Enumeration of L-convex polyominoes. Bijection and area.

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Abstract

We consider the class of L-convex polyominoes, i.e. (convex) polyominoes in which any two cells can be connected by a path of cells in the polyomino that switches direction between the vertical and the horizontal at most once - such paths with one change of direction look like the letter L in one of its four cyclic orientations, hence the name. In this paper we prove that the number f_n of L-convex polyominoes with perimeter $2(n+2)$ satisfies the linear recurrence relation $f_{n+2} = 4f_{n+1} - 2f_n$, by determining a coding of such polyominoes in terms of words of a regular language over four letters, thus giving a bijection with the class of 2-compositions (a simple generalization of the ordinary compositions) with sum equal to n. Moreover we study some combinatorial properties of 2-compositions. In the last section we determine the area generating function of L-convex polyominoes.

1 L-convex polyominoes: basic definitions

A *polyomino* is a finite union of elementary cells of the lattice $\mathbb{Z} \times \mathbb{Z}$, whose interior is connected (see Fig. 1 (a)). A polyomino is *h-convex* (resp. *v-convex*) if every row (resp. column) is connected. A polyomino is *hv-convex*, or simply *convex*, if it is both h-convex and v-convex (see Fig. 1 (b)). In a polyomino the *semi-perimeter* is half the length of the border, while the *area* is the number of its cells.

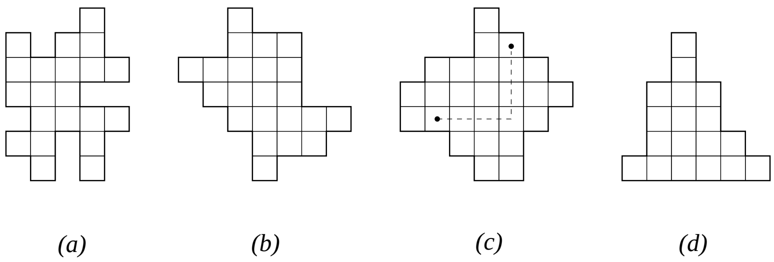

Figure 1: (a) a polyomino; (b) a convex polyomino; (c) a L-convex polyomino; (d) a stack polyomino.

In a polyomino we will define a *path* as a self-avoiding sequence of unitary steps of four types: *north* $(0,1)$, *south* $(0,-1)$, *east* $(1,0)$, and *west* $(-1,0)$. A path connecting two distinct cells A and B, starts from the center of A, and ends in the center of B (see Fig. 2 (a)). We say that a path is *monotone* if it is constituted only of steps of at most two types (see Fig. 2 (b)). Given a path $w = u_1 \dots u_k$, each pair of steps $u_i u_{i+1}$ such that $u_i \neq u_{i+1}$, $0 < i < k$, is called a *change of direction*.

In [4] the authors observe that a polyomino P is convex if and only if every pair of cells is connected by a monotone path. Hence, taking into account the minimum number of changes of

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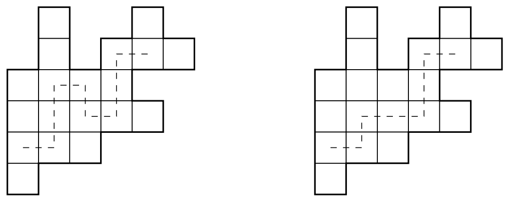


(a) (b)

Figure 2: (a) a path between two cells in a polyomino; (b) a monotone path made only of north and east steps.

direction in their monotone paths, they give a classification of convex polyominoes. In particular, they call *k-convex* a convex polyomino such that every pair of cells can be connected by a monotone path with at most k changes of direction. For $k = 1$ we have the *L-convex polyominoes* i.e. the class of polyominoes such that each two cells can be connected by a path with at most one change of direction (see Fig. 1 (c)). In an L-convex polyomino the *horizontal basis* (resp. *vertical basis*) is the set of rows (resp. columns) having maximal length; by definition, both the horizontal and the vertical basis are rectangles.

In this paper we will also deal with the well-known class of *stack polyominoes* [8] [12, p. 76] [13] (see Fig. 1 (d)).

Let us denote by L_n the set of L-convex polyominoes having semi-perimeter $n+2$. In [3], using the ECO method, it was proved that the numbers $f_n = |L_n|$ satisfy the recurrence relation:

$$f_{n+2} = 4f_{n+1} - 2f_n \qquad (n \geq 1) \tag{1}$$

with $f_0 = 1$, $f_1 = 2$, $f_2 = 7$, giving the sequence 1,2,7,24,82,280,956,3264, ... (sequence A003480 in [11]).

