ON TRIANGULATIONS WITH HIGH VERTEX DEGREE

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Abstract. We solve three enumerative problems concerning families of planar maps. More precisely, we establish algebraic equations for the generating function of non-separable triangulations in which all vertices have degree at least d, for a certain value d chosen in $\{3,\ 4,\ 5\}$.
The originality of the problem lies in the fact that degree restrictions are placed both on vertices and faces. Our proofs first follow Tutte's classical approach: we decompose maps by deleting the root and translate the decomposition into an equation satisfied by the generating function of the maps under consideration. Then we proceed to solve the equation obtained using a recent technique that extends the so-called *quadratic method*.

Résumé: Nous énumérons trois familles de cartes planaires. Plus précisément, nous démontrons des résultats d'algébricité pour les familles de triangulations non-séparables dont le degré des sommets est au moins égal à une certaine valeur d choisie parmi $\{3,\ 4,\ 5\}$.
L'originalité de nos résultats tiens au fait que les restrictions de degré portent simultanément sur les faces et les sommets. Nous adoptons, dans un premier temps, la démarche classique de Tutte : nous décomposons nos cartes par suppression de la racine et traduisons cette décomposition en une équation portant sur la série génératrice correspondante. Nous résolvons ensuite l'équation obtenue en utilisant des techniques récentes qui généralisent la *méthode quadratique*.

1. Introduction

The enumeration of planar maps (or *maps* for short) has received a lot of attention in the combinatorists community for nearly fifty years. Originally motivated by the four-color problem, W.Tutte introduced the concept of map in the late fifties and considered a great number of map families corresponding to various constraints on face or vertex degrees. These seminal works, based on basic decomposition techniques allied to a generating function approach, gave rise to many explicit results ([9] - [11]). Fifteen years later, some physicists became interested in the subject and developed their own tools [5] for tackling the problem. Their techniques based on matrix integrals (see [15] for an introduction) proved very powerful [4]. More recently, a more bijective approach based on conjugacy classes of trees emerged providing new insights on the subject ([8], [3], [1], [7]).
However, when one considers a map family defined by both face and vertex constraints, each of the mentioned methods seems relatively ineffective and very few enumerative results are known. There are however two major exceptions. The enumeration of bipartite (i.e. faces have even degree) cubic (i.e. vertices have degree 3) maps was first performed by Tutte by a classic generating function approach ([10],[13]). And, more recently, the enumeration of all bipartite maps according to the degree distribution of their vertices was accomplished using conjugacy classes of trees [1].

Date: 6th May 2005.

In this paper, we shall consider triangulations (i.e. faces have degree 3) constrained by vertex degree conditions. We first follow Tutte's classical approach, which consists in trying to translate the decomposition obtained by deletion of the root into a functional equation satisfied by the generating function. It is not clear at first sight why this approach should work, but it does up to the condition of relaxing some of the constraints at this stage of the resolution. This process requires to take into account in our generating function, beside the size of the map, the degree of its root-face. We end up with a polynomial equation for the (bivariate) generating function in which the variable counting the degree of the root-face cannot be trivially eliminated. We then use a recent generalization of the quadratic method [2] to get rid of this extra variable and compute an algebraic equation characterizing the univariate generating function.

We begin by some vocabulary on maps. A map is a proper embedding of a connected graph into the two-dimensional Riemann sphere, considered up to continuous deformations. A map is *rooted* if one of its edges is distinguished as the *root* and attributed an orientation. Unless otherwise specified, all maps under consideration in this paper are rooted. The face at the right of the root is called the *root-face* and the other faces are said *internal*. Similarly, the vertices incident to the root-face are said *external* and the others are said *internal*. Graphically, the root-face is usually represented as the infinite face when the map is projected on the plane (see Figure 1). The endpoints of the root are distinguished as its *origin* and *end* according to the orientation of the root. A map is *separable* if it can be decomposed into two parts (not reduced to a vertex) whose intersection is reduced to a vertex. It is *non-separable* otherwise. For instance, the map in Figure 1 is non-separable. Lastly, a map is a *triangulation* (resp. *near-triangulation*) if all its faces (resp. all its internal faces) have degree 3. For instance, the map of Figure 1 is a near-triangulation with root-face of degree 5.

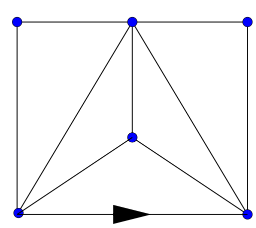

FIGURE 1. A non-separable near-triangulation.

In the sequel, we shall enumerate 3 families of non-separable triangulations. We recall some basic facts about these maps. Observe that a non-separable map (not reduced to an edge) cannot have loops nor isthmuses. Non-separable maps cannot have vertices of degree 1 either. Moreover, it is well known that planar graphs have at least one vertex of degree less than 6. We prove a stronger property: any triangulation has a vertex not incident to the root of degree less than 6. Indeed, consider a triangulation with f faces, e edges and v vertices. The incidence relation between faces and edges shows that $2e = 3f$ and reporting this identity in the Euler relation gives $e = 3v - 6$. If all vertices not incident to the root have degree at least 6 we have the inequality $2e \geq 6(v - 2) + 2$ which contradicts the previous identity.

