RIBBON TABLEAUX, RIGGED CONFIGURATIONS AND HALL-LITTLEWOOD FUNCTIONS AT ROOTS OF UNITY

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Abstract: Hall-Littlewood functions indexed by rectangular partitions, specialized at primitive roots of unity, can be expressed as plethysms. We propose a combinatorial proof of this formula using Schilling's bijection between ribbon tableaux and rigged configurations [13].

Résumé: La spécialisation aux racines de l'unité des fonctions de Hall-Littlewood indexées par des partitions rectangulaires peut s'exprimer à l'aide de pléthysmes. On propose une preuve combinatoire de cette formule en utilisant la bijection de Schilling entre les tableaux de rubans et les configurations [13].

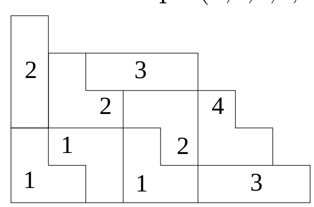
1. Introduction

In [7, 8], Lascoux, Leclerc and Thibon proved a formula for Hall-Littlewood functions, when the parameter is set to a root of unity.
This formula implies a combinatorial interpretation of the plethysms $l_k^{(j)}[h_\lambda]$ and $l_k^{(j)}[e_\lambda]$ where h_λ, e_λ are respectively products of complete and elementary symmetric functions, and $l_k^{(j)}$ the Frobenius characteristics of representations induced by a transitive cyclic subgroup of $\mathfrak{S}_k$.
However, the combinatorial interpretation of the plethysms of Schur functions $l_k^{(j)}[s_\lambda]$ would be far more interesting. This question led the same authors to introduce a new basis $H_\lambda^{(k)}(X;q)$ of symmetric functions, depending on an integer $k \geq 1$ and a parameter q, which interpolate between Schur functions ($k = 1$) and Hall-Littlewood functions $Q'_\lambda(X;q)$ (for $k \geq l(\lambda)$). These were conjectured to behave similarly under specialization at root of unity, and to provide a combinatorial expression of the expansion of the plethysm $l_k^{(j)}[s_\lambda]$ in the Schur basis for suitable values of the parameters. This conjecture has been proved only in two cases: the stable case, which reduces to the previous result on Hall-Littlewood functions, and $k = 2$ which gives the symmetric and antisymmetric squares $h_2[s_\lambda]$ and $e_2[s_\lambda]$.
The proof given in [1] relies upon the study of diagonal classes of domino tableaux, i.e. sets of domino tableaux having the same diagonals. Carré and Leclerc proved that the spin polynomial of such a class has the form $(1+q)^a q^b$, and from this obtained the specialization $H_{\lambda\cup\lambda}^{(2)}(X;-1)$.
The aim of this note is to provide a similar proof for the stable case, that is, to show that the result on Hall-Littlewood functions at roots of unity follows from an explicit formula for the spin polynomials of certain diagonal classes of ribbon tableaux, which turn out to have a very simple characterization through Schilling's bijection [13].

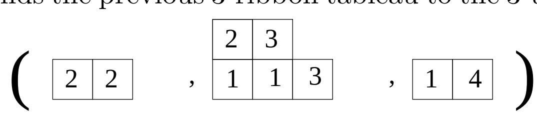
2. Basic definitions on ribbon tableaux

For a partition $\lambda = (\lambda_1, \ldots, \lambda_p)$, we write $l(\lambda)$ its length, $|\lambda|$ its weight and ${}^t\lambda$ its conjugate. With λ is associated a k-core $\lambda_{(k)}$ and a k-quotient $\lambda^{(k)}$. The k-core is the unique partition obtained by removing successively k-ribbons from λ, and the k-quotient is a sequence of k partitions derived from λ (see [3]). A k-ribbon is a connected skew diagram of weight k which does not contain a 2×2 square. The first (north-west) cell of a k-ribbon is called the *head* and the last one (south-east) the *tail*. A k-ribbon tableau of shape λ and weight μ is a tiling of the skew diagram $\lambda/\lambda_{(k)}$ by labelled k-ribbons such that the head of a ribbon labelled i must not be on the right of a ribbon labelled $j > i$ and its tail must not be on the top of a ribbon labelled $j \geq i$. We denote by $Tab_k(\lambda, \mu)$ the set of all k-ribbon tableaux of shape λ and weight μ.