The main results of the paper are the following:

1. we prove that the class of 2-compositions (a natural extension of the ordinary compositions) is enumerated by the sequence $(f_n)_{n\geq 0}$, and then we obtain several other properties of such a sequence;
2. we determine a bijection between 2-compositions and L-convex polyominoes, thus giving a combinatorial explanation that L-convex polyominoes satisfy the recurrence in (1);
3. finally we find the generating function for L-convex polyominoes according to the area.

2 2-compositions

A *composition* of a natural number n is an ordered partition of n, that is a k-tuple $(x_1, \ldots, x_k)$ of positive integers such that $x_1 + \cdots + x_k = n$ (see [5]).

We now extend the definition of composition to the 2-dimensional case. For any positive integer k, a *2-composition* of *length* k is a $2 \times k$ matrix whose entries are nonnegative integers, such that each column has at least one non null element; the sum of the elements in a 2-composition M is called the *sum* of M. Let U_n be the class of 2-compositions with sum equal to n and let $u_n = |U_n|$. For instance

$$U_1 = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}, \qquad U_2 = \left\{ \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}$$

and $u_1 = 2$, $u_2 = 7$. In particular U_0 contains only the empty 2-composition, with length 0, and $u_0 = 1$.

In what follows we will study 2-compositions. Some of their properties are easy to prove and for brevity they will only be stated.

Proposition 1 *The numbers* u_n *satisfy the recurrence* $u_{n+2} = 4u_{n+1} - 2u_n$ *for* $n \geq 1$, *with the initial values* $u_0 = 1$, $u_1 = 2$, $u_2 = 7$.

Proof. Let $n \geq 1$. The 2-compositions in U_{n+2} can be all obtained by performing the following operations on each 2-composition $M \in U_{n+1}$:

1. add a column $\begin{bmatrix}1\\0\end{bmatrix}$ on the left of M;
2. add a column $\begin{bmatrix}0\\1\end{bmatrix}$ on the left of M;
3. increase by one the first element on the first row of M;
4. increase by one the first element on the second row of M.

By performing the four operations on the 2-compositions of U_{n+1} we obtain a set of $4u_{n+1}$ elements of U_{n+2}. However, some 2-compositions are obtained twice, and they are precisely those containing no null elements in the first column, that is:

1. those whose first column is $\begin{bmatrix}1\\1\end{bmatrix}$;
2. those whose first column is $\begin{bmatrix}x+1\\y+1\end{bmatrix}$, with x, $y \geq 0$ and $(x,y) \neq (0,0)$.

Since the number of elements in each class is clearly given by u_n it follows that $u_{n+2} = 4u_{n+1} - 2u_n$. Finally the initial values have been already determined in the initial examples. □

We have then the remarkable fact that the number of the L-convex polyominoes with semi-perimeter $n+2$ is equal to the number of the 2-compositions of n. In Section 3 we will determine a simple bijection between these two classes.

Let $u_{n,k}$ be the number of the elements of U_n having length k. The first terms of $u_{n,k}$ are presented in the table (a) of Fig. 3.

Proposition 2 *The numbers* $u_{n,k}$ *satisfy the recurrence relations*

$$u_{n+2,k+1} = 2u_{n+1,k+1} + 2u_{n+1,k} - u_{n,k+1} - u_{n,k}$$

$$u_{n+1,k+1} = u_{n,k+1} + 2u_{n,k} + u_{n-1,k} + \ldots + u_{0,k}\,.$$

In particular the infinite lower triangular matrix $[u_{n,k}]_{n,k\geq 0}$ *is a Riordan matrix with spectrum*

$$\left(1,\ \frac{2x - x^2}{(1-x)^2}\right).$$

See [10] for the theory of Riordan matrices.

Proposition 3 *The numbers* u_n *have the Pisot property:*

$$u_{n+1}^2 - u_{n+2}\,u_n = 2^{n-1} \qquad (n \geq 1) \tag{2}$$

(which reassembles the well-known Cassini's identity for Fibonacci numbers [7]).

Since every 2-composition can be viewed as the concatenation of its columns, it follows that the set $\mathbb{U}$ of all 2-compositions is the language $\mathcal{A}^*$ on the infinite alphabet

$$\mathcal{A} = \{\, a_{10},\ a_{01},\ a_{20},\ a_{11},\ a_{02},\ \ldots\},$$

where the letter a_{ij} corresponds to the column $\begin{bmatrix}i\\j\end{bmatrix}$. Then the generating series of $\mathbb{U}$ is

$$u(x_{10}, x_{01}, x_{20}, x_{11}, x_{02}, \ldots) = \frac{1}{1 - x_{10} - x_{01} - x_{20} - x_{11} - x_{02} - \cdots}$$

n/k	0	1	2	3	4	5	6
0	1						
1	0	2					
2	0	3	4				
3	0	4	12	8			
4	0	5	25	36	16		
5	0	6	44	102	96	32	
6	0	7	70	231	344	240	64

n/k	0	1	2	3	4	5	6
0	1						
1	1	1					
2	2	3	2				
3	4	8	8	4			
4	8	20	26	20	8		
5	16	48	76	76	48	16	
6	32	112	208	252	208	112	32

Figure 3: (a) Table of the numbers $u_{n,k}$; (b) table of the numbers $v_{n,k}$.

(see [9]). In particular, for $x_{ij} = x^{i+j}y$ we obtain the generating series

$$u(x,y) = \sum_{n,k \geq 0} u_{n,k}\, x^n y^k = \frac{1}{1 - xh(x)\, y}, \tag{3}$$

where

$$h(x) = \sum_{n \geq 0} (n+2)x^n = \frac{2-x}{(1-x)^2}\,.$$

This also proves the second part of Proposition 2. From (3) it follows the identity $u(x,y) = 1 + xy\, h(x)\, u(x,y)$ which implies the recurrence

$$u_{n+1,k+1} = \sum_{i=0}^{n} (i+2) u_{n-i,k}\,.$$

Finally, expanding (3) we can obtain the following explicit formula (for n , $k \geq 1$):

$$u_{n,k} = \sum_{j=0}^{k} \binom{k}{j} \binom{n+k-j-1}{2k-1} (-1)^j\, 2^{k-j}\,.$$

For $y = 1$ in (3), we reobtain the generating series $u(x)$ for the numbers u_n . We also retrieve that $u(x)$ is the *quasi-inversion* of the series $xh(x)$ as pointed out in [2], in a completely different study. Moreover, since $u(x) = 1 + xh(x)$, it follows that

$$u_{n+1} = \sum_{k=0}^{n} (k+2) u_{n-k}\,.$$

Another interesting statistic can be obtained in the following way. For any $n \geq 1$, the *projection* (here the term is used in the sense of the discrete tomography [6]) of the 2-composition

$$M = \begin{bmatrix} x_1 & x_2 & \dots & \dots & x_k \\ y_1 & y_2 & \dots & \dots & y_k \end{bmatrix} \in U_n$$

is the 2-composition

$$\pi(M) = \begin{bmatrix} x_1 + x_2 + \ldots + x_k \\ y_1 + y_2 + \ldots + y_k \end{bmatrix}.$$

Clearly $\pi(M)$ is still an element of U_n. Moreover, for any $M \in U_n$ let us define

$$[M] = \{Q \in U_n : \pi(Q) = \pi(M)\}\,.$$

For instance:

$$\left[\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right] = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

One can easily observe that for any $n \geq 0$, there are $n+1$ distinct classes $[M]$ in U_n. For $0 \leq k \leq n$, let $v_{n,k}$ be the number of elements of U_n whose projection is equal to $\begin{bmatrix} n-k \\ k \end{bmatrix}$. The first terms of $v_{n,k}$ are presented in the table in Fig. 3 (b) (sequence A059576 in [11]).

Proposition 4 *The generating series for the numbers* $v_{n,k}$ *is*

$$v(x,y) = \sum_{n,k\geq 0} v_{n,k}\, x^n y^k = \frac{1-x-xy+x^2y}{1-2x-2xy+2x^2y}\,.$$

In particular

$$\sum_{n\geq 0} v_{n,0}\, x^n = \frac{1-x}{1-2x}\,, \qquad \sum_{n\geq k} v_{n,k}\, x^n = \frac{2^{k-1}(x-x^2)^k}{(1-2x)^{k+1}} \quad (k\geq 1)\,.$$

Moreover the numbers $v_{n,k}$ *satisfy the recurrence*

$$v_{n+2,k+1} = 2v_{n+1,k+1} + 2v_{n+1,k} - 2v_{n,k}$$

and (for $(n,k)\neq(0,0)$ *)*

$$v_{n,k} = \sum_{j=0}^{\min(k,n-k)} \binom{k}{j}\binom{n-j}{k}(-1)^j\, 2^{n-j-1}\,.$$

3 A bijection between U_n and L_n

In this section we will present a bijection between L-convex polyominoes with semi-perimeter equal to $n+2$ and 2-compositions with sum n. In order to do this, we need first to represent L-convex polyominoes in terms of *2-colored stacks.* A stack polyomino is *2-colored* when its rows are colored black or white and satisfy the following priority conditions:

1. if a row is white then all the other rows of the same length above it (if any) have the same color;
2. the rows having maximal length are colored white.

Starting from an L-convex polyomino, we give the black color to the rows placed below the horizontal basis, and then vertically translate them above the basis respecting condition 1. (see Fig. 4 (b)). We observe that by the definition of L-convexity, the obtained polyomino is actually a 2-colored stack polyomino. Conversely, to each 2-colored stack polyomino there corresponds a unique L-convex polyomino.

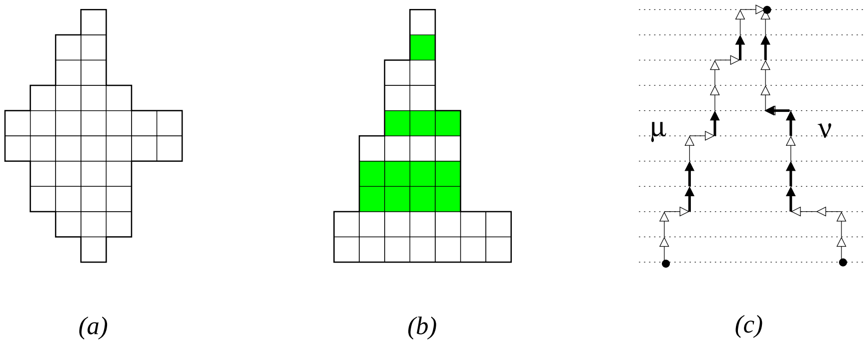


Figure 4: (a) an L-convex polyomino; (b) the corresponding 2-colored stack polyomino; (c) the paths μ and ν; for simplicity we represent the north, east and west steps by means of 2-colored arrows.

The boundary of a 2-colored stack polyomino is uniquely determined by two non-intersecting (except at the end points) lattice paths μ and ν (see Fig. 4 (c)):

1. μ runs from the leftmost point having minimal ordinate to the rightmost point having maximal ordinate in the polyomino, and uses 2-colored north and east unitary steps, that are black (resp. white) when it meets a black (resp. white) cell;

2. ν runs from the rightmost point having minimal ordinate to the rightmost point having maximal ordinate in the polyomino, and uses 2-colored north and west unitary steps, that are black (resp. white) when it meets a black (resp. white) cell.

By definition, both μ and ν start with a white north step, and μ ends with an east step.

Now we give a coding of 2-colored stacks in terms of words of a regular language over the alphabet $\{a, b, c, d\}$. The word representation of the polyomino is obtained by following the two paths, μ and ν, level by level from the bottom to the top of the polyomino. At each level one can meet:

1. a pair of north steps, one in μ, and the other in ν; in this case we write a (resp. d) if the steps are white (resp. black);
2. a sequence of east steps in μ, and, on the same horizontal line, a sequence of west steps in ν; in this case we write a b for each east step, and a c for each west step. By convention, we assume that, at the same level, we read east steps before west steps.

Using such a coding we have that any L-convex polyomino having semi-perimeter $n+2$ can be represented as a word in the alphabet $\{a, b, c, d\}$, having the same length. The language of all such words will be referred to as $\mathcal{K}$. For example, the word corresponding to the polyomino in Fig. 4 (a) is $aabccddabdcaabdab$. The number of rows (resp. columns) of the polyomino is given by the number of a plus the number of d (resp. the number of b plus the number of c) in the corresponding word of $\mathcal{K}$.

The words of $\mathcal{K}$ are characterized by the property that they begin with an a and end with a b, and contain neither the factor ad nor the factor cb. These simple observations lead us to state that $\mathcal{K}$ is a regular language, whose regular expression is:

$$a\left(a + b + c^+a + bd^+ + c^+d^+\right)^* b\,. \tag{4}$$

Notice that using the same coding we can represent stack polyominoes in terms of words on the alphabet $\{a, b, c\}$, beginning with an a and ending with a b, and not containing the factor cb. A coding of L-convex polyominoes in terms of a regular language has been also considered in [1] in order to investigate about ordering properties of polyominoes.

Let $l_{i,j}$ be the number of L-convex polyominoes with $i+1$ rows and $j+1$ columns, as shown in the table of Fig.5. From (4), removing the first and the last letter, we can obtain the generating function for these numbers, as described in [9], after setting $a = d = x$ and $b = c = y$:

$$l(x,y) = \sum_{i,j \geq 0} l_{i,j}\, x^i y^j = \frac{1}{1 - x - y - \dfrac{xy}{1-x} - \dfrac{xy}{1-y} - \dfrac{xy}{(1-x)(1-y)}}$$

that is

$$l(x,y) = \frac{(1-x)(1-y)}{1 - 2x - 2y + x^2 + y^2}\,. \tag{5}$$

Hence, it follows that the numbers $l_{i,j}$ satisfy the recurrence

$$l_{i+2,j+2} = 2l_{i+1,j+2} + 2l_{i+2,j+1} - l_{i,j+2} - l_{i+2,j}\,.$$

Letting $x = y$ in (5) we reobtain the generating function $f(x) = l(x,x)$ of $\mathcal{K}$ i.e. the generating function of L-convex polyominoes according to the semi-perimeter.

To conclude our bijection, we now give a representation of the words of $\mathcal{K}$ of length $n+2$ in terms of 2-compositions of U_n. First we observe that each word of $\mathcal{K}$ can be uniquely factorized into the factors:

$$c^h a, \quad bd^j, \quad c^r d^s, \qquad h, j \geq 0, \quad r, s \geq 1.$$

Let w be the word of $\mathcal{K}$ corresponding to a polyomino $P \in L_n$, from which we have removed the first and the last symbol; we use the following coding:

$$c^h a \to \begin{bmatrix} h+1 \\ 0 \end{bmatrix}, \qquad bd^k \to \begin{bmatrix} 0 \\ k+1 \end{bmatrix} \quad (h, k \geq 0), \qquad c^r d^s \to \begin{bmatrix} r \\ s \end{bmatrix} \quad (r, s \geq 1),$$

i/j	0	1	2	3	4	5	6
0	1	1	1	1	1	1	1
1	1	5	11	19	29	41	55
2	1	11	42	110	235	441	756
3	1	19	110	402	1135	2709	5740
4	1	29	235	1135	4070	11982	30618
5	1	41	441	2709	11982	42510	128534
6	1	55	756	5740	30618	128534	452900

Figure 5: Table of the numbers $l_{i,j}$ of L-convex polyominoes with $i+1$ rows and $j+1$ columns.

thus obtaining a 2-composition. For instance, the word *aabccddabdcaabdab* (corresponding to the polyomino in Fig. 4 (a)) is translated into the 2-composition

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 2 & 0 & 0 & 2 & 0 \end{bmatrix}.$$

The reader can easily observe that using the above coding is also easy to pass from a 2-composition to a word of $\mathcal{K}$, and then to an L-convex polyomino, which completes the bijection. Fig. 6 shows the bijection between L_2 and U_2.

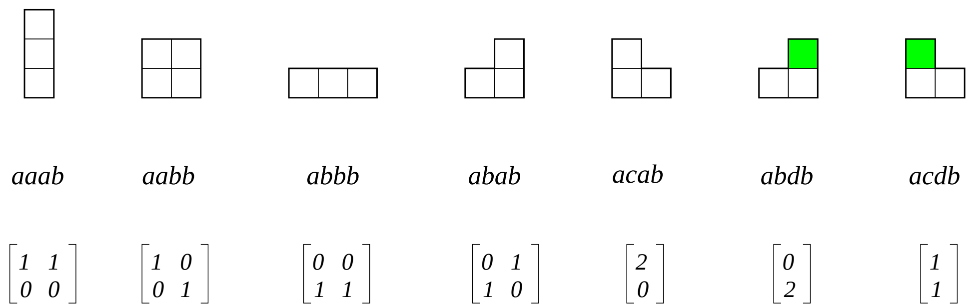


Figure 6: The bijection between L_2 and U_2.

Using the previously defined bijection, one natural question is how to interpret in terms of L-convex polyominoes the various properties determined in Section 2.

For instance, let us now consider the statistic in the table (b), Fig. 3. In terms of the word representation of a 2-composition M, the two entries of $\pi(M)$ are given by the number of a plus the number of c, and by the number of b plus the number of d, respectively.

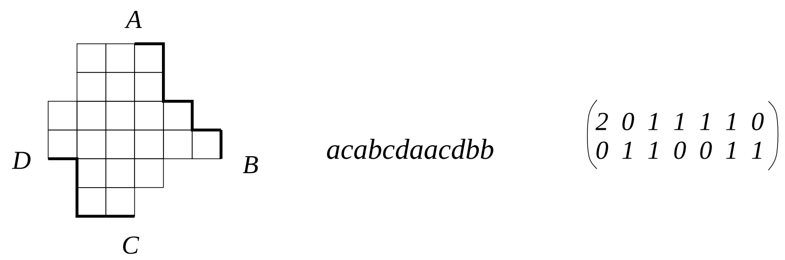


Figure 7: A polyomino P, the corresponding 2-composition M, with projection $\pi(M) = \binom{6}{4}$; the two entries of $\pi(M)$ are given by the lengths of the paths AB and CD minus one, respectively.

It is also possible to read the previous statistic in terms of L-convex polyominoes. Let P be an L-convex polyomino, and $M(P)$ (briefly, M) the corresponding 2-composition. Let us consider the following discrete points on P (see Fig. 7):

i) A is rightmost point having maximal ordinate of the vertical basis;

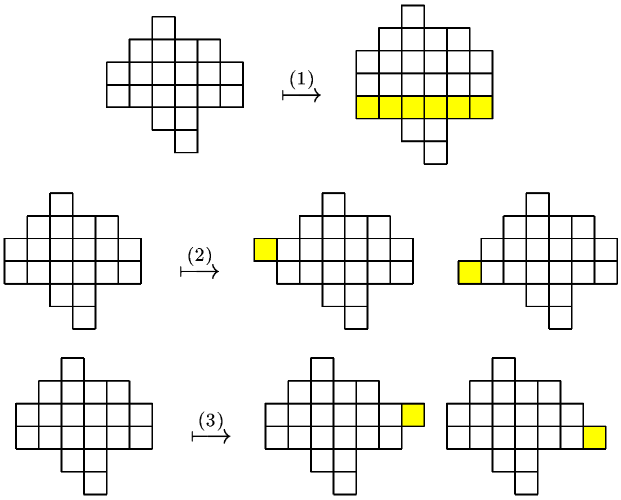


Figure 8: Generation of L-convex polyominoes.

ii) B is the rightmost point having minimal ordinate of the horizontal basis;

iii) C is the rightmost point having minimal ordinate of the vertical basis;

iv) D is the leftmost point having minimal ordinate of the horizontal basis.

Now, the element in the first row of $\pi(M)$ is given by the number of steps in the path connecting A to B, minus one. Analogously, the element in the second row of $\pi(M)$ is the number of steps in the path connecting C to D, minus one.

It would be also worth studying the interpretation of some parameters defined on a 2-composition (for instance the length of the composition), with respect to the correspondent L-convex polyomino through the bijection we have described above. On the other side, it would be interesting to investigate how some parameters defined on L-convex polyominoes (for example, the number of rows and columns, the area) can be interpreted on the corresponding 2-compositions.

4 Enumeration according to the area

In this section we determine the generating function of L-convex polyominoes according to the area, solving a problem posed in [3]. In order to do this, we first observe that every L-convex polyomino can be obtained with a sequence of the following operations starting from the polyomino formed by one cell:

1. add a new row of maximal length;
2. add a new cell on the left of a row of maximal length;
3. add a new cell on the right of a row of maximal length.

(See Fig. (8) for an example.) In this way, however, every polyomino with exactly one row of maximal length with a cell protruding on the left and a cell protruding on the right can be obtained two times applying operations 2. and 3. first in this order and then in the inverse order.

Let $L_{n,i,j}$ be the set of all L-convex polyominoes with semi-perimeter $n+2$ and $i+1$ rows of maximal length $j+1$ and let

$$a_{n,i,j}(q) = \sum_{P \in L_{n,i,j}} q^{a(\pi)}$$

where $a(P)$ is the area (i.e. the number of cells) of P. It follows that

$$a_{n+1,i+1,j}(q) = q^{j+1} a_{n,i,j}(q) \tag{6}$$

$$a_{n+2,0,j+2}(q) = \sum_{k=0}^{n+1} (k+1)(2q a_{n+1,k,j+1}(q) - q^2 a_{n,k,j}(q)). \tag{7}$$

Consider now the generating series

$$a_{i,j}(q;x) = \sum_{n\geq 0} a_{n,i,j}(q)\; x^n\,, \qquad a_j(q;x,y) = \sum_{n,i\geq 0} a_{n,i,j}(q)\; x^n y^i\,.$$

Let us consider L-convex polyomino whose maximal rows have length $j+1$. It may be reduced to an L-convex polyomino having a single row with that length. Otherwise, if it has several such rows, by removing one of them we can obtain a new L-convex polyomino of the same type. Fig. 9 depicts this decomposition.

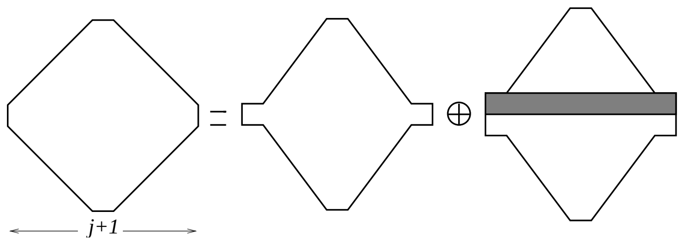


Figure 9: The decomposition of L-convex polyominoes whose maximal rows have length $j+1$

It follows that

$$a_j(q;x,y) = \frac{b_j(q;x)}{1 - q^{j+1}\,x\,y} \tag{8}$$

where $b_j(q;x) = a_{0,j}(q;x)$.

From equation (7) it follows

$$\mathcal{R}_x^2 b_{j+2}(q;x) = 2q\,[\,\mathcal{R}_x(\theta_y+1)a_{j+1}(q;x,y)\,]_{y=1} - q^2\,[\,(\theta_y+1)a_j(q;x,y)\,]_{y=1} \tag{9}$$

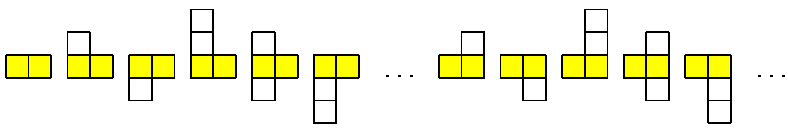
where $\mathcal{R}_x$ is the operator defined by $\mathcal{R}_x f(x) = (f(x)-f(0))/x$ and $\theta_y = y\frac{\mathrm{d}}{\mathrm{d}y}$. From equation (8) it follows that

$$[\theta_y a_j(q;x,y)]_{y=1} = \frac{q^{j+1} x b_j(q;x)}{(1-q^{j+1}x)^2}\,.$$

Since $b_{0,j+2}(q) = a_{0,0,j+2}(q) = 0$ and $b_{1,j+2}(q) = a_{1,0,j+2}(q) = 0$ for every j, equation (9) becomes

$$b_{j+2}(q;x) = \frac{2qx}{(1-q^{j+2}x)^2}\, b_{j+1}(q;x) - \frac{q^2x^2}{(1-q^{j+1}x)^2}\, b_j(q;x)\,. \tag{10}$$

Finally we need the initial values. For $j=0$ there is only the L-convex polyomino □ and hence $b_0(q;x) = q$. For $j=1$ we have all the following polyominoes



and then

$$b_1(q;x) = 2\sum_{h,k\geq 0} q^{h+k+2} x^{h+k+1} - q^2 x = \frac{q^2x(1+2qx-q^2x^2)}{(1-qx)^2}\,.$$

Recurrence (10), with the given initial values, completely determines the sequence $b_j(q;x)$ and easily allow to find that

$$b_j(q;x) = \frac{q^{j+1}x^j f_j(q;x)}{(1-qx)^2(1-q^2x)^2\cdots(1-q^jx)^2} \tag{11}$$

for suitable polynomials $f_j(q;x)$. Substituting the expression of $b_j(q;x)$ given by (11) in (10), it follows that the polynomials $f_j(q;x)$ satisfy