Let **S** be the set of non-separable rooted near-triangulations. As observed above, the vertices of maps in **S** have degree at least 2. We consider three sub-families **T**, **U**, **V** of **S**. The set **T** (resp. **U**, **V**) is the subset of non-separable near-triangulations in which any internal vertex has degree at least 3 (resp. 4, 5). For each of the families $\mathbf{R} = \mathbf{S},\ \mathbf{T},\ \mathbf{U},\ \mathbf{V}$, we consider the bivariate generating function $\mathbb{R}(x,z)$, where z counts the size (the number of edges) and x the degree of the root-face minus 2. That is to say, $\mathbb{R}(x) \equiv \mathbb{R}(x,z) = \sum_{n,d} a_{n,d}x^d z^n$ where $a_{n,d}$ is the number of maps in **R** with size n and root-face of degree $d+2$. Note that the degree of the root-face is less than the number of edges. Therefore $\mathbb{R}(x,z)$ is a power series in the main variable z with polynomial coefficients in the secondary variable x. For each family $\mathbf{R} = \mathbf{S},\ \mathbf{T},\ \mathbf{U},\ \mathbf{V}$, we will characterize the generating function $\mathbb{R}(x)$ as the unique power series solution of a functional equation (see Equations (2),(11),(13),(14)).
We also consider the set **F** of non-separable rooted triangulations and three subsets **G** ,**H** ,**K**. The set **G** (resp. **H**, **K**) is the subset of non-separable triangulations in which any vertex not incident to the root has degree at least 3 (resp. 4, 5). As observed above, the subset of non-separable triangulations in which any vertex not incident to the root has degree at least 6 is empty. Note that, given the incidence relation between faces and edges, any triangulation has a size (number of edges) multiple of 3. To each of the families $\mathbf{L} = \mathbf{F},\ \mathbf{G},\ \mathbf{H},\ \mathbf{K}$, we associate the univariate generating function $\mathbb{L}(t) = \sum_n a_n t^n$ where a_n is the number of maps in **L** of size $3n$. For each family $\mathbf{L} = \mathbf{F},\ \mathbf{G},\ \mathbf{H},\ \mathbf{K}$ we will give an algebraic characterization of $\mathbb{L}(t)$ (see Equations (4),(17),(18) and Proposition 6).

There is a simple connection between the generating functions $\mathbb{F}(t)$ (resp. $\mathbb{G}(t)$, $\mathbb{H}(t)$, $\mathbb{K}(t)$) and $\mathbb{S}(x) \equiv S(x,z)$ (resp. $\mathbb{T}(x)$, $\mathbb{U}(x)$, $\mathbb{V}(x)$). This connection relies on a very simple bijection between non-separable triangulations and non-separable near-triangulations rooted on a digon (i.e. the root-face has degree 2). The bijection consists in deleting the external edge which is not the root in a near-triangulation rooted on a digon (see Figure 2). This bijection establishes a one-to-one correspondence between the set of triangulations **F** (resp. **G** ,**H** ,**K**) and the set of near-triangulations in **S** (resp. **T**, **U**, **V**) rooted on a digon.

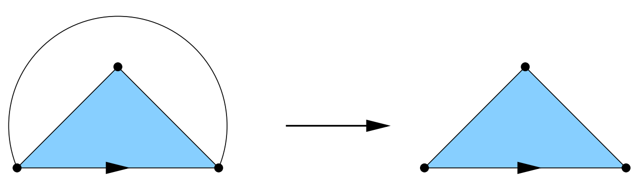

FIGURE 2. Near-triangulations rooted on a digon and triangulations.

Observe that for $\mathbf{R} \in \{\mathbf{S},\ \mathbf{T},\ \mathbf{U},\ \mathbf{V}\}$, the power series $\mathbb{R}(0) = \mathbb{R}(0,z)$ is the generating function of near-triangulations in **R** rooted on a digon. Thus, we have the relations:

$$\mathbb{S}(0) = z\mathbb{F}(z^3), \qquad \mathbb{T}(0) = z\mathbb{G}(z^3), \qquad \mathbb{U}(0) = z\mathbb{H}(z^3), \qquad \mathbb{V}(0) = z\mathbb{K}(z^3). \quad (1)$$

This paper is organized as follows. In Section 2, we recall the decomposition scheme on maps due to W.T. Tutte. We apply it to the set **S** of unconstrained

non-separable near-triangulation and recall some known results about the generating functions $\mathbb{S}(x)$ and $\mathbb{F}(t)$. In Section 3, we apply the same decomposition scheme to the sets of near-triangulations **T**, **U** and **V** . This allows to write functional equations concerning the generating functions $\mathbb{T}(x)$, $\mathbb{U}(x)$, $\mathbb{V}(x)$. In the equations obtained, the variable x appears and cannot be trivially eliminated. This is precisely why we introduced this variable in our generating functions: it allows us to write the functionnal equations. In Section 4, we use techniques generalizing the *quadratic method* in order to get rid of the variable x. We obtain algebraic equations characterizing the generating function of the subsets of triangulations **G** , **H** , **K**. At this point, only the degree of the endpoints of the root remains unconstrained. That is, we have an algebraic characterization of triangulations for which any vertex *not incident to the root* has degree at least 3,4 or 5. In Section 5, we show that we can constrain, *a posteriori*, the degree of the endpoints of the root in the two first cases. This provides an algebraic characterization for triangulations in which *any* vertex has degree at least 3 or 4. However, no similar result is found for the set of triangulations in which *any* vertex has degree at least 5. We also study the singularities of our series and deduce the asymptotic behavior of the number of maps in each family.
Some of the results concerning triangulations in which any vertex has degree at least 3 were already proved in [6] via a compositional approach. We give here an alternative proof.