Example: A 3-ribbon tableau of shape (8,7,6,5,1) and weight (3,3,2,1)



In [14], Stanton and White first introduced in the standard case (weight $\mu = (1, \ldots, 1)$) a correspondence between k-ribbon tableaux and k-tuples of standard Young tableaux. In the following, we will denote this bijection by sw. This map sends the previous 3-ribbon tableau to the 3-tuple of tableaux:



The spin of a k-ribbon R is defined by $sp(R) = \frac{h(R)-1}{2}$ where $h(R)$ is the height of R. The spin of a k-ribbon tableau is the sum of the spins of all its ribbons, and the cospin is the associated co-statistic into $Tab_k(\lambda, \mu)$. We define spin and cospin polynomials as generating polynomials of $Tab_k(\lambda, \mu)$ with spin or cospin statistics:

$$G^{(k)}_{\lambda,\mu}(q) = \sum_{T \in Tab_k(\lambda,\mu)} q^{sp(T)} \quad \text{and} \quad \tilde{G}^{(k)}_{\lambda,\mu}(q) = \sum_{T \in Tab_k(\lambda,\mu)} q^{cosp(T)} .$$

Example: In $Tab_3((8,7,6,5,1),(3,3,2,1))$, these polynomials are:

$$G^{(3)}_{(8,7,6,5,1),(3,3,2,1)}(q) = 3q^2 + 17q^3 + 33q^4 + 31q^5 + 18q^6 + 5q^7$$

$$\tilde{G}^{(3)}_{(8,7,6,5,1),(3,3,2,1)}(q) = 3q^5 + 17q^4 + 33q^3 + 31q^2 + 18q + 5 .$$

By definition, the Hall-Littlewood functions Q'_λ can be written as:

$$Q'_\lambda(X;q) = \prod_{i<j}(1 - qR_{ij})^{-1} s_\lambda(X)$$

where R_{ij} is the raising operator such that $R_{ij} \cdot s_\lambda = s_{R_{ij}\cdot\lambda}$, with

$$R_{ij} \cdot \lambda = (\lambda_1, \ldots, \lambda_i + 1, \ldots, \lambda_j - 1, \ldots, \lambda_p) .$$

In [9], Lascoux, Leclerc and Thibon showed that Hall-Littlewood functions can be expressed in terms of ribbon tableaux by:

$$Q'_\lambda(X;q) \;=\; \sum_{T\in Tab_p(p\lambda)} q^{sp(T)} X^T = \sum_\mu G^{(k)}_{p\lambda,\mu}(q)\; m_\mu \;.$$

The following specialization, with n a positive integer and ζ a primitive k-th root of unity, is proved in [7]:

$$Q'_{n^k}(X;\zeta) = (-1)^{(k-1)n} p_k \circ h_n(X)\;.$$

We shall give a combinatorial proof of this formula using the bijection between ribbon tableaux and rigged configurations given in [13].

3. Diagonal classes and rigged configurations

Let T be a k-ribbon tableau in $Tab_k(\lambda,\mu)$ and $(\Lambda^{(1)},\ldots,\Lambda^{(k)}) = sw(T)$. By writing $\alpha = -\lfloor\frac{\lambda_1}{k}\rfloor$ and $\beta = \lfloor\frac{l(\lambda)}{k}\rfloor$, for all $i\in\{\alpha,\ldots,\beta\}$ we define d_i as the word obtained by concatenation of the i-th diagonals of all the tableaux $\Lambda^{(j)}$ for j in $\{1\ldots k\}$ (we recall that the i-th diagonal of a Young tableau consists of all the cells with coordinates (x,y) such that $y-x=i$). We call *diagonal vector* of T the vector $d_T = (d_\alpha,\ldots,d_\beta)$. Two k-ribbon tableaux T and T' in $Tab_k(\lambda,\mu)$ are said to be *equivalent* if for all i in $\{\alpha,\ldots,\beta\}$ the i-th sorted word in d_T and $d_{T'}$ are the same. A diagonal class in $Tab_k(\lambda,\mu)$ is the set $D^{(k)}_{\lambda,\mu,d}$ of all equivalent ribbon tableaux with diagonal vector d. The set of all diagonal classes is denoted by $\Delta^{(k)}_{\lambda,\mu}$. We also define $G^{(k)}_{\lambda,\mu}(q,d)$ (resp. $\tilde G^{(k)}_{\lambda,\mu}(q,d)$) as the spin (resp. cospin) polynomials of the diagonal class $D^{(k)}_{\lambda,\mu,d}$.

