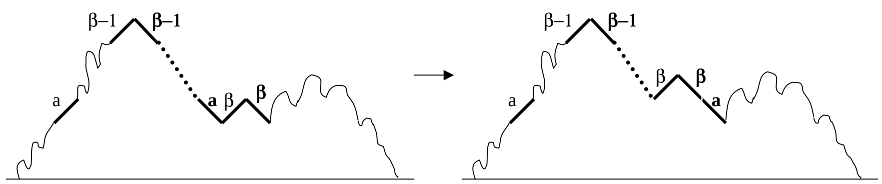
Proof. Suppose that P_π, P_ρ are the Dyck paths associated with π, ρ, respectively. The fact that $P_\pi \prec P_\rho$ in $\mathcal{D}_n$ means that P_ρ is obtained from P_π by replacing a valley with a peak. In the context of noncrossing partitions this amounts to moving the minimum a of a block, since the down step of a valley is the last step of a descent. The element a is moved into the block containing the element corresponding to the down step matched with the up step of the valley. It follows directly from the above bijection that such a down step has label equal to $\beta = \max B + 1$, where B is the block containing a in π. The following figure illustrates these facts.



Now, what happens with the element β? There are essentially two different cases. If the up step of the valley in P_π is followed by another up step, then β is not the maximum of its block in π, and it is easy to check that in ρ it becomes a singleton block (since in P_ρ the corresponding step is preceded and followed by up steps).



If the up step of the valley is followed by a down step, then β is the maximum of its block in π, and it remains in the same block also in ρ, as illustrated in the next figure.



■

Example. Given the partition $2|54|631 \in NC(6)$, there are precisely two partitions covering it, which are $3|54|621$ (2 is moved and 3 is not the maximum of its block) and $2|5|6431$ (4 is moved and 6 is the maximum of its block).

It is interesting to observe that the two "instructions" 1. and 2. in the previous theorem have a striking analogy with the definition of a *filler point* given in [DS]. Indeed, a filler point is produced whenever a valley preceded by an up step is changed into a peak in the associated Dyck path. Thus a filler point in a noncrossing partition corresponds to a down step preceded by a long ascent in the associated Dyck path (where a long ascent is a sequence of two or more consecutive up steps). Therefore, the number of noncrossing partitions of an n-set having k filler points coincides with the number $T_{n,k}$ of Dyck paths of length $2n$ having k long ascents, namely (see [Sl]):

$$T_{n,k} = \frac{1}{n+1}\binom{n+1}{k}\sum_{j=0}^{n-2k}\binom{k+j-1}{k-1}\binom{n+1-k}{n-2k-j}.$$

Our next result is a criterion to compare two given noncrossing partitions. In order to properly state it, we need to introduce a technical definition. Consider a noncrossing partition $\pi \in NC(n)$. We define the *max-vector* of π to be the vector $\max(\pi) = (\mu_1, \ldots, \mu_n)$ such that μ_i is the maximum of the first i elements of π. So, for instance, if $\pi = 2|31|54$, then $\max(\pi) = (2,3,3,5,5)$. We invite the reader to check that the max-vector uniquely determines its associated noncrossing partition. This fact will be very important in the sequel.

Theorem 3.2 *(Characterization of the Bruhat order of NC) Let $\pi, \rho \in NC(n)$. Then $\pi \leq \rho$ if and only if $\max(\pi) \leq \max(\rho)$ in the coordinatewise order.*