2. The decomposition principle

In the following, we adopt Tutte's classical approach for enumerating maps. That is, we decompose maps by deleting their root and translate this combinatorial decomposition into an equation satisfied by the corresponding generating function. Let us illustrate this classic approach on the problem of enumerating unconstrained non-separable triangulations (this was first done in [12]). We recall that **S** denotes the set of non-separable near-triangulations and $\mathbb{S}(x) = \mathbb{S}(x, z)$ the corresponding generation function. By convention, we exclude the map reduced to a vertex from **S**. Thus, the smallest map in **S** is the map reduced to a straight edge (see Figure 3). This map is called the *link-map* and denoted L. Its contribution to the generating function is z, hence $\mathbb{S}(x) = z + o(z)$.

FIGURE 3. The link-map L.

We decompose maps distinct from L by deleting the root. Note that, if M is a non-separable triangulation distinct from L, the face at the left of the root is internal (otherwise M would be separable) thus it has degree 3. Moreover, since M has no loop, the three vertices incident to this face are distinct. Let v be the vertex not incident to the root. When examining what can happen to M when deleting its root, one is led to distinguish two cases (see Figure 4).

Either the vertex v was incident to the root-face, in which case the map obtained by deletion of the root is separable (see Figure 5). Or v was not incident to the root-face and the map obtained by deletion of the root is a non-separable near-triangulation (see Figure 6). In the first case, the map obtained is in correspondence with an ordered pair of non-separable near-triangulations. This correspondence is bijective, that is, any ordered pair is the image of exactly one near-triangulation. In the

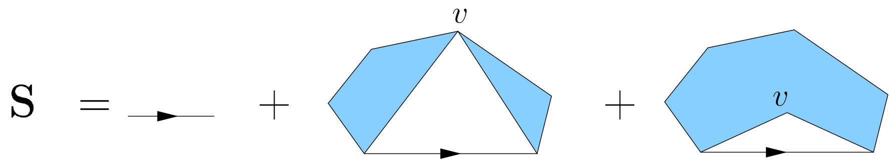

FIGURE 4. Decomposition of non-separable near-triangulations.

second case the degree of the root-face is increased by one. Hence the root-face of the near-triangulation obtained has degree at least 3. Here again, any near-triangulation for which the root-face has degree at least 3 is the image of exactly one near-triangulation.

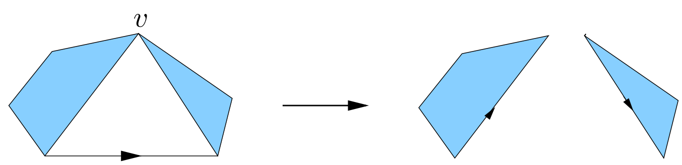

FIGURE 5. Case 1. The vertex v was incident to the root-face.

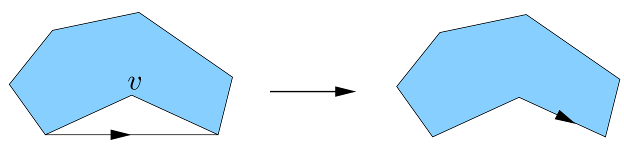

FIGURE 6. Case 2. The vertex v was not incident to the root-face.

We want to translate this analysis into a functional equation. Observe that the degree of the root-face appears in this analysis. This is why we are *forced* to introduce the variable x counting this parameter in our generating function $\mathbb{S}(x, z)$. For this reason, following Zeilberger's terminology [14], the secondary variable x is said to be *catalytic*: we need it to write the functional equation, but we shall try to get rid of it later.

In our case, the decomposition translates into the following equation:

$$\mathbb{S}(x, z) = z + xz\mathbb{S}(x, z)^2 + \frac{z}{x}\left(\mathbb{S}(x, z) - \mathbb{S}(0, z)\right) \ . \tag{2}$$

We shall explain this equation later. The first summand in the right-hand side accounts for the link map, the second summand corresponds to the case in which the vertex v is incident to the root-face, and the third summand corresponds to the case in which v is not incident to the root-face.

It is an easy exercise to check that this equation defines the series $\mathbb{S}(x, z)$ uniquely as a power series in z. By resolutions techniques presented in Section 4, we can derive from equation (2) a polynomial equation satisfied by the series $\mathbb{S}(0, z)$ where the extra variable x does not appear anymore. This equation reads

$$\mathbb{S}(0, z) \ = \ z - 27z^4 + 36z^3\mathbb{S}(0, z) - 8z^2\mathbb{S}(0, z)^2 - 16z^4\mathbb{S}(0, z)^3 \ . \tag{3}$$

Knowing that $\mathbb{S}(0,z)=z\mathbb{F}(z^3)$, we deduce the algebraic equation

$$\mathbb{F}(t) \;= 1-27t+36t\mathbb{F}(t)-8t\mathbb{F}(t)^2-16t^2\mathbb{F}(t)^3 \;, \tag{4}$$

characterizing $\mathbb{F}(t)$ uniquely as a power series in t. From this equation we can deduce the asymptotic behavior of the coefficients of $\mathbb{F}(t)$, that is, the number of non-separable triangulations of a given size.