Let $\nu = (\nu^{(1)},\ldots,\nu^{(p)})$ be an increasing p-tuple of partitions and J be such that: for all a in $\{1,\ldots,p-1\}$, $J^{(a)}$ is a $l(\nu^{(a)})$-tuple of partitions $(J^{(a)}_1,\ldots,J^{(a)}_{l(\nu^{(a)})})$ with $l(J^{(a)}_i)\le \nu^{(a)}_i - \nu^{(a)}_{i+1}$ and each part of $J^{(a)}_i$ less than $\nu^{(a+1)}_i - \nu^{(a)}_i$. A *rigged configuration* of shape ν, written (ν,J), is defined by: for all a, top cells of each column of $\nu^{(a)}$ which are in the i-th line are filled with parts of the partition $J^{(a)}_i$. For two partitions μ and δ, we define by $RC(\mu,\delta)$ the set of all the rigged configurations (ν,J) such that $\nu^{(p)} = {}^t\delta$ and $|\nu^{(a)}| = \mu_1+\ldots+\mu_a$ for all a in $\{1..p\}$ (definition as in [13] rather than in [4, 5, 6]).

In the following, $\lambda = (\lambda_1,\ldots,\lambda_p)$ is a partition with its k-core $\lambda_{(k)}$ empty and its k-quotient $\lambda^{(k)}$ equal to a k-tuple of single rows. We also set $m = \max(|\Lambda^{(1)}|,\ldots,|\Lambda^{(k)}|)$. In this special case, Schilling gives in [13] a bijection Ψ between $Tab_k(\lambda,\mu)$ and rigged configurations $RC(\mu,\delta)$, with $\delta_i = |\lambda^{(k)}_i|$. She also defined a co-statistic on these rigged configurations which corresponds to cospin under Ψ. Consequently, by enumeration of

$RC(\mu, \delta)$, she obtains:

$$\tilde{G}^{(k)}_{\lambda,\mu}(q) = \sum_{\{\nu_{\mu,\delta}\}} q^{\Phi(\nu)} \prod_{\substack{1\leq a\leq n-1\\ 1\leq i\leq \mu_1}} \begin{bmatrix} \nu_i^{(a+1)} - \nu_{i+1}^{(a)} \\ \nu_i^{(a)} - \nu_{i+1}^{(a)},\ \nu_i^{(a+1)} - \nu_i^{(a)} \end{bmatrix} \qquad (1)$$

where $\{\nu_{\mu,\delta}\}$ represents the set of all shapes appearing in $RC(\mu, \delta)$ and

$$\Phi(\nu) = \sum_{\substack{1\leq a\leq n-1\\ 1\leq i\leq \mu_1}} \nu_{i+1}^{(a)} (\nu_i^{(a+1)} - \nu_i^{(a)}) \ . \quad (1')$$

In the following, we will be mainly interested in the shapes of rigged configurations. We shall therefore propose a simpler but similar algorithm for finding only the shape of the rigged configuration $\Psi(T) = (\nu_T, J_T)$ with T in $Tab_k(\lambda, \mu)$. We construct an $m \times l(\mu)$ matrix M^T with the following rule:

$$M^T_{i,j} = \text{number of cells labelled } j \text{ in } d_{-i+1} \ .$$

Then, we construct a matrix N^T where each column $N^T_{\cdot,j}$ is defined by:

$$N^T_{\cdot,j} = \sum_{l\leq j} M^T_{\cdot,l} \ .$$

The j-th column is then equal to the j-th partition of ν_T.

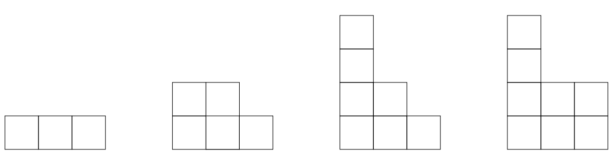
Example: For the 3-ribbon tableau corresponding to the following 3-tuple:

$$\left(\boxed{1\,|\,2\,|\,3\,|\,3} \ , \ \boxed{1\,|\,4} \ , \ \boxed{1\,|\,2} \right)$$

we construct the matrices M^T and N^T :

$$M^T = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad N^T = \begin{pmatrix} 3 & 3 & 3 & 3 \\ 0 & 2 & 2 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} .$$

The shape of the rigged configuration $\Psi(T)$ is:



as can be read from N^T.

We can remark an additive property of this construction. From a k-tuple of tableaux Λ, we can construct a k-ribbon tableau T_h, which is the Stanton-White inverse image of the k-tuple of tableaux formed by the h-th element of each tableau of Λ.

