

# THE HIVE MODEL AND THE FACTORISATION OF KOSTKA COEFFICIENTS

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ABSTRACT. The hive model is used to explore the properties of both Kostka coefficients and stretched Kostka coefficient polynomials. It is shown that both of these may factorise, and that they can then be expressed as products of certain primitive coefficients and polynomials, respectively. It is further shown how to determine a sequence of linear factors  $(t + m)$  of the primitive polynomials, where  $t$  is the stretching parameter, as well as a bound on their degree in the form of simple formula which is conjectured to be exact.

RÉSUMÉ. Nous utilisons le modèle des ruches pour étudier les propriétés des coefficients de Kostka et des polynômes associés aux coefficients de Kostka dilatés. Nous montrons que les uns et les autres peuvent se factoriser : ils s'écrivent comme des produits de coefficients (resp. polynômes) primitifs. En outre, nous montrons comment établir une suite de facteurs linéaires  $(t + m)$  des polynômes primitifs ( $t$  est le paramètre de dilatation), et proposons une formule simple donnant une borne supérieure de leur degré.

## 1. INTRODUCTION

There is no doubt that Kostka coefficients,  $K_{\lambda\mu}$  are interesting combinatorial objects. They are indexed by pairs of partitions  $\lambda$  and  $\mu$ , and are non-zero if and only if these partitions have the same weight and  $\lambda$  precedes  $\mu$  with respect to the dominance partial order on partitions [JK]. They count the number of semistandard Young tableaux of shape determined by  $\lambda$  and of weight determined by  $\mu$ , see for example [L, M, S2]. They also count Gelfand-Tsetlin patterns, as described for example in [GT, S2]. These patterns are in bijective correspondence with semistandard Young tableaux and also with certain K-hives, introduced comparatively recently [KTT] as a variation on the hives used to calculate Littlewood-Richardson coefficients [KT, KTW, B]. These K-hives are triangular arrays of non-negative integers with borders of length  $n$  labelled by the parts of the partitions  $0$ ,  $\lambda$  and  $\mu$ , where  $0 = (0, 0, \dots, 0)$ . Counting such K-hives gives  $K_{\lambda\mu}$ .

Multiplying all the parts of the partitions  $\lambda$  and  $\mu$  by a stretching parameter  $t$ , with  $t$  a positive integer, gives new partitions  $t\lambda$  and  $t\mu$ . Since the weights of  $t\lambda$  and  $t\mu$  are simply those of  $\lambda$  and  $\mu$  multiplied by  $t$ , and this scaling preserves dominance partial ordering, it follows that  $K_{t\lambda, t\mu}$  is non-zero if and only if  $K_{\lambda\mu}$  is non-zero.

By way of a non-trivial example, for  $n = 9$ ,  $\lambda = (9, 5, 2, 2, 2, 2, 1)$  and  $\mu = (5, 5, 5, 3, 1, 1, 1, 1, 1)$  it is found quite typically that

$$(1.1) \quad K_{t\lambda, t\mu} = \frac{1}{24}(t+1)^2(t+2)(5t+2)(t^2+3t+6)$$

These functions  $K_{t\lambda, t\mu}$  are necessarily quasi-polynomials in  $t$  since they enumerate integer points of rational polytopes subject to a scaling by  $t$  [E, S1]. However, contrary to initial expectations [BK], these rational polytopes may, and indeed sometimes do, possess non-integral vertices [KTT, DeLM]. Despite this, it has been proved [KR, K, BGR] that  $P_{\lambda\mu}(t) = K_{t\lambda, t\mu}$  is always a polynomial in  $t$ .

A study of such K-polynomials has revealed a number of interesting features that are illustrated in the above example. In particular, it appears that the coefficients in the expansion of  $P_{\lambda\mu}(t)$  are all positive rational numbers. This remains a conjecture [KTT].

Secondly,  $P_{\lambda\mu}(t)$  often contains a sequence of factors  $(t + m)$  for  $m = 1, 2, \dots, M$ , with  $M$  some non-negative integer. In the above example  $M = 2$ . More generally, both the value of  $M$  and the

degree,  $d$  of the K-polynomial appear to be difficult to predict from a knowledge of  $n$ ,  $\lambda$  and  $\mu$ . Although the hive model does give an immediate bound,  $d \leq (n-1)(n-2)/2$ , on the degree, it can be seen in the above  $n = 9$  example that we have  $d = 6$ , so that the bound is far from being saturated.

It is also found that under certain circumstances there exists a factorisation of the form

$$(1.2) \quad P_{\lambda\mu}(t) = P_{\sigma\zeta}(t) P_{\tau\eta}(t)$$

for some  $\sigma$ ,  $\tau$ ,  $\zeta$  and  $\eta$  such that  $\lambda = (\sigma, \tau)$  and  $\mu = (\zeta, \eta)$ , where the notation is intended to signify that the list of parts of  $\lambda$  are simply those of  $\sigma$  followed by those of  $\tau$ , and similarly for  $\mu$ . In the above example we have  $P_{9522221,555311111}(t) = P_{9522,5553}(t) P_{221,11111}(t)$  with

$$(1.3) \quad P_{9522,5553}(t) = \frac{1}{2}(t+1)(5t+2) \quad \text{and} \quad P_{221,11111}(t) = \frac{1}{12}(t+1)(t+2)(t^2+3t+6).$$

In this presentation, our intention is to exploit the hive model to study the properties of stretched Kostka coefficient polynomials. We first derive combinatorially the precise conditions under which such K-polynomials factorise as products of certain primitive K-polynomials. In the case of any primitive K-polynomial we then show how to determine the precise range of values,  $1 \leq m \leq M$ , such that the K-polynomial contains a factor  $(t+m)$ . This is done by giving an interpretation of  $P_{\lambda\mu}(t)$  for negative integer values of  $t$ . Finally, we obtain a formula for an explicit bound on the degree  $d$  of a primitive K-polynomial which we conjecture is always saturated.

Our analysis covers not only the Kostka coefficients  $K_{\lambda\mu}$  and the K-polynomials  $P_{\lambda\mu}(t)$  in which  $\mu$  is a partition, but also  $K_{\lambda\beta}$  and  $P_{\lambda\beta}(t)$  in which  $\beta$  is, more generally, a weight. The combinatorial proof of the factorisation theorem in this case is presented in an Appendix.