Proof. Let $\omega_1 = \omega_1(U,D)$ and $\omega_2 = \omega_2(U,D)$ be the two Dyck paths corresponding to π and ρ, respectively. Then it is clear that $\omega_1 \leq \omega_2$ if and only if every prefix of ω_1 contains at least as many D's as the corresponding prefix of ω_2. This can be translated on partitions using max-vectors. Indeed, if $\max(\pi) = (\mu_1, \ldots, \mu_n)$ and $\max(\rho) = (\nu_1, \ldots, \nu_n)$, consider the two vectors $(\overline{\mu_1}, \ldots, \overline{\mu_n})$ and $(\overline{\nu_1}, \ldots, \overline{\nu_n})$, where $\overline{\mu_i} = \mu_i + i$ and $\overline{\nu_i} = \nu_i + i$. Then, it is not difficult to observe that $\overline{\mu_i}$ and $\overline{\nu_i}$ encode the position of the i-th D in the corresponding Dyck path. From the hypotheses, we have that the i-th D of ω_1 occurs before the i-th D of ω_2, and so $\overline{\mu_i} \leq \overline{\nu_i}$. Since this holds for every $i \leq n$, the thesis follows. ■

Example. Let $\pi = 2|43|51|6$, $\rho = 43|52|61 \in NC(6)$. We easily find $\max(\pi) = (2,4,4,5,5,6)$ and $\max(\rho) = (4,4,5,5,6,6)$. It is immediate to see that $\max(\pi) \leq \max(\rho)$, whence $\pi \leq \rho$.

Remark. Observe that, if $\pi \prec \rho$, then $\max(\pi)$ and $\max(\rho)$ differ precisely in one position.

It is known [FP2] that Dyck lattices possess a rank function (simply because they are distributive lattices) which is essentially given by the *area* bounded by a Dyck path and the x-axis. More precisely, if $A(P)$ is the area of a Dyck path P of length n, then the rank of P inside its Dyck lattice is given by $r(P) = \frac{A(P)-n}{2}$. Our next goal is to translate the parameter "area under Dyck paths" into a parameter on noncrossing partitions, in order to define a rank on the Bruhat noncrossing partition lattices.

Our first result is a formula for the area of Dyck paths in terms of its peaks and valleys. Since we have not found such a formula in the literature, we also propose a proof for the reader's convenience.

Lemma 3.1 *Let P be a Dyck path. Let p_i and v_j denote the height of the i-th peak and the j-th valley of P, respectively. Then*

$$A(P) = \sum_i (p_i^2 - v_i^2). \tag{1}$$

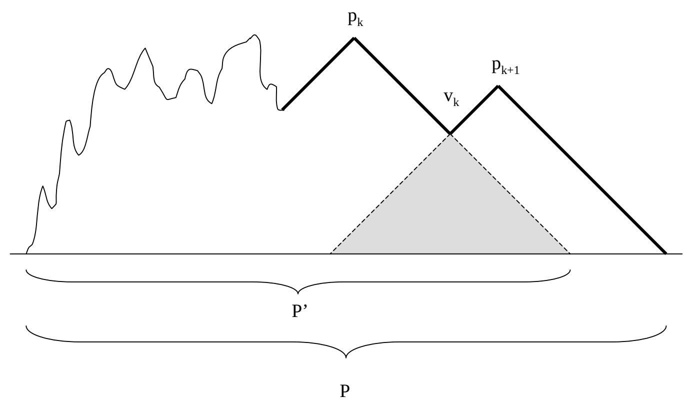


Figure 4: How P' is obtained from P.

Proof. We proceed by induction on the number of peaks. If a Dyck path P has only one peak, then it is the maximum of its Dyck lattice, and the formula immediately follows. Now suppose that P has $k+1$ peaks. Consider the path P' obtained by P by removing the last peak, i.e. coinciding with P up to the k-th peak and then ending with a sequence of down steps (see Figure 4).

It is now easy to see that

$$A(P) = A(P') + p_{k+1}^2 - v_k^2,$$

whence, thanks to the induction hypothesis:

$$A(P) = \sum_i (p_i^2 - v_i^2). \quad \blacksquare$$

Now we are ready to find a formula to express the rank of a partition in the Bruhat noncrossing partition lattice. The proof of the next theorem is left to the reader.

Theorem 3.3 *$NC(n)$ is a distributive lattice, and therefore it is ranked. More precisely, if $\pi = B_1|\dots|B_k \in NC(n)$, then its rank is given by:*

$$r_n(\pi) = \frac{A(\pi) - n}{2}, \tag{2}$$

where

$$A(\pi) = \sum_{i=1}^{k} \left(|B_i| \left(2b_i - 2\sum_{j=1}^{i-1} |B_j| - |B_i| \right) \right) \tag{3}$$

(here $b_i = \max B_i$).

4 Relationship with the strong Bruhat order on permutations

The last formula given for the rank of a noncrossing partition inside its Bruhat lattice is not as easy to understand as the rank function for Dyck paths. In order to find a better way to express this parameter, we make use of the concept of (generalized) pattern avoiding permutation. What we obtain is yet another description of Bruhat noncrossing partition lattices which provides some important information on the (strong) Bruhat order of the symmetric groups.

Proposition 4.1 *Removing the bars in noncrossing partitions defines a bijection between $NC(n)$ and the set $S_n(312)$ of 312-avoiding permutations of $[n]$, for any $n \in \mathbf{N}$.*

Proof. First observe that, for any $n \in \mathbf{N}$, $S_n(312) = S_n(31-2)$, since it is known that these two finite sets are both enumerated by Catalan numbers and obviously $S_n(312) \subseteq S_n(31-2)$. Now, if a pattern 31-2 appears in a noncrossing partition, then, denoting by $b < c < a$ the three elements corresponding to such a pattern, a and b must belong to the same block, and the maximum d of the block containing c must be larger than a (since the maximum of a block in a noncrossing partition is larger than every element preceding it). Thus, the four elements a, b, c, d would constitute a crossing, against the hypothesis. ■

Remark. In the rest of this section we will make an extensive use of the above described canonical bijection. In particular, we will freely switch from a noncrossing partition to its associated 312-avoiding permutation without stating it explicitly. Moreover, we will always use the same Greek letters $(\pi, \rho, \sigma, \ldots)$ to denote both a noncrossing partition and its associated 312-avoiding permutation. Finally, observe that each maximum of a block of a noncrossing partition corresponds to a left-to-right maximum in the corresponding permutation, that is an element which is greater than every other element on its left.

Observe that the composition of the bijection between Dyck paths and noncrossing partitions with the above one between noncrossing partitions and 312-avoiding permutations is precisely the bijection considered in [BK] and in [F]. More specifically, in [BK] the authors show that the area of a Dyck path corresponds to the inversion number of the associated permutation. Since the rank function of the strong Bruhat order on permutations is precisely the inversion number, we are led to conjecture a close relation between our noncrossing partition lattices and the subposets induced by the Bruhat order on 312-avoiding permutations.

Theorem 4.1 *Let $(S_n(312); \leq)$ be the poset obtained by transferring the structure of the Bruhat noncrossing partition lattice $NC(n)$ along the previous bijection. This is precisely the subposet induced on $S_n(312)$ by the strong Bruhat order of the symmetric group S_n. Therefore $S_n(312)$ is a distributive sublattice of S_n endowed with the strong Bruhat order.*

Proof. What we have to show is that the Hasse diagram of the Bruhat noncrossing partition lattice is isomorphic to that of $S_n(312)$ with the induced strong Bruhat order. To do this, it is enough to prove that the sets of elements covering a noncrossing partition and its associated 312-avoiding permutation coincide, via the bar-removing bijection.

Let π, ρ be noncrossing partitions, and suppose that $\pi \prec \rho$ in the Bruhat noncrossing partition lattice. This means that ρ is obtained by π using one of the two rules described in Theorem 3.1. In both cases, ρ is obtained from π by interchanging the minimum a of a block B with $\beta = \max B + 1$. On permutations this means that the inversion number of ρ is larger than that of π (since $a < \beta$). Now to conclude that $\pi \prec \rho$ in $S_n(312)$ it remains only to show that the above transposition does not generate other inversions, or, equivalently, that all the entries between a and β in π are either smaller than a or larger than β. Indeed, $\beta - 1$ is the maximum of B, so it appears before a in π. Hence, if there is an element x such that $a < x < \beta$ and x is between a and β in π, then we would have a pattern 312, which is excluded. Therefore we have shown that, if $\pi \prec \rho$ in $NC(n)$, then also $\pi \prec \rho$ in $S_n(312)$.

To conclude the proof we will show that, if $\pi \prec \rho$ in $S_n(312)$, then necessarily ρ is obtained by π as in Theorem 3.1. From the hypothesis it follows that ρ differs from π by a transposition of a pair of elements a and β. Suppose that $a < \beta$ and so a appears before β in π. If a was not a minimum in the noncrossing partition associated with π, then there would be an entry $x < a$ appearing after a, and so in ρ the elements β, x, a would show a pattern 312. Therefore a must be the minimum of its block B in the noncrossing partition π. Now set $b = \max B$. We claim that $\beta = b + 1$. Indeed, if it is not, then $\beta - 1$ could not appear between a and β in π (since otherwise ρ would contain too many inversions). Clearly $\beta - 1$ can not appear before b too, since every entry before b must be smaller than b. Thus $\beta - 1$ lies necessarily on the right of β in π. But in this case the permutation ρ would contain a pattern 312 in the entries $\beta, a, \beta - 1$, a contradiction. Therefore $\beta = b + 1$, and the theorem is finally proved. ■

At this stage it is worth mentioning the following, remarkable corollary.

Corollary 4.1 *For any $n \in \mathbf{N}$, the Dyck lattice $\mathcal{D}_n$ is isomorphic to the lattice $S_n(312)$ with the strong Bruhat order.*

Our next goal is to find a synthetic description of the meet and join operations in the Bruhat lattices of 312-avoiding permutations.

Let $\pi = \pi_1 \cdots \pi_n$, $\rho = \rho_1 \cdots \rho_n \in S_n(312)$. Define the permutation $\pi \vee \rho = \sigma_1 \cdots \sigma_n$ by setting σ_i equal to the largest element among those smaller than or equal to $\max\{\pi_1, \ldots, \pi_i, \rho_1, \ldots, \rho_i\}$ not yet appeared in some previous positions. Analogously, the permutation $\pi \wedge \rho = \tau_1 \cdots \tau_n$ is defined by setting τ_i equal to the smallest element among those larger than or equal to $\min\{\pi_1, \ldots, \pi_i, \rho_1, \ldots, \rho_i\}$ not yet appeared in some previous positions. For instance, given $\pi = 32657481, \rho = 24378651$ we get $\pi \vee \rho = 34678521$ and $\pi \wedge \rho = 23457681$. In the following proposition we show that the above defined operations actually coincide with the join and meet operations in $S_n(312)$.

Proposition 4.2 *For any $\pi, \rho \in S_n(312)$, the permutations $\pi \vee \rho$ and $\pi \wedge \rho$ are respectively the join and the meet of π and ρ in the Bruhat lattice $S_n(312)$.*

Proof. Let $\max(\pi)$ and $\max(\rho)$ be the max-vectors of the noncrossing partitions associated with π and ρ, respectively. The join of the two Dyck paths associated with π and ρ corresponds to the Dyck path determined by the coordinatewise join of $\max(\pi)$ and $\max(\rho)$, say $\max(\pi) \vee \max(\rho)$, which is then the max-vector of the join of π and ρ in $S_n(312)$. There is a unique 312-avoiding permutation associated with $\max(\pi) \vee \max(\rho)$, which can be obtained as follows: the i-th entry of the permutation is the largest element among those smaller than or equal to the i-th component of the max-vector not yet appeared in the permutation. This corresponds precisely to our definition of $\pi \vee \rho$. The argument for the meet is completely analogous, and so the proof is complete. ■

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