Lemma 1.

$$N^T = \sum_{h=1}^{m} N^{T_h} \ .$$

Proof: Let $M^T_{i,\cdot}$ be the matrix which has only the i-th line of M^T and zero everywhere else. Thus, by definition of M^{T_i} we have:

$$M^T = \sum_{i=1}^{m} M^T_{i,\cdot} = \sum_{i=1}^{m} M^{T_i} \ .$$

Then, we can write for all j:

$$\sum_{l\leq j} M^T_l = \sum_{i=1}^{m}\sum_{l\leq j} M^{T_i}_l \ .$$

Consequently,

$$N^T_j = \sum_{i=1}^{m} N^{T_i}_j \ . \ \square$$

Proposition 1. *For a given shape λ and weight μ, there is a bijection Γ between $\{\nu_{\lambda,\delta}\}$ and $\Delta^{(k)}_{\lambda,\mu}$ compatible with the statistics. Hence, the explicit expression for the cospin polynomial of a diagonal class is:*

$$\tilde{G}^{(k)}_{\lambda,\mu}(q,d) = q^{\Phi(\nu)} \prod_{\substack{1\leq a\leq n-1\\ 1\leq i\leq \mu_1}} \begin{bmatrix} \nu_i^{(a+1)}-\nu_{i+1}^{(a)} \\ \nu_i^{(a)}-\nu_{i+1}^{(a)},\ \nu_i^{(a+1)}-\nu_i^{(a)} \end{bmatrix} \qquad (2)$$

where ν is the shape corresponding to the diagonal class.

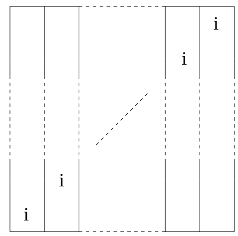
Proof: As the k-quotient consists of single rows, each diagonal class $D^{(k)}_{\lambda,\mu,d}$ is stable under permutation of cells which are in the same positions in each tableau. By construction, this property implies that, for all $l \in \{1\ldots,m\}$, $M^{T_l} = M^{T'_l}$. Then $N^{T_l} = N^{T'_l}$ and $N^T = N^{T'}$, so $\Psi(T)$ and $\Psi(T')$ have the same shape. Consequently, as map Ψ is a bijection, $D^{(k)}_{\lambda,\mu,d}$ is embedded into $\{\nu^T_{\lambda,\mu}\}$. Conversely, let T and T' be two tableaux in $Tab_k(\lambda,\mu)$ which are not in the same diagonal class. Thus, there exists j in $\{1,\ldots,m\}$ such that $\Lambda^T_j \neq \Lambda^{T'}_j$. This implies that $M^T_{\cdot,j} \neq M^{T'}_{\cdot,j}$ and $N^T \neq N^{T'}$ and consequently $\{\nu^T_{\lambda,\mu}\} \neq \{\nu^{T'}_{\lambda,\mu}\}$. Finally, we conclude that

$$\Psi(D^{(k)}_{\lambda,\mu,d}) = \{\nu^T_{\lambda,\mu}\} \ \text{ for all diagonal classes.}$$

The expression of cospin polynomials of diagonal classes in terms of q-supernomial coefficients follows immediately from the properties of Ψ. $\square$

In the following, we consider k-ribbon tableaux of shape $\lambda = (kn)^k$ for some $n \geq 1$. This implies that the image of these tableaux by the Stanton-White map is a k-tuple of semi-standard Young tableaux with the same single row partition of length n as shape.

Corollary 1. *Diagonal classes with only one element correspond to ribbon tableaux which are filled with $k \times k$ blocks of type:*



and the cospin of such a tableau is divisible by k.

Proof: A diagonal class $D^{(k)}_{\lambda,\mu,d}$ has an unique element if and only if there is an unique way to fill $\lambda^{(k)}$ according to the vector d. This implies that, for all i in $\{1,\dots,k\}$, all letters of d_i are the same. With this property, all the T_i's are the $k\times k$ blocks of the statement. For filling identically each position of d_i, the weight μ must be of the form $\mu=(k\cdot s_1,\dots,k\cdot s_p)$. Then, we construct the matrices M^T and N^T as :

$$M^T=\begin{pmatrix} k & 0 & \dots & \dots & \dots & 0\\ \vdots & \vdots & & & & \vdots\\ k & 0 & \dots & \dots & \dots & 0\\ 0 & k & 0 & \dots & \dots & 0\\ \vdots & \vdots & \vdots & & & \vdots\\ 0 & k & 0 & \dots & \dots & 0\\ \vdots & & & & & \vdots\\ 0 & \dots & \dots & \dots & 0 & k\\ \vdots & & & & \vdots & \vdots\\ 0 & \dots & \dots & \dots & 0 & k \end{pmatrix}$$