## 2. KOSTKA COEFFICIENTS

Let  $n$  be a fixed positive integer. For any  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$  let  $|\alpha| = \alpha_1 + \dots + \alpha_n$  be the weight of  $\alpha$ , let  $\#\alpha = n$  be the number of components of  $\alpha$  and let the symmetric group  $S_n$  act naturally on the components of  $\alpha$ . Let  $\lambda = (\lambda_1, \dots, \lambda_n)$  be a partition of length  $\ell(\lambda) \leq n$  and weight  $|\lambda|$  with  $\lambda_1 \geq \dots \geq \lambda_n \geq 0$ . Then  $|\lambda| = \lambda_1 + \dots + \lambda_n$  with  $\lambda_i > 0$  for  $i \leq \ell(\lambda)$  and  $\lambda_i = 0$  for  $i > \ell(\lambda)$ . It is sometimes convenient to write  $\lambda$  in terms of its distinct parts, that is to set  $\lambda = (\kappa_1^{v_1}, \dots, \kappa_m^{v_m})$  with  $\kappa_1 > \dots > \kappa_m \geq 0$  and  $v_j > 0$  for  $j = 1, \dots, m$ . Then  $|\lambda| = v_1\kappa_1 + \dots + v_m\kappa_m$  with  $v_1 + \dots + v_m = n$ , and  $\ell(\lambda) = n$  if  $\kappa_m > 0$  and  $\ell(\lambda) = n - v_m$  if  $\kappa_m = 0$ .

**Definition 2.1.** Let  $\mathbf{x} = (x_1, \dots, x_n)$ . Then to each partition  $\lambda$  with  $\ell(\lambda) \leq n$  there corresponds a Schur function  $s_\lambda(\mathbf{x})$  defined by

$$(2.1) \quad s_\lambda(\mathbf{x}) = \frac{\left| x_i^{n+\lambda_j-j} \right|_{1 \leq i, j \leq n}}{\left| x_i^{n-j} \right|_{1 \leq i, j \leq n}}.$$

Since  $s_\lambda(\mathbf{x})$  is a ratio of two alternants, it is a symmetric polynomial in the components  $x_1, \dots, x_n$  of  $\mathbf{x}$ . Indeed, the Schur functions  $\{s_\lambda(\mathbf{x}) \mid \ell(\lambda) \leq n\}$  constitute a linear basis of the algebra of symmetric polynomials in the set of variables  $\{x_1, \dots, x_n\}$ .

**Definition 2.2.** The expansion of each Schur function in terms of monomials takes the form

$$(2.2) \quad s_\lambda(\mathbf{x}) = \sum_{\beta} K_{\lambda\beta} \mathbf{x}^\beta,$$

where the summation is over all  $\beta = (\beta_1, \dots, \beta_n)$  and  $\mathbf{x}^\beta = x_1^{\beta_1} \dots x_n^{\beta_n}$ . The coefficients  $K_{\lambda\beta}$  are known as Kostka coefficients.

The fact that  $s_\lambda(\mathbf{x})$  is symmetric implies that the coefficients  $K_{\lambda\beta}$  are insensitive to the permutations of the components of  $\beta$ , that is they possess:

**Property 2.3.** For all  $w \in S_n$  we have  $K_{\lambda, w(\beta)} = K_{\lambda\beta}$ , and there must exist a partition  $\mu$  such that  $w(\beta) = \mu$  for some  $w \in S_n$ , in which case  $K_{\lambda\beta} = K_{\lambda\mu}$ .

To specify those partitions  $\lambda$  and weights  $\beta$  for which  $K_{\lambda\beta}$  is non-zero it is convenient to introduce the notion of partial sums of the parts of partitions and weights, and the dominance partial order.

**Definition 2.4.** For any partition  $\nu = (\nu_1, \nu_2, \dots, \nu_n)$  and any weight  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ , their partial sums are defined by

$$(2.3) \quad ps(\nu)_i = \nu_1 + \nu_2 + \dots + \nu_i \quad \text{and} \quad ps(\alpha)_i = \alpha_1 + \alpha_2 + \dots + \alpha_i \quad \text{for all } i = 1, 2, \dots, n.$$

More generally, for any subset  $I = \{i_1, i_2, \dots, i_r\}$  of  $N = \{1, 2, \dots, n\}$  of cardinality  $\#I = r$  with  $1 \leq r \leq n$  let

$$(2.4) \quad ps(\nu)_I = \nu_{i_1} + \nu_{i_2} + \dots + \nu_{i_r} \quad \text{and} \quad ps(\alpha)_I = \alpha_{i_1} + \alpha_{i_2} + \dots + \alpha_{i_r}.$$

**Definition 2.5.** Given partitions  $\lambda$  and  $\mu$  of lengths  $\ell(\lambda), \ell(\mu) \leq n$ , then  $\lambda$  is said to dominate  $\mu$ , and we write  $\lambda \succeq \mu$ , or more precisely  $\lambda \succeq_n \mu$ , if

$$(2.5) \quad ps(\lambda)_i \geq ps(\mu)_i \quad \text{for all } i = 1, 2, \dots, n \quad \text{and} \quad |\lambda| = |\mu|.$$

Moreover,  $\lambda$  is said to strongly dominate  $\mu$ , and we write  $\lambda \succ \mu$ , or more precisely  $\lambda \succ_n \mu$ , if

$$(2.6) \quad ps(\lambda)_i > ps(\mu)_i \quad \text{for all } i = 1, 2, \dots, n-1 \quad \text{and} \quad |\lambda| = |\mu|.$$

We now have the following important condition for the non-vanishing of Kostka coefficients:

**Theorem 2.6.** Let  $\lambda$  and  $\mu$  be partitions of lengths  $\ell(\lambda), \ell(\mu) \leq n$ , and let  $N = \{1, 2, \dots, n\}$ . Then

$$(2.7) \quad K_{\lambda\mu} > 0 \iff \lambda \succeq_n \mu$$

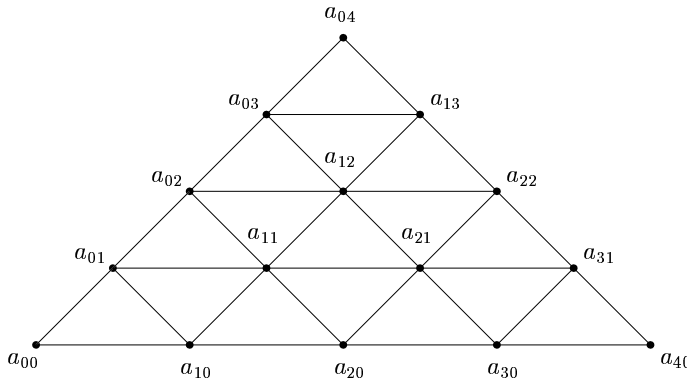
More generally, let  $\lambda$  be a partition of length  $\ell(\lambda) \leq n$  and let  $\beta$  be a weight with  $\#\beta = n$ . Then

$$(2.8) \quad K_{\lambda\beta} > 0 \iff |\lambda| = |\beta| \quad \text{and} \quad ps(\lambda)_i \geq ps(\beta)_I \quad \text{for all } I \subseteq N \quad \text{with } i = \#I > 0.$$

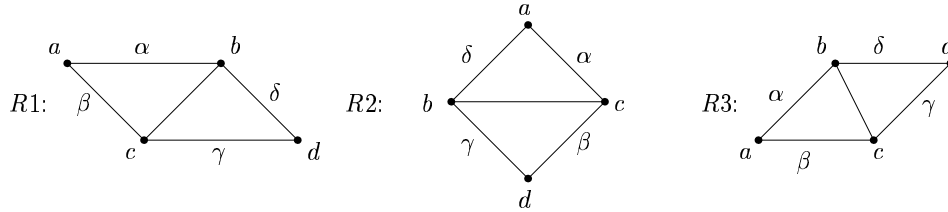
The first set of conditions (2.7) is well-known (see for example [JK]p44), and the second set of conditions (2.8) is a simple corollary following from the fact that if  $\mu = w(\beta)$  for some  $w \in S_n$  then  $ps(\mu)_i = ps(\beta)_I$  for some  $I \subseteq N$  with  $i = \#I$ , and  $ps(\mu)_i \geq ps(\beta)_J$  for all  $J \subseteq N$  with  $i = \#J$ .

### 3. THE HIVE MODEL

An  $n$ -hive is an array of numbers  $a_{ij}$  with  $0 \leq i, j \leq n$  placed at the vertices of an equilateral triangular graph. Typically, for  $n = 4$  their arrangement is as shown below:



Such an  $n$ -hive is said to be an integer hive if all of its entries are non-negative integers. Neighbouring entries define three distinct types of rhombus, each with its own constraint condition.



In each case, with the labelling as shown, the hive condition takes the form:

$$(3.1) \quad b + c \geq a + d$$

In what follows we make use of edge labels more often than vertex labels. Each edge in the hive is labelled by means of the difference,  $\epsilon = q - p$ , between the labels,  $p$  and  $q$ , on the two vertices connected by this edge, with  $q$  always to the right of  $p$ . In all the above cases, with this convention, we have  $\alpha + \delta = \beta + \gamma$ , and the hive conditions take the form:

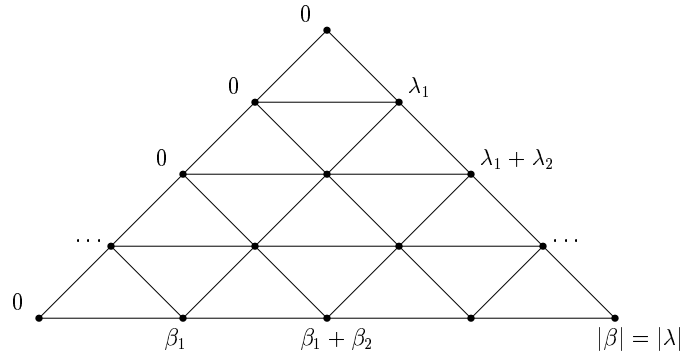
$$(3.2) \quad \alpha \geq \gamma \quad \text{and} \quad \beta \geq \delta,$$

where, of course, either one of the conditions  $\alpha \geq \gamma$  or  $\beta \geq \delta$  is sufficient to imply the other.

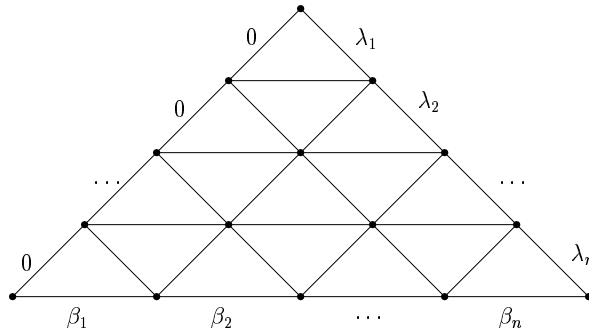
Although, for completeness, we have included hive conditions for all three types of rhombus that appear in a general hive, it is only the type R1 and R2 hive conditions that apply to what we call K-hives. For such K-hives the type R3 hive conditions will, in general, be violated.

**Definition 3.1.** *A K-hive is an integer hive satisfying the hive conditions (3.1), or equivalently (3.2) for all its constituent rhombi of type R1 and R2 (but not R3), with border labels determined by the zero partition or weight  $0 = (0, 0, \dots, 0)$  with  $\#0 = n$ , a partition  $\lambda$  with  $\ell(\lambda) \leq n$  and a weight  $\beta$  with  $\#\beta = n$ , satisfying the constraint  $|\lambda| = |\beta|$ , in such a way that  $a_{0i} = 0$  for  $i = 0, 1, \dots, n$ ,  $a_{j, n-j} = ps(\lambda)_j = \lambda_1 + \lambda_2 + \dots + \lambda_j$  for  $j = 1, 2, \dots, n$ ,  $a_{k,0} = ps(\beta)_k = \beta_1 + \beta_2 + \dots + \beta_k$  for  $k = 1, 2, \dots, n$ .*

Schematically, we have



Alternatively, in terms of edge labels we have:



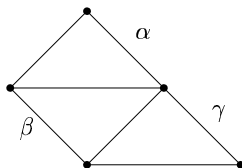
**Proposition 3.2.** [KTT] *The Kostka coefficient  $K_{\lambda\beta}$  is the number of  $K$ -hives with border labels determined as above by  $\lambda$  and  $\beta$ .*

For example, if  $\lambda = (3, 2)$  and  $\beta = (2, 1, 2)$  the corresponding  $K$ -hives take the form:

$$(3.3) \quad \begin{array}{cccc} & & 0 & \\ & & 0 & 3 \\ & 0 & a & 5 \\ 0 & 2 & 3 & 5 \end{array}$$

where for the sake of clarity all the hive edges have been omitted. Since the only integer values of  $a$  satisfying the hive conditions (3.1) for all the constituent rhombi of type R1 and R2 are  $a = 2$  and  $a = 3$ , it follows that  $K_{32,212} = 2$ .

When expressed in terms of edge labels, the hive conditions (3.2) for all constituent rhombi of types R1 and R2 imply that in every  $K$ -hive the edge labels along any line parallel to the right-hand edge of the hive are weakly decreasing from top left to bottom right. This can be seen from the following 5-vertex sub-diagram.



The edge conditions on the rhombi of type R1 and R2 in the above diagram give  $\beta \geq \gamma$  and  $\alpha \geq \beta$ , respectively, so that  $\alpha \geq \gamma$ , as claimed. This is of course consistent with the edges of the right-hand boundary of each  $K$ -hive being specified by a partition  $\lambda$ .

#### 4. STRETCHED COEFFICIENTS

**4.1. Polynomial conjectures.** Now we are in a position to define and evaluate stretched Kostka coefficients. The partition obtained from  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_p)$  by multiplying all of its parts by the same positive integer,  $t$ , is denoted by  $t\lambda = (t\lambda_1, t\lambda_2, \dots, t\lambda_p)$ . With this notation, we refer to  $K_{t\lambda, t\beta}$  as stretched Kostka coefficients, where  $t$  is said to be the stretching parameter. It is not difficult to evaluate these stretched coefficients, particularly through the use of the hive model, for a range of positive integer values of  $t$ .

For example, for  $\lambda = (3, 2)$  and  $\mu = (1, 1, 1, 1, 1)$  the corresponding stretched Kostka coefficients are given by

$$(4.1) \quad K_{t\lambda, t\mu} = \frac{1}{2}(t+1)(t^2 + 2t + 2).$$

The generating function for these coefficients takes the form:

$$(4.2) \quad F_{\lambda\mu}(z) = \sum_{t=0}^{\infty} K_{t\lambda, t\mu} z^t = \frac{1+z+z^2}{(1-z)^4}.$$

On the basis of this and many other examples we were led to the following:

**Conjecture 4.1.** [KTT] *For all partitions  $\lambda$  and  $\mu$  such that  $K_{\lambda\mu} > 0$  there exists a polynomial  $P_{\lambda\mu}(t)$  in  $t$  with positive rational coefficients such that  $P_{\lambda\mu}(0) = 1$  and  $P_{\lambda\mu}(t) = K_{t\lambda, t\mu}$  for all positive integers  $t$ .*

**Conjecture 4.2.** [KTT] *Given that the degree of the polynomial  $K_{t\lambda, t\mu}$  is  $d$ , the generating function for  $K_{t\lambda, t\mu}$  takes the form  $F_{\lambda\mu}(z) = G_{\lambda\mu}(z)/(1-z)^{d+1}$ , with  $G_{\lambda\mu}(z)$  a polynomial of degree  $\leq d$  having non-negative integer coefficients.*

Much of these conjectures has now been proved. In particular the fact that  $P_{\lambda\mu}(t)$  is polynomial appears to have been proved first by Kirillov and Reshetekin [KR], with a more recent proof provided by Billey, Guillemin and Rassart [BGR]. In fact all that remains to be answered are the questions of the positivity of the coefficients in  $P_{\lambda\mu}(t)$  and  $G_{\lambda\mu}(z)$ .

Before looking in more detail at the nature of the polynomials  $P_{\lambda\mu}(t)$  we note:

**Saturation Condition 4.3.** [KTT]

$$(4.3) \quad K_{t\lambda, t\mu} > 0 \iff K_{\lambda\mu} > 0.$$

**Proof** This can be seen immediately from Theorem 2.6 by noting that  $|t\lambda| = t|\lambda|$  and  $ps(t\lambda)_r = tps(\lambda)_r$  for all partitions  $\lambda$  and all  $r$ . It follows that

$$(4.4) \quad K_{\lambda\mu} > 0 \iff |\lambda| = |\mu| \text{ and } \lambda \succeq \mu \iff |t\lambda| = |t\mu| \text{ and } t\lambda \succeq t\mu \iff K_{t\lambda, t\mu} > 0,$$

as required.  $\square$

Turning to the Conjecture 4.1 itself, the following key component has been established:

**Theorem 4.4.** [K, BGR] *Let  $\lambda$  and  $\mu$  be partitions of lengths  $\ell(\lambda), \ell(\mu) \leq n$  such that  $K_{\lambda\mu} > 0$ . Then  $P_{\lambda\mu}(t) = K_{t\lambda, t\mu}$  is a polynomial of degree at most  $(n-1)(n-2)/2$  in  $t$ .*

Clearly, Property 2.3 then implies as a corollary of Theorem 4.4 that if  $K_{\lambda\beta} > 0$  then  $P_{\lambda\beta}(t) = K_{t\lambda, t\beta}$  is also a polynomial in  $t$  with the same upper bound on its degree.

## 5. FACTORISATION

It was noted in the work of Berenstein and Zelevinsky [BZ] that Kostka coefficients factorise. Here we give a combinatorial proof of this factorisation property based on the use of K-hives.

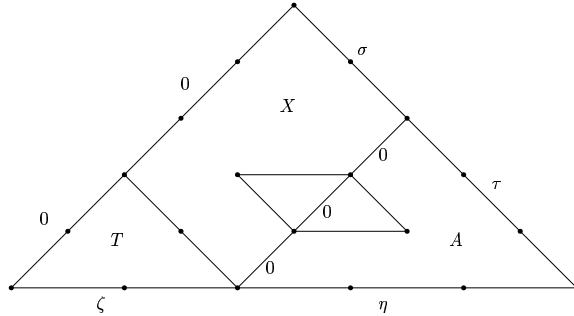
First we deal with the case  $K_{\lambda\mu}$  where  $\lambda$  and  $\mu$  are partitions of the same weight,  $|\lambda| = |\mu|$ , with  $\ell(\lambda), \ell(\mu) \leq n$ . Then following Berenstein and Zelevinsky, we say that  $K_{\lambda\mu}$  is *primitive* if  $\lambda \succ_n \mu$ . If  $K_{\lambda\mu}$  is not primitive then we have the following factorisation:

**Theorem 5.1.** *Let  $\lambda, \mu$  be partitions such that  $|\lambda| = |\mu|$  and  $\ell(\lambda), \ell(\mu) \leq n$ , with  $\lambda \succeq_n \mu$  and  $ps(\lambda)_r = ps(\mu)_r$  for some  $r$  for which  $1 \leq r < n$ . Let  $\lambda = (\sigma, \tau)$  and  $\mu = (\zeta, \eta)$ , with  $\ell(\sigma) = \ell(\zeta) = r$  and  $\ell(\tau) = \ell(\eta) = n - r$ , then*

$$(5.1) \quad K_{\lambda\mu} = K_{\sigma\zeta} K_{\tau\eta}.$$

**Proof** First it should be noted that the hypothesis  $\lambda \succeq_n \mu$  implies that  $K_{\lambda\mu} > 0$ . Moreover, we have  $ps(\sigma)_i = ps(\lambda)_i \geq ps(\mu)_i = ps(\zeta)_i$  for  $i = 1, 2, \dots, r-1$  and  $|\sigma| = ps(\sigma)_r = ps(\lambda)_r = ps(\mu)_r = ps(\zeta)_r = |\zeta|$ . Thus  $\sigma \succeq_r \zeta$  and hence  $K_{\sigma\zeta} > 0$ . In addition we have  $ps(\tau)_i = ps(\lambda)_{r+i} - ps(\lambda)_r = ps(\lambda)_{r+i} - |\sigma| \geq ps(\mu)_{r+i} - |\zeta| = ps(\mu)_{r+i} - ps(\mu)_i = ps(\eta)_i$  for  $i = 1, 2, \dots, n-r$ , and  $|\tau| = |\lambda| - |\sigma| = |\mu| - |\zeta| = |\eta|$ . Thus  $\tau \succeq_{n-r} \eta$  and hence  $K_{\tau\eta} > 0$ .

Now consider the K-hive with boundary determined by  $\lambda = (\sigma, \tau)$

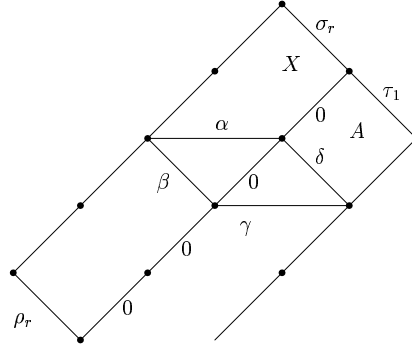


The rules about edge sums are such that on the boundary of the triangular region  $T$  we have  $|\zeta| = |0| + |\rho| = |\rho|$ . Since  $|\zeta| = |\sigma|$  we have  $|\rho| = |\sigma|$ . The same rules applied to the parallelogram  $X$  then imply that the sum of the edge lengths on the boundary between  $X$  and  $A$  must be  $|0| + |\sigma| - |\rho| = 0$ . Since  $X$  can be viewed as a collection of rhombi of type R2, applying the hive condition  $\beta \geq \delta$  to each of these rhombi implies that the edge lengths on the lower right boundary of  $X$  must be non-negative. Since their sum is zero, they must all be zero, as indicated in the diagram. The  $\alpha \geq \gamma$  hive condition for R2 may then be applied to these same rhombi constituting  $X$ , yielding the constraints  $\rho_i \leq \sigma_i$  for  $i = 1, 2, \dots, r$ . Since  $|\rho| = |\sigma|$ , we must have  $\rho_i = \sigma_i$  for  $i = 1, 2, \dots, r$ , that is  $\rho = \sigma$ .

Thus each K-hive  $H$  with boundary  $\lambda$  and  $\mu$  consists of a parallelogram  $X$  in which all edge lengths are fixed, together with a K-hive  $T$  with boundary  $\sigma$  and  $\zeta$ , and a second K-hive  $A$  with boundary  $\tau$  and  $\eta$ . The hive conditions for  $H$  imply those appropriate to  $T$  and  $A$ .

In order to prove the required factorisation (5.1) it only remains to show that combining all possible hives  $T$  and  $A$  with a parallelogram  $X$  of the appropriate boundary gives a hive  $H$  in which all the hive conditions corresponding to rhombi crossing the boundaries of  $T$  with  $X$ , and  $A$  with  $X$ , are automatically satisfied. Since, for K-hives we only use the rhombi of type R1 and R2, it is clear that no such rhombus crosses the boundary between  $T$  and  $X$ , while only rhombi of type R1 cross the boundary between  $A$  and  $X$ . One such rhombus has been shown above.

We may look in more detail at such a rhombus by means of the following diagram which shows a strip one edge length wide on either side of the  $A - X$  boundary.



With the edge labels as shown, the R2 rhombus condition applied in the region  $A$  just below the  $A - X$  boundary gives  $\delta \leq \tau_1$ , while the same rhombus condition applied in the region  $X$  just above the  $A - X$  boundary gives  $\rho_r \leq \beta \leq \sigma_r$ . Since  $\rho_r = \sigma_r$  and  $\tau_1 = \lambda_{r+1} \leq \lambda_r = \sigma_r$ , we have  $\delta \leq \tau_1 \leq \sigma_r = \beta$ . However, this is just what is required to satisfy the R1 condition (3.2).

This completes the proof of the K-hive factorisation theorem.  $\square$

Repeated use of this theorem allows any non-primitive Kostka coefficient  $K_{\lambda\mu}$  to be expressed as a unique product of primitive Kostka coefficients  $K_{\sigma\zeta}$ . It is only necessary at each stage to factor out the term meeting the hypotheses of the above Theorem 5.1 with the smallest possible value of  $r$ .

Applying the above argument to stretched Kostka coefficients immediately gives the following:

**Theorem 5.2.** *Let  $\lambda, \mu$  be partitions such that  $|\lambda| = |\mu|$  and  $\ell(\lambda), \ell(\mu) \leq n$ , with  $\lambda \succeq_n \mu$  and  $ps(\lambda)_r = ps(\mu)_r$  for some  $r$  for which  $1 \leq r < n$ . Let  $\lambda = (\sigma, \tau)$  and  $\mu = (\zeta, \eta)$ , with  $\ell(\sigma) = \ell(\zeta) = r$  and  $\ell(\tau), \ell(\eta) \leq n - r$ , then for all positive integers  $t$  we have*

$$(5.2) \quad P_{\lambda\mu}(t) = P_{\sigma\zeta}(t) P_{\tau,\eta}(t).$$

An example of this type has been provided in the introduction in (1.3).

It is quite instructive from a combinatorial point of view, although not strictly necessary by virtue of Property 2.3, to extend the above analysis to the case of Kostka coefficients  $K_{\lambda\beta}$  for which the weight  $\beta$  is not necessarily a partition. In this case  $K_{\lambda\beta}$  is said to be primitive if  $|\lambda| = |\beta|$  and  $ps(\lambda)_r \geq ps(\beta)_I$  for any proper subset  $I$  of  $N = \{1, 2, \dots, n\}$  with  $r = \#I$ .

In the non-primitive case, we have:

**Theorem 5.3.** *Let  $\lambda$  be a partition with  $\ell(\lambda) \leq n$ , and let  $\beta = (\beta_1, \beta_2, \dots, \beta_n)$  be a weight such that  $|\lambda| = |\beta|$  and  $ps(\lambda)_i \geq ps(\beta)_I$  for any  $I \subset N$ , with  $i = \#I$ . Then  $K_{\lambda\beta}$  is not primitive if there exists a proper subset  $I = \{i_1, i_2, \dots, i_r\}$  of  $N$  such that  $ps(\lambda)_r = ps(\beta)_I$  with  $r = \#I$  for some  $r$  for which  $1 \leq r < n$ . Let the complement of  $I$  in  $N$  be denoted by  $\bar{I} = \{j_1, j_2, \dots, j_{n-r}\}$ . In such a case let  $\lambda = (\sigma, \tau)$ ,  $\zeta = (\beta_{i_1}, \beta_{i_2}, \dots, \beta_{i_r})$  and  $\eta = (\beta_{j_1}, \beta_{j_2}, \dots, \beta_{j_{n-r}})$ , so that  $\beta = \zeta \cup \eta$ , with  $\zeta$  and  $\eta$  weights, not necessarily partitions, with  $\#\zeta = r$  and  $\#\eta = n - r$ , satisfying the partial sum constraints  $|\zeta| = |\sigma|$  and  $|\eta| = |\tau|$ . Then*

$$(5.3) \quad K_{\lambda\beta} = K_{\sigma\zeta} K_{\tau\eta}.$$

A complete combinatorial proof of this theorem, based on the use of hives, has not been included due to space limitations.

Once again scaling everything by  $t$  is straightforward. If  $K_{\lambda\beta}$  is primitive then so is  $K_{t\lambda,t\beta}$  since all the partial sum conditions are scaled by the same factor  $t$ . By the same token the factorisation occurs in the stretched non-primitive case just as it does in the unstretched non-primitive case – all boundary edges are simply scaled by  $t$ , and as we have shown, it is these boundary edges that completely determine the factorisation. Thus under the hypotheses of Theorem 5.3 we have

$$(5.4) \quad P_{\lambda\beta}(t) = P_{\sigma\zeta}(t) P_{\tau\eta}(t).$$

This is illustrated for  $n = 6$  and  $r = 3$  by  $\lambda = (9, 6, 4, 4, 2, 0)$  and  $\beta = (2, 6, 3, 7, 6, 1)$ , for which  $ps(\lambda)_3 = 19 = ps(\beta)_I$  with  $I = \{2, 4, 5\}$ . In this case  $\sigma = (9, 6, 4)$ ,  $\tau = (4, 2, 0)$ ,  $\zeta = (6, 7, 6)$  and  $\eta = (2, 3, 1)$ . Correspondingly we find

$$(5.5) \quad P_{96442,263761} = (t+1)(2t+1), \quad P_{964,676} = (2t+1), \quad P_{42,231} = (t+1),$$

thereby exemplifying the factorisation (5.4).

It is interesting to note that the above factorisation of Kostka coefficients can be extended to the case of  $q$ -dependent Kostka-Foulkes polynomials,  $K_{\lambda\mu}(q)$ . These have a combinatorial definition in terms of the charge statistic on semistandard Young tableaux [M]p242. By way of illustration, in the case of the above example we find

$$(5.6) \quad K_{t(96442),t(766321)}(q) = q^{3t} \frac{q^{t+1} - 1}{q - 1} \frac{q^{2t+1} - 1}{q - 1}$$

$$(5.7) \quad K_{t(964),t(766)}(q) = q^{2t} \frac{q^{2t+1} - 1}{q - 1}$$

$$(5.8) \quad K_{t(42),t(321)}(q) = q^t \frac{q^{t+1} - 1}{q - 1}.$$

## 6. THE ZEROS OF STRETCHED KOSTKA POLYNOMIALS

As has been noted the stretched Kostka polynomials  $P_{\lambda\mu}(t)$  contain factors  $(t + m)$  for some sequence of values  $m = 1, 2, \dots, M$  for some positive integer  $M$ . This is no accident. In this section we describe a method of determining  $M$  for stretched Kostka polynomials. It will be recalled that the Kostka coefficients are defined in terms of Schur functions by (2.2). It then follows from (2.1) that  $P_{\lambda\mu}(t) = K_{t\lambda,t\mu}$  is the coefficient of  $x_1^{t\mu_1} x_2^{t\mu_2} \dots x_n^{t\mu_n}$  in the expansion of

$$(6.1) \quad s_{\lambda}(\mathbf{x}) = \frac{|x_i^{t\lambda_j+n-j}|}{|x_i^{n-j}|}$$

This may be readily extended to the case  $t = -m$  with  $m$  a positive integer. Then  $P_{\lambda\mu}(-m) = K_{-m\lambda,-m\mu}$  is the coefficient of  $x_1^{-m\mu_1} x_2^{-m\mu_2} \dots x_n^{-m\mu_n}$  in the expansion of  $s_{-m\lambda}(\mathbf{x})$ . However

$$(6.2) \quad s_{-m\lambda}(\mathbf{x}) = \frac{|x_i^{-m\lambda_j+n-j}|}{|x_i^{n-j}|} = \frac{|x_i^{-m\lambda_{n-k+1}-n+k}|}{|x_i^{-n+k}|},$$

where first  $x_i^{n-1}$  has been extracted as a common factor from the  $i$ th row of each determinant for  $i = 1, 2, \dots, n$  and cancelled from numerator and denominator, and then  $j$  replaced by  $k = n - j + 1$  with an appropriate reversal of order of the columns in both determinants. If we now set  $\bar{x}_i = x_i^{-1}$  for  $i = 1, 2, \dots, n$  and  $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)$ , this gives

$$(6.3) \quad s_{-m\lambda}(x_1, x_2, \dots, x_n) = \frac{|\bar{x}_i^{m\lambda_{n-k+1}+n-k}|}{|\bar{x}_i^{n-k}|} = s_{m\lambda_n, \dots, m\lambda_2, m\lambda_1}(\bar{\mathbf{x}}).$$

Since  $x_1^{-m\mu_1} x_2^{-m\mu_2} \dots x_n^{-m\mu_n} = \bar{x}_1^{m\mu_1} \bar{x}_2^{m\mu_2} \dots \bar{x}_n^{m\mu_n}$  it follows that  $P_{\lambda\mu}(-m) = K_{-m\lambda,-m\mu}$  is the coefficient of  $\bar{x}_1^{m\mu_1} \bar{x}_2^{m\mu_2} \dots \bar{x}_n^{m\mu_n}$  in the expansion of the right hand side. It then follows, replacing the dummy variables  $\bar{x}_i$  by  $x_i$  for  $i = 1, 2, \dots, n$ , that  $P_{\lambda\mu}(-m) = K_{-m\lambda,-m\mu}$  is the coefficient of



$x_1^{m\mu_1} x_2^{m\mu_2} \cdots x_n^{m\mu_n}$  in the expansion of  $s_{m\tilde{\lambda}}(\mathbf{x})$ , where  $\tilde{\lambda} = (\lambda_n, \dots, \lambda_2, \lambda_1)$  is the weight obtained by reversing the order of the parts of  $\lambda$ , including any trailing zeros. Then to exploit the above it is only necessary to standardise

$$(6.4) \quad s_{m\tilde{\lambda}}(\mathbf{x}) = s_{m\lambda_n, \dots, m\lambda_2, m\lambda_1}(x_1, x_2, \dots, x_n) = \frac{|x_i^{m\lambda_n - k + 1 + n - k}|}{|x_i^{n - k}|}.$$

This is carried out in the usual way [L] by reordering the columns of the numerator determinant. However there are two quite different possible outcomes:

$$(6.5) \quad s_{m\tilde{\lambda}}(\mathbf{x}) = \begin{cases} 0 & \text{case (i);} \\ \eta_\rho s_\rho(\mathbf{x}) \text{ with } \eta_\rho = \pm 1 & \text{case (ii).} \end{cases}$$

For example, in the case  $n = 5$  and  $\lambda = (4, 2, 1, 0, 0)$  we have  $\tilde{\lambda} = (0, 0, 1, 2, 4)$ . For  $m = 1$  this gives  $s_{m\tilde{\lambda}}(\mathbf{x}) = s_{0,0,1,2,4}(\mathbf{x}) = 0$ . Similarly for  $m = 2$  we have  $s_{m\tilde{\lambda}}(\mathbf{x}) = s_{0,0,2,4,8}(\mathbf{x}) = 0$ . However, for  $m = 3$  we obtain  $s_{m\tilde{\lambda}}(\mathbf{x}) = s_{0,0,3,6,12}(\mathbf{x}) = -s_{8,4,3,3,3}(\mathbf{x})$ , while for  $m = 5$  we have  $s_{m\tilde{\lambda}}(\mathbf{x}) = s_{0,0,4,8,16}(\mathbf{x}) = -s_{12,6,4,3,3}(\mathbf{x})$ .

More generally, the case (i) result 0 arises if and only if

$$(6.6) \quad m\lambda_{n-k+1} + n - k = m\lambda_{n-l+1} + n - l$$

for some  $k$  and  $l$  such that  $1 \leq l < k \leq n$ . In all other cases the formula given in case (ii) applies for some partition  $\rho$  of length at most  $n$  and weight  $m|\lambda|$ , and  $\eta_\rho = \pm 1$  is a sign factor recording the number of transpositions of columns of the numerator determinant of (6.4) required to standardise the Schur function.

It follows that we can expect there to be a possibility of two types of zero of  $P_{\lambda\mu}(t)$  for  $t = -m$ : type (i) associated with case (i) of (6.5), and type (ii) associated under certain conditions on  $\rho$  and  $m\mu$  with case (ii). Indeed adopting the notation of (6.5) we have the following:

**Proposition 6.1.** *Let  $\lambda$  and  $\mu$  be such that  $K_{\lambda\mu}$  is primitive. Then  $P_{\lambda\mu}(t) = K_{t\lambda, t\mu}$  contains a factor  $(t + m)$  if and only if either case (i) applies and*

$$(6.7) \quad m = (j - i)/(\lambda_i - \lambda_j)$$

for some  $i$  and  $j$  such that  $1 \leq i < j \leq n$ , or case (ii) applies and

$$(6.8) \quad ps(\rho)_k < m ps(\mu)_k$$

for some  $k$  such that  $1 \leq k < n$ .

**Proof** We have already noted that  $P_{\lambda\mu}(-m) = K_{-m\lambda, -m\mu}$  is the coefficient of  $x_1^{m\mu_1} x_2^{m\mu_2} \cdots x_n^{m\mu_n}$  in  $s_{m\tilde{\lambda}}(\mathbf{x}) = s_{m\lambda_n, \dots, m\lambda_2, m\lambda_1}(x_1, x_2, \dots, x_n)$ . Thus if case (i) of (6.5) applies then  $P_{\lambda\mu}(-m) = 0$ , while if case (ii) applies then  $P_{\lambda\mu}(-m) = \eta_\rho K_{\rho, m\mu}$ . The conditions for case (i) to apply are given in (6.6). However, setting  $i = n - k + 1$  and  $j = n - l + 1$  leads immediately to the required case (i) conditions (6.7) for  $P_{\lambda\mu}(-m) = 0$ . Turning to case (ii) of (6.5)  $P_{\lambda\mu}(-m) = \eta_\rho K_{\rho, m\mu}$  will be zero by virtue of Theorem 2.6 if any one of the partial sum conditions  $ps(\rho)_k \geq m ps(\mu)_k$  for all  $k = 1, 2, \dots, n$  is violated. Since  $|\rho| = m|\mu|$  this leaves precisely the required case (ii) conditions (6.8) for  $P_{\lambda\mu}(-m) = 0$ .  $\square$

We may extend our earlier example with  $n = 5$  and  $\lambda = (4, 2, 1, 0, 0)$  by taking  $\mu = (3, 1, 1, 1, 1)$  for illustrative purposes. We have already seen that there are type (i) zeros, independent of  $\mu$  that arise for  $m = 1$  and  $m = 2$ . These arise from (6.7) for the pairs  $(i, j) = (3, 4)$  and  $(3, 5)$ , respectively, leading to  $(j - i)/(\lambda_i - \lambda_j) = (4 - 3)/(1 - 0) = 1$  and  $(5 - 3)/(1 - 0) = 2$ . For  $m = 3$  we have a type (ii) zero since  $P_{\lambda\mu}(-3) = \eta_\rho K_{\rho, m\mu} = -K_{84333, 93333} = 0$  by virtue of the fact that  $ps(\rho)_1 = 8 < 9 = ps(m\mu)_1$ . On the other hand for  $m = 4$  we have  $P_{\lambda\mu}(-4) = \eta_\rho K_{\rho, m\mu} = -K_{126433, 124444} = -3$  which is non-zero, and the same will be true for all  $m \geq 4$ .

In view of our earlier remarks about the sequence of zeros of  $P_{\lambda\mu}(t)$  for  $t = -m$  with  $m = 1, 2, \dots, M$ , it comes as no surprise that we are able to elaborate on the above and establish that the zeros are indeed consecutive and that for any primitive  $K_{\lambda\mu}$  we can always find a finite  $M > 0$  such that  $P_{\lambda\mu}(-m) = 0$  for all  $m = 1, 2, \dots, M$ , but thereafter  $P_{\lambda\mu}(-m) \neq 0$  for all  $m > M$ .

All this is borne out in our example with  $n = 5$ ,  $\lambda = (4, 2, 1, 0, 0)$  and  $\mu = (3, 1, 1, 1, 1)$  for which

$$(6.9) \quad P_{\lambda\mu}(t) = K_{t\lambda,t\mu} = \frac{1}{60}(t+1)(t+2)(t+3)(3t^2+7t+10),$$

for which  $M = 3$  and we have consecutive zeros at  $t = -m$  with  $m = 1, 2, 3$ . Of these, as we have seen, the first two are type (i) and the third is type (ii). The first non-zero case occurs with  $t = -4$  and we obtain from the explicit formula (6.9) the result  $P_{\lambda\mu}(-4) = -3$  that we had found earlier. In fact for  $t = -m$  it is clear that (6.9) gives  $P_{\lambda\mu}(-m) < 0$  for all  $m > M = 3$ .

## 7. THE DEGREES OF STRETCHED KOSTKA POLYNOMIALS

As we have seen the calculation of stretched Kostka polynomials may be reduced to that of calculating these polynomials in primitive cases only. Even so the task may be combinatorially formidable. In any given case a knowledge of the degree of the polynomial would be extremely advantageous. Here we establish an upper bound on this degree by means of the following rather innocuous looking result. Let the edge labelling of a particular 5-vertex subset of a K-hive be as shown below in the left hand diagram, with two identical labels  $\alpha$ .

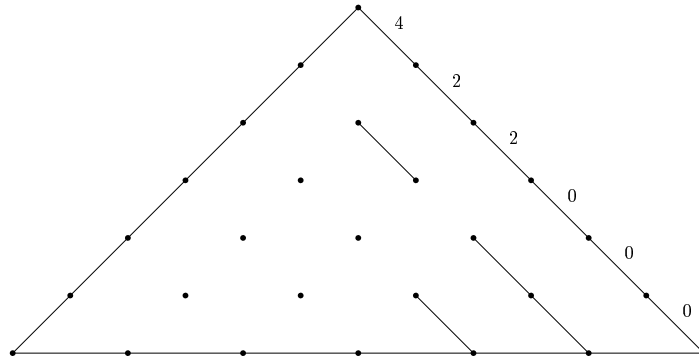


This diagram contains rhombi of type R1 and R2. The hive conditions (3.2) for the former imply  $\alpha \leq \beta$ , while those for the latter give  $\beta \leq \alpha$ . It follows that  $\beta = \alpha$ . This result is displayed more simply by deleting all the edges except those sharing the same label  $\alpha$ , and suppressing the label itself. This gives the right hand diagram where the equality of a pair of edge labels in a linear sequence forces an identical edge label in a neighbouring line.

Applying these notions to the case of K-hives with boundaries of length  $n$  and with border labels determined by  $0, \lambda$  and  $\mu$ , it follows from the above that any equalities of successive parts of  $\lambda$  propagate as equalities of edge labels within each possible K-hive. To be more precise let all the  $\lambda$ -boundary edges be labelled by the parts of  $\lambda$ . If any sequence of parts of  $\lambda$  share the same value, say  $\alpha$ , then we can identify an equilateral sub-hive having the sequence of equally labelled edges as one boundary, with its other boundaries parallel to the  $0$  and  $\mu$ -boundaries of the original hive. Within this sub-hive all the vertices along lines parallel to the  $\lambda$ -boundary are to be connected by edges indicating that in any K-hive the differences in values between neighbouring entries along these lines are all  $\alpha$ .

This process is to be repeated for all sequences of equal edge labels along the  $\lambda$ -boundary. Finally, all neighbouring vertices on all three boundaries are to be connected by edges. In this way we arrive at a graph  $G_{n,\lambda}$  that depends only upon  $n$  and  $\lambda$ .

For example, for  $n = 6$  and  $\lambda = (4, 2, 2, 0, 0, 0)$  the graph  $G_{n,\lambda}$  takes the form:



With this notation we have:

**Proposition 7.1.** *Let  $\lambda$  and  $\mu$  be partitions such that  $K_{\lambda\mu} > 0$ . Let  $\deg(P_{\lambda\mu}(t))$  be the degree of the corresponding stretched K-polynomial  $P_{\lambda\mu}(t) = K_{t\lambda,t\mu}$ . Let  $d(G_{n;\lambda})$  be the number of connected components of the graph  $G_{n;\lambda}$  that are not connected to the boundary. Then*

$$(7.1) \quad \deg(P_{\lambda\mu}(t)) \leq d(G_{n;\lambda}).$$

Moreover, if  $\lambda = (w_1^{v_1}, w_2^{v_2}, \dots, w_t^{v_t})$ , with  $w_1 > w_2 > \dots > w_t \geq 0$ ,  $v_s \geq 1$  for all  $s = 1, 2, \dots, t$ , with  $t \geq 2$ , and  $v_1 + v_2 + \dots + v_t = n$ , then

$$(7.2) \quad d(G_{n;\lambda}) = \frac{1}{2}(n-1)(n-2) - \sum_{s=1}^t \frac{1}{2}v_s(v_s-1).$$

**Proof** Any labelling of the vertices of the hive must be such that the difference  $\alpha$  in values along each interior edge of the graph  $G_{n;\lambda}$  is, by construction of those edges, equal to a corresponding edge value on the parallel  $\lambda$ -boundary, that is  $\alpha = \lambda_i = w_s$  for some  $i$  and a corresponding  $s$ . For each connected component this fixes the values of all vertex labels in terms of that of any one vertex of that component. In the case of a component connected to the boundary, the value at the boundary vertex fixes all the others in that component. The maximum number of degrees of freedom in assigning entries to the corresponding set of K-hives with given boundary is then  $d(G_{n;\lambda})$ .

The application of the stretching parameter  $t$  leaves  $G_{n;\lambda}$  unaltered, that is  $G_{n;t\lambda} = G_{n;\lambda}$ , so that the number of degrees of freedom in assigning entries to the stretched K-hives is still  $d(G_{n;\lambda})$ . For each connected component of  $G_{n;\lambda}$  that is not connected to the boundary we may select any one convenient vertex. The values of its label are not fixed by the hive constraints. They must satisfy a set of linear inequalities that are all scaled by  $t$  as both the boundary vertex and edge labels are scaled by the stretching parameter  $t$ . It follows that the range of allowed values of the label must be either independent of  $t$  or linear in  $t$ , thereby giving rise to a corresponding contribution to  $P_{\lambda\mu}(t)$  that is at most linear in  $t$ . The number of degrees of freedom,  $d(G_{n;\lambda})$ , that we have identified therefore gives an upper bound on the degree of the corresponding stretched K-polynomial in  $t$ .

The number of internal vertices of  $G_{n;\lambda}$  is  $(n-1)(n-2)/2$ . For given  $n$  and arbitrary  $\lambda$  this provides a preliminary upper bound on the degree of  $P_{\lambda\mu}(t)$ . However, for  $\lambda = (w_1^{v_1}, w_2^{v_2}, \dots, w_t^{v_t})$ , each  $s$  such that  $v_s > 1$  specifies a sequence of components of  $\lambda$  having the same value, namely  $w_s$ . There is a corresponding set of  $v_s(v_s-1)/2$  interior edges within  $G_{n;\lambda}$ . Whether or not these interior edges reach the boundary, as they do in the two cases  $s=1$  and  $s=t$ , their introduction reduces the number of connected components not linked to the boundary by  $v_s(v_s-1)/2$ . The result (7.2) then follows by noting that for different  $s$  the sets of vertices linked by the interior edges are disjoint, so that their contributions to the reduction of  $d(G_{n;\lambda})$  are independent.  $\square$

In the case of our example with  $n=6$  and  $\lambda = (4, 2, 2, 0, 0, 0) = (4, 2^2, 0^3)$  we can see from the graph of  $G_{n;\lambda}$  that  $d(G_{n;\lambda}) = 6$ , in agreement with the formula (7.2) that gives  $d(G_{n;\lambda}) = 5 \cdot 4/2 - 2 \cdot 1/2 - 3 \cdot 2/2 = 6$ . Hence for all  $\mu$  the degree of the corresponding stretched Kostka coefficient polynomial  $P_{\lambda\mu}(t)$  must satisfy

$$(7.3) \quad \deg(P_{\lambda\mu}(t)) \leq 6.$$

By way of example, for  $\mu = (3, 1, 1, 1, 1, 1)$  we find

$$(7.4) \quad P_{\lambda\mu}(t) = \frac{1}{72}(t+1)(t+2)(t+3)(t+4)(t^2+2t+3) \quad \text{so that} \quad F_{\lambda\mu}(z) = \frac{1+3z+6z^2}{(1-z)^7},$$

while for  $\mu = (2, 2, 1, 1, 1, 1)$  we have

$$(7.5) \quad P_{\lambda\mu}(t) = \frac{1}{60}(t+1)(t+2)(t+3)(3t+5)(t^2+2t+2) \quad \text{so that} \quad F_{\lambda\mu}(z) = \frac{1+9z+19z^2+7z^3}{(1-z)^7}.$$

On the other hand if  $\mu = (3, 2, 1, 1, 1, 0)$  we find

$$(7.6) \quad P_{\lambda\mu}(t) = \frac{1}{24}(t+1)(t+2)(t+3)(t+4) \quad \text{so that} \quad F_{\lambda\mu}(z) = \frac{1}{(1-z)^5}.$$

The third case in which  $\deg(P_{\lambda\mu}(t)) = 4 < 6$  is one for which  $K_{\lambda\mu}$  is not primitive. In both the other cases  $K_{\lambda\mu}$  is primitive, and we have  $\deg(P_{\lambda\mu}(t)) = 6$  so that the bound (7.1) is saturated. On the basis of very many calculations of this type we are led to the following:

**Conjecture 7.2.** *If  $K_{\lambda,\mu} > 0$  is primitive and  $P_{\lambda\mu}(t) = K_{t\lambda,t\mu}$ , then  $\deg(P_{\lambda\mu}(t)) = d(G_{n;\lambda})$  for all  $\mu$ .*

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