3. Functional equations

In this section, we apply the decomposition principle presented in the previous section to the families $\mathbf{T},\ \mathbf{U},\ \mathbf{V}$ of non-separable near-triangulations in which all internal vertices have degree at least 3, 4, 5. We obtain functional equations satisfied by the corresponding generating functions $\mathbb{T}(x),\ \mathbb{U}(x),\ \mathbb{V}(x)$.
When we delete the root of a map, the degree of its endpoints is lowered by one. In order to translate the decomposition of maps into equations, we are forced to relax the constraints on the degree of external vertices and to control some parameters. Let $\mathbf{R}$ be one of the set $\mathbf{S},\ \mathbf{T},\ \mathbf{U},\ \mathbf{V}$. We define $\mathbf{R}_k$ as the set of maps in $\mathbf{R}$ such that the root-face has degree at least 3 and the origin of the root has degree k. We also define the set $\mathbf{R}_\infty$ as the image of $\mathbf{R}^2$ by the mapping ϕ represented in Figure 7. The mapping ϕ takes an ordered pair of maps and glues the end of the root of the first map to the origin of the second. The new root is chosen to be the root of the second map. Lastly, we write $\mathbf{R}_{\geq k} \triangleq \mathbf{R}_\infty \cup \bigcup_{j\geq k} \mathbf{R}_j$.

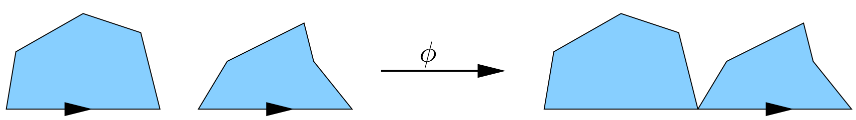

Figure 7. The mapping ϕ.

We shall use the symbols $\mathbb{R}_k(x,z)$, $\mathbb{R}_\infty(x,z)$ and $\mathbb{R}_{\geq k}(x,z)$ for the bivariate generating functions corresponding to the sets $\mathbf{R}_k$, $\mathbf{R}_\infty$ and $\mathbf{R}_{\geq k}$ respectively. In these series, as in $\mathbb{R}(x,z)$, the contribution of a map of size n and root-face degree d is $x^{d-2}z^n$.

We are now ready to prove our first results. Let us write L for the link-map (see Figure 3) and consider a near-triangulation M in $\mathbf{R}-\{L\}$. As observed before, the face at the left of the root is an internal face incident to three distinct vertices. Let v be the vertex not incident to the root. If v is external, the deletion of the root produces a map in $\mathbf{R}_\infty$. If v is internal and M is in $\mathbf{S}$ (resp. $\mathbf{T},\ \mathbf{U},\ \mathbf{V}$) then v has degree at least 2 (resp. 3, 4, 5) and the map obtained by deletion of the root is in $\bigcup_{k\geq 2}\mathbf{S}_k$ (resp. $\bigcup_{k\geq 3}\mathbf{T}_k$, $\bigcup_{k\geq 4}\mathbf{U}_k$, $\bigcup_{k\geq 5}\mathbf{V}_k$). Therefore, the mapping goes from $\mathbf{S}-\{L\}$ (resp. $\mathbf{T}-\{L\}$, $\mathbf{U}-\{L\}$, $\mathbf{V}-\{L\}$) to $\mathbf{S}_{\geq 2}$ (resp. $\mathbf{T}_{\geq 3}$, $\mathbf{U}_{\geq 4}$, $\mathbf{V}_{\geq 5}$). And this mapping is clearly bijective. Moreover, the map obtained after deletion of the root has size lowered by one and root-face degree increased by one. This analysis translates into the following equations:

$$\mathbb{S}(x) = z + \frac{z}{x}\mathbb{S}_{\geq 2}(x) , \quad (5)$$

$$\mathbb{T}(x) = z + \frac{z}{x}\mathbb{T}_{\geq 3}(x) , \quad (6)$$

$$\mathbb{U}(x) = z + \frac{z}{x}\mathbb{U}_{\geq 4}(x) , \quad (7)$$

$$\mathbb{V}(x) = z + \frac{z}{x}\mathbb{V}_{\geq 5}(x) . \quad (8)$$

In view of Equation (5), we will obtain a non-trivial equation for $\mathbb{S}(x)$ if we can express $\mathbb{S}_{\geq 2}(x)$ in terms of $\mathbb{S}(x)$. Similarly, we will obtain a non-trivial equation for $\mathbb{T}(x)$ if we can express $\mathbb{T}_{\geq 2}(x)$ and $\mathbb{T}_2(x)$ in terms of $\mathbb{T}(x)$. For $\mathbb{U}(x)$ (resp. $\mathbb{V}(x)$) we need to express $\mathbb{U}_{\geq 2}(x)$, $\mathbb{U}_2(x)$ and $\mathbb{U}_3(x)$ (resp. $\mathbb{V}_{\geq 2}(x)$, $\mathbb{V}_2(x)$, $\mathbb{V}_3(x)$ and $\mathbb{V}_4(x)$). Our first task will thus be to evaluate $\mathbb{R}_{\geq 2}(x)$ for $\mathbb{R}$ in $\{\mathbb{S}, \mathbb{T}, \mathbb{U}, \mathbb{V}\}$.
Let the set $\mathbf{R}$ be one of $\{\mathbf{S}, \mathbf{T}, \mathbf{U}, \mathbf{V}\}$. By definition, $\mathbf{R}_\infty$ is in bijection with $\mathbf{R}^2$. This bijection translates into the following functional equation

$$\mathbb{R}_\infty(x) = x^2\mathbb{R}(x)^2 .$$

Observe that $\bigcup_{k\geq 2} \mathbf{R}_k$ is the set of maps in $\mathbf{R}$ for which the root-face has degree at least 3. We thus have the set identity $\bigcup_{k\geq 2} \mathbf{R}_k = \mathbf{R} - \{M \in \mathbf{R} \ / \ d(M) = 2\}$. And from this, we deduce

$$\sum_{k\geq 2} \mathbb{R}_k(x) = \mathbb{R}(x) - \mathbb{R}(0) ,$$

since $\mathbb{R}(0)$ is the generating function of maps in $\mathbf{R}$ rooted on a digon.
Now, since

$$\mathbf{R}_{\geq 2} = \mathbf{R}_\infty \cup \bigcup_{k\geq 2} \mathbf{R}_k ,$$

we have

$$\mathbb{R}_{\geq 2}(x) = x^2\mathbb{R}(x)^2 + (\mathbb{R}(x) - \mathbb{R}(0)) \quad \text{for } \mathbb{R} \text{ in } \{\mathbb{S}(x), \mathbb{T}(x), \mathbb{U}(x), \mathbb{V}(x)\} . \quad (9)$$

Equations (5) and (9) already allow us to recover Equation (2) announced in Section 2:

$$\mathbb{S}(x) = z + xz\mathbb{S}(x)^2 + z\left(\frac{\mathbb{S}(x) - \mathbb{S}(0)}{x}\right) .$$

In order to go further, we will need to express $\mathbb{T}_2(x)$, $\mathbb{U}_2(x)$, $\mathbb{U}_3(x)$, $\mathbb{V}_2(x)$, $\mathbb{V}_3(x)$ and $\mathbb{V}_4(x)$ (see Equations (5-8)). We begin by the equation concerning $\mathbb{R}_2(x)$ for $\mathbb{R}$ in $\{\mathbb{S}, \mathbb{T}, \mathbb{U}, \mathbb{V}\}$.
Observe that if the set $\mathbf{R}$ is one of $\{\mathbf{S}, \mathbf{T}, \mathbf{U}, \mathbf{V}\}$, then the set $\mathbf{R}_2$ is in bijection with $\mathbf{R}$ by the mapping illustrated in Figure 8. Consequently we can write

$$\mathbb{R}_2(x) = xz^2\mathbb{R}(x) \quad \text{for } \mathbb{R} \text{ in } \{\mathbb{S}(x), \mathbb{T}(x), \mathbb{U}(x), \mathbb{V}(x)\} . \quad (10)$$

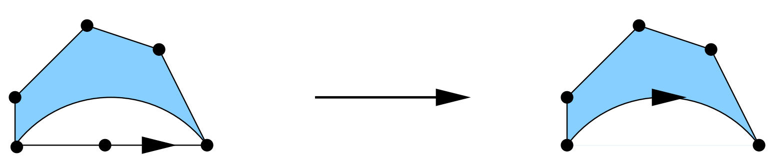

FIGURE 8. A bijection between $\mathbf{R}_2$ and $\mathbf{R}$.

We put immediately this equation to contribution in order to write an equation for **T**:

$$
\begin{aligned}
\mathbb{T}(x) &= z + \frac{z}{x}\mathbb{T}_{\geq 3}(x) && \text{by equation (6)}\\
&= z + \frac{z}{x}\left(\mathbb{T}_{\geq 2}(x) - \mathbb{T}_2(x)\right)\\
&= z + \frac{z}{x}\left(x^2\mathbb{T}(x)^2 + (\mathbb{T}(x) - \mathbb{T}(0)) - xz^2\mathbb{T}(x)\right) && \text{by equations (9) and (10).}
\end{aligned}
$$

Proposition 1. *The generating function $\mathbb{T}(x)$ of non-separable near-triangulations in which all internal vertices have degree at least 3 satisfies:*

$$\mathbb{T}(x) \;=\; z \;+\; xz\mathbb{T}(x)^2 \;+\; z\left(\frac{\mathbb{T}(x) - \mathbb{T}(0)}{x}\right) \;-\; z^3\mathbb{T}(x)\ . \tag{11}$$

We go a step further to find an equation concerning the set **U**. We have to express $\mathbb{U}_3(x)$ in terms of $\mathbb{U}(x)$. Let M be a map in $\mathbf{U}_3$ and n the origin of its root. By definition, n has degree 3 and the root-face of M has degree at least 3. Observe that, since the map is non-separable, the vertex a preceding n on the root-face is distinct from the vertex b following n (see Figure 9). Let us denote by v the third vertex adjacent to n. Since there can be no loop, it is clear that v is distinct from a and b. With these considerations, it is clear that maps in $\mathbf{U}_3$ are in bijection with maps in $\mathbf{U}_{\geq 3}$ by the mapping illustrated in Figure 9. Indeed, the vertex v must be either incident to the root-face (in which case the result is in $\mathbf{U}_\infty$) or of degree $d \geq 4$ (in which case the result is in $\mathbf{U}_{d-1}$).
The bijection translates into the identity:

$$\mathbb{U}_3(x) = z^3\mathbb{U}_{\geq 3}(x) = z^3(\mathbb{U}_{\geq 2}(x) - \mathbb{U}_2(x))\ . \tag{12}$$

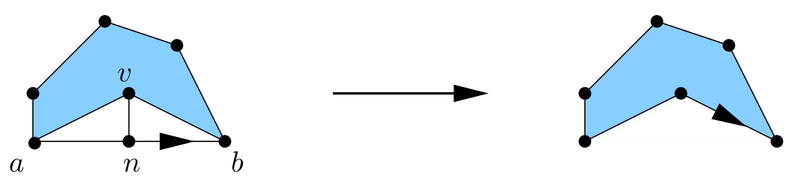

FIGURE 9. A bijection between $\mathbf{U}_3$ and $\mathbf{U}_{\geq 3}$.

We are now ready to establish an equation for **U**:

$$
\begin{aligned}
\mathbb{U}(x) &= z + \frac{z}{x}\mathbb{U}_{\geq 4}(x) && \text{by equation (7)}\\
&= z + \frac{z}{x}\left(\mathbb{U}_{\geq 2}(x) - \mathbb{U}_2(x) - \mathbb{U}_3(x)\right)\\
&= z + \frac{z(1-z^3)}{x}\left(\mathbb{U}_{\geq 2}(x) - \mathbb{U}_2(x)\right) && \text{by equation (12)}\\
&= z + \frac{z(1-z^3)}{x}\left(x^2\mathbb{U}(x)^2 + (\mathbb{U}(x) - \mathbb{U}(0)) - xz^2\mathbb{U}(x)\right) && \text{by (9) and (10).}
\end{aligned}
$$

Proposition 2. *The generating function $\mathbb{U}(x)$ of non-separable near-triangulations in which all internal vertices have degree at least 4 satisfies:*

$$\mathbb{U}(x) = z + xz(1-z^3)\mathbb{U}(x)^2 + z(1-z^3)\left(\frac{\mathbb{U}(x) - \mathbb{U}(0)}{x}\right) - z^3(1-z^3)\mathbb{U}(x)\ . \tag{13}$$

It is possible to use the same approach to establish an equation concerning the set **V**. Unfortunately, the decomposition happens to be considerably more entangled and the proof quite heavy. We spare the reader the details of these calculations and simply state the following proposition.

Proposition 3. *The generating function* $\mathbb{V}(x) = VF(x,z)$ *of non-separable near-triangulations in which all internal vertices have degree at least 5 satisfies:*

$$\begin{aligned} x^2z^5(z^3-1)\mathbb{V}(x)^3 - z(-x+2z^4+xz^9-2xz^6-2z^7+xz^{12})\mathbb{V}(x)^2 \\ +\frac{(2x^2z^6+xz-2x^2z^3-z^5+z^8-xz^{10}-xz^{13}-x^2z^{12}+x^2z^{15}-2x^2z^9+2xz^7-x^2)}{x^2}\mathbb{V}(x) \\ +\frac{z(-x+z^4+xz^9-2xz^6-z^7+xz^{12})}{x^2}\mathbb{V}(0) - \frac{z^5(z^3-1)}{x}[x]\mathbb{V}(x) + z(z^3+1) &= 0, \end{aligned} \tag{14}$$

where $[x]V(x)$ *denotes the coefficient of* x *in* $\mathbb{V}(x)$.

4. Algebraic equations for triangulations with high degree

In the previous section, we have exhibited functional equations concerning the families of near-triangulations $\mathbf{T}, \mathbf{U}, \mathbf{V}$. We now solve these equations and establish algebraic characterization for the families $\mathbf{F}, \mathbf{G}, \mathbf{H}$ of triangulations in which vertices not incident to the root have degree at least $3, 4, 5$ respectively. As observed in the introduction, the generating functions $\mathbb{F}(t), \mathbb{G}(t), \mathbb{H}(t)$ are related to the series $\mathbb{T}(0), \mathbb{U}(0), \mathbb{V}(0)$ (see Equation (1)).

Let us look at Equations (11), (13) and (14) satisfied by the series $\mathbb{T}(x)$, $\mathbb{U}(x)$ and $\mathbb{V}(x)$ respectively. We begin with Equation (11). Multiplying each side of this equation by x, we obtain a polynomial equation in the two variables x and z and the two unknown series $\mathbb{T}(x)$ and $\mathbb{T}(0)$. It is easily seen that this equation allows us to compute the coefficients of $\mathbb{T}(x)$ (hence those of $\mathbb{T}(0)$) iteratively. In particular, Equation (11) defines the series $\mathbb{T}(0)$ uniquely as a power series in z. The same property holds for Equations (13) and (14): reducing both sides of the equation to the same denominator we obtain a polynomial equation. In the case of Equation (14) there is a third unknown series, $[x]\mathbb{V}(x)$ appearing. But in both cases, it is easily seen that the equation defines the unknown series $\mathbb{U}(0)$, $\mathbb{V}(0)$ uniquely as a power series in z.
So, in a sense, these equations solve the enumeration problems. However we want to find algebraic equations for the series $\mathbb{T}(0)$, $\mathbb{U}(0)$ and $\mathbb{V}(0)$ in which the extra variable x do not appear anymore. Techniques for performing such manipulations appear many times in the literature. In the cases of Equation (11) and (13) we can routinely apply the so called *quadratic method*. This method allows one to solve any quadratic equation in which there is one unknown bivariate series and one unknown univariate series. However, Equation (14) has two unknown univariate series and is moreover cubic in the bivariate series. Very recently, a new formalism due to Bousquet-Mélou and Jehanne has emerged allowing one to solve this kind of equation [2]. We adopt this formalism.

Let us begin with Equation (11) concerning $\mathbb{T}(0)$. We define the polynomial

$$P(T, T_0, X, Z) = XZ + X^2ZT^2 + ZT - ZT_0 - XZ^3T - XT \ .$$

Equation (11) can be written as

$$P(\mathbb{T}(x), \mathbb{T}(0), x, z) = 0 \ . \tag{15}$$

Let us consider the equation $P_1'(\mathbb{T}(x), \mathbb{T}(0), x, z) = 0$, where P_1' denotes the first derivative of P with respect to its first variable. This equation can be written as

$$x = z + 2x^2z\mathbb{T}(x) - xz^3 \ .$$

This equation is not satisfied for a generic x. However, considered as an equation in x, it is straightforward to show that it admits a unique power series solution $X(z)$. Taking the derivative of Equation (15) with respect to x one obtains

$$\frac{\partial \mathbb{T}(x)}{\partial x} \cdot P_1'(\mathbb{T}(x), \mathbb{T}(0), x, z) + P_3'(\mathbb{T}(x), \mathbb{T}(0), x, z) = 0$$

where P_3' denotes the first derivative of P with respect to its third variable. Substituting the series $X(z)$ for x in that equation, we see that the series $X(z)$ is also solution of equation $P_3'(\mathbb{T}(x), \mathbb{T}(0), x, z) = 0$. Hence, we have a system of three equations

$$\begin{array}{rcl} P(\mathbb{T}(X(z)), \mathbb{T}(0), X(z), z) &=& 0\ , \\ P_1'(\mathbb{T}(X(z)), \mathbb{T}(0), X(z), z) &=& 0\ , \\ P_3'(\mathbb{T}(X(z)), \mathbb{T}(0), X(z), z) &=& 0\ , \end{array} \tag{16}$$

for the three unknown series $\mathbb{T}(X(z))$, $\mathbb{T}(0)$ and $X(z)$. This polynomial system can be solved by elimination techniques using either resultant calculations or Gröbner bases. Performing these eliminations one obtains an algebraic equation for $\mathbb{T}(0)$:

$$\mathbb{T}(0) = z - 24z^4 + 3z^7 + z^{10} + (32z^3 + 30z^6 - 4z^9 - z^{12})\mathbb{T}(0) - 8z^2(1+z^3)^2\mathbb{T}(0)^2 - 16z^4\mathbb{T}(0)^3.$$

Using the fact that $\mathbb{T}(0) = z\mathbb{G}(z^3)$ we get the following theorem.

Theorem 4. *Let* $\boldsymbol{G}$ *be the set of non-separable triangulations for which any vertex not incident to the root has degree at least 3 and let* $\mathbb{G}(t)$ *be its generating function. The series* $\mathbb{G}(t)$ *is uniquely defined as a power series in* t *by the algebraic equation:*

$$\begin{array}{rl} 1 - 24t + 3t^2 + t^3 \ - & (1+t)(1 - 33t + 3t^2 + t^3)\mathbb{G}(t) \\ & -\ 8t(1+t)^2\mathbb{G}(t)^2 \ - \ 16t^2\mathbb{G}(t)^3 \ = \ 0\ . \end{array} \tag{17}$$

The same manipulations lead to a similar result concerning the set **H**.

Theorem 5. *Let* $\boldsymbol{H}$ *be the set of non-separable triangulations for which any vertex not incident to the root has degree at least 4 and let* $\mathbb{H}(t)$ *be its generating function. The series* $\mathbb{H}(t)$ *is uniquely defined as a power series in* t *by the algebraic equation:*

$$\begin{array}{c} 1 - 24t + 54t^2 - 32t^3 + 3t^5 - t^6 + 8(1-t)^2(1+t+t^2)^2 t\mathbb{H}(t)^2 \\ -(1+t-t^2)(1 - 33t + 72t^2 - 41t^3 + 3t^5 - t^6)\mathbb{H}(t) - 16t^2(t-1)^4\mathbb{H}(t)^3 = 0. \end{array} \tag{18}$$

For the Equation (14) concerning $\mathbb{V}(0)$ the method is almost identical. We see that there is a polynomial $Q(V, V_0, V_1, x, z)$ such that Equation (14) can be written as $Q(\mathbb{V}(x), \mathbb{V}(0), [x]\mathbb{V}(x), x, z) = 0$. But we can show that there are exactly *two* series $X_1(z)$, $X_2(z)$ such that $Q_1'(\mathbb{V}(X(z)), \mathbb{V}(0), [x]\mathbb{V}(x), X(z), z) = 0$. Thus, we obtain a system of 6 equations

$$\begin{array}{rcl} Q(\mathbb{V}(X_i(z)), \mathbb{V}(0), [x]\mathbb{V}(x), X_i(z), z) &=& 0 \\ Q_1'(\mathbb{V}(X_i(z)), \mathbb{V}(0), [x]\mathbb{V}(x), X_i(z), z) &=& 0 \qquad i = 1,\ 2 \\ Q_3'(\mathbb{V}(X_i(z)), \mathbb{V}(0), [x]\mathbb{V}(x), X_i(z), z) &=& 0 \end{array} \tag{19}$$

for the 6 unknown series $\mathbb{V}(X_1(z))$, $\mathbb{V}(X_2(z))$, $X_1(z)$, $X_2(z)$, $\mathbb{V}(0)$ and $[x]\mathbb{V}(x)$. This system can be solved via elimination techniques. The calculus involved are really heavy, and the result is too big to fit in here. However, we have the following theorem.

Theorem 6. *Let* $\boldsymbol{K}$ *be the set of non-separable triangulations for which any vertex not incident to the root has degree at least 5 and let* $\mathbb{K}(t)$ *be its generating function. The series* $\mathbb{K}(t)$ *is algebraic of degree 6.*

5. Constraining the vertices incident to the root

We have established algebraic equations for triangulations in which any vertex *not incident to the root* has degree at least 3 (resp. 4). We will now establish equations for triangulations in which *any* vertex has degree at least 3 (resp. 4). This can be done by expressing the generating function of triangulations in which any vertex has degree at least 3, 4 in terms of the series $\mathbb{G}(t)$, $\mathbb{H}(t)$.

Theorem 7. *Let* $\boldsymbol{G}^*$ *be the set of non-separable triangulations for which* any *vertex has degree at least 3 and let* $\mathbb{G}^*(t)$ *be its generating function. The series* $\mathbb{G}$ *and* $\mathbb{G}^*$ *are related by the identity*

$$\mathbb{G}^*(t) = (1-2t)\mathbb{G}(t)\ . \tag{20}$$

Theorem 8. *Let* $\boldsymbol{H}^*$ *be the set of non-separable triangulations for which* any *vertex has degree at least 4 and let* $\mathbb{H}^*(t)$ *be its generating function. The series* $\mathbb{H}$ *and* $\mathbb{H}^*$ *are related by the identity*

$$\mathbb{H}^*(t) = \frac{1-5t+5t^2-3t^3}{1-t}\mathbb{H}(t)\ . \tag{21}$$

The proofs of Theorems 7 and 8 are reminiscent of the manipulations practiced in Section 3. We do not detail them here.

Plugging Equation (20) (resp. (21)) in Equation (17) (resp. (18)) one obtains algebraic characterizations of the series $\mathbb{G}^*$ (resp. $\mathbb{H}^*$). In particular, it is possible to compute the first coefficients of these series. We find:

$$\mathbb{G}^*(t) = t^2 + 3t^3 + 19t^4 + 128t^5 + 909t^6 + 6737t^7 + 51683t^8 + o(t^8),$$
$$\mathbb{H}^*(t) = t^4 + 3t^5 + 12t^6 + 59t^7 + 325t^8 + 1875t^9 + 11029t^{10} + o(t^{10}).$$

In the expansion of $\mathbb{G}^*(t)$, the smallest non-zero coefficient t^2 corresponds to the tetrahedron. In the expansion of $\mathbb{H}^*(t)$, the smallest non-zero coefficient t^4 corresponds to the octahedron (see Figure 10).
We were unable to find an equation that would permit to count non-separable triangulations in which *any* vertex has degree at least 5. However, we can use the algebraic equation satisfied by the series $\mathbb{K}(t)$ and discover that the first non-zero coefficient corresponds to the icosahedron (see Figure 10).

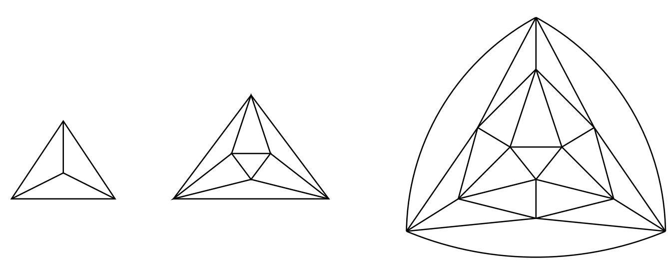

FIGURE 10. The platonic solids: tetrahedron, octahedron, icosahedron.

The algebraic equations for $\mathbb{G}^*(t)$, $\mathbb{H}^*(t)$ allow one to study their dominant singularity and deduce the asymptotic behavior of the number of triangulations in which any vertex has degree at least 3, 4. It can be shown that the number of triangulations in which any vertex has degree at least 5 has the same asymptotic behavior (up to a constant factor) as the number of triangulations in which any vertex *not incident to the root* has degree at least 5. Therefore, this behavior can be deduced from the algebraic equation concerning $\mathbb{K}(t)$. As expected, the number f_n, g_n, h_n, k_n of triangulations in which any vertex has degree at least 2,3,4,5 respectively have the generic form: $f_n, g_n, h_n, k_n \sim \alpha n^{-5/2}\mu^n$. And the exponential factors are approximately equal to $\mu_F = 13.5$, $\mu_G = 10.20$, $\mu_H = 7.03$, $\mu_K = 4.06$.

6. Concluding remarks

We studied three families of triangulations. We were able to establish algebraic equations for the generating functions of non-separable triangulations in which vertices not incident to the root have degree at least 3, 4, 5. It was then possible to obtain algebraic equations for non-separable triangulations in which any vertex has degree at least 3, 4. However, no similar result was found for degree 5.
The algebraic equations can be converted into differential equations (using for instance the *algeqtodiffeq* Maple command) from which one is able to compute the coefficients of the series in linear time. Thus our equations allow to compute efficiently the number of maps of a given size for each of the mentioned families. Moreover, asymptotic results for the number of such maps can also be found routinely from the algebraic equations.
We proved our results using basic decomposition techniques allied with a generating function approach. Alternatively, it is possible to obtain some of these results by a compositional approach. The equation concerning non-separable triangulations in which any vertex has degree at least 3 was proved by this method in [6]. It is also possible to recover the equation concerning non-separable triangulations in which any vertex has degree at least 4. However, the results concerning non-separable triangulations in which vertices not incident to the root have degree at least 5 seem hard to obtain by this method.
The approach adopted in this paper is classic except for some manipulations of equations relying on very recent techniques. Still, it is quite surprising that our approach works since the maps under consideration were constrained by degree conditions placed both on vertices and faces. This relies on the possibility to relax some of the constraints (the degree of the root-face and external vertices) until the last steps of the resolution.
In this paper, we have focused on non-separable triangulations but it is possible to practice the same kind of manipulations for all (that is to say *possibly separable*) triangulations. The method should also apply to some other families of maps, in particular to quadrangulations. Thus, a whole new class of map families is expected to have algebraic generating functions.

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