$$N^T=\begin{pmatrix} k & k & \dots & \dots & \dots & k\\ \vdots & \vdots & & & & \vdots\\ k & k & \dots & \dots & \dots & k\\ 0 & k & k & \dots & \dots & k\\ \vdots & \vdots & \vdots & & & \vdots\\ 0 & k & k & \dots & \dots & k\\ \vdots & & & & & \vdots\\ 0 & \dots & \dots & \dots & 0 & k\\ \vdots & & & & \vdots & \vdots\\ 0 & \dots & \dots & \dots & 0 & k \end{pmatrix}$$

where k occurs s_i times in the i-th column of the matrix M^T. Then the i-th partition in the shape $\nu_{\Psi(T)}$ is the rectangle $k^{s_1+\dots+s_i}$. This is why each term in the expression (1') is zero or a multiple of k. □

Proposition 2. *For such a shape λ, k-th primitive roots of unity are roots of cospin polynomials for all diagonal classes with strictly more than one element.*

Proof: Let T be a tableau in a diagonal class $D^{(k)}_{\lambda,\mu,d}$ and $\nu = (\nu^{(1)}, \dots, \nu^{(p)})$ be the shape of $\Psi(T)$, which is the same for all tableaux in this diagonal class. Let $\Lambda = sw(T)$ and h be the last position such that the diagonal vector d_h has at least two different elements. Then the $(h+1)$-th partition in ν_T is a rectangle of width k and height $s \leq r$. The last part of $\nu^{(h)} = (\nu^{(h)}_1, \dots, \nu^{(h)}_l)$ is equal to a with $a < h$ and the following coefficient appears:

$$\begin{bmatrix} \nu_l^{(h+1)} - \nu_{l+1}^{(h)} \\ \nu_l^{(h)} - \nu_{l+1}^{(h)},\ \nu_l^{(h+1)} - \nu_l^{(h)} \end{bmatrix} = \begin{bmatrix} k \\ a-0,\ k-a \end{bmatrix} .$$

Consequently, by a known property of the q-binomial $\begin{bmatrix} k \\ a,\ k-a \end{bmatrix}$, all k-th primitive roots of unity annihilate the diagonal class polynomials. □

Theorem 1. *We have the specialization:*

$$Q'_{n^k}(X;\zeta) = (-1)^{(k-1)n} p_k \circ h_n(X) .$$

Proof: We use functions H and $\tilde{H}$ as defined in [9]:

$$H^{(k)}_{n^k}(X;q) = \sum_{T \in Tab_k(kn^k)} q^{sp(T)} X^T \quad \text{and} \quad \tilde{H}^{(k)}_{n^k}(X;q) = \sum_{T \in Tab_k(kn^k)} q^{cosp(T)} X^T .$$

Let ζ be a k-th primitive root of unity. When q is set to ζ^{-1} in the expression of $\tilde{H}$, by Proposition 2 one is left with

$$\tilde{H}^{(k)}_{n^k}(X;\zeta^{-1}) = \sum_T (\zeta^{-1})^{cosp(T)} X^T$$

where T ranges now over k-ribbon tableaux as described in Corollary 1. By definition, these tableaux have maximum spin because they are only constructed with vertical ribbons, so their cospin is zero. Then, if we set $\Lambda_T = sw(T)$ we have

$$\tilde{H}^{(k)}_{n^k}(X;\zeta^{-1}) = \sum_T \underbrace{X^{\Lambda_T^{(1)}} \dots X^{\Lambda_T^{(1)}}}_{k \text{ times}} = \sum_S \underbrace{X^S \dots X^S}_{k \text{ times}}$$

where S ranges over all semi-standard Young tableaux with shape a single row of length n. We obtain

$$\tilde{H}^{(k)}_{n^k}(X;\zeta^{-1}) = p_k \circ h_n(X) .$$

Using relation between H and $\tilde{H}$ given in [9], we have

$$H^{(k)}_{n^k}(X;\zeta) = \zeta^{\frac{k(k-1)n}{2}} \tilde{H}^{(k)}_{n^k}(X;\zeta^{-1}) = (-1)^{(k-1)n} p_k \circ h_n(X) . \ \square$$

Remark: In the case where $\lambda = (k^{c \cdot k})$ there is a similar bijection between k-ribbon tableaux of shape λ and evaluation μ (see [13]) that allows to prove with the same method the following specialisation:

$$H^{(k)}_{\lambda}(X;\zeta) = (-1)^{(k-1)c} p_k \circ e_c(X) .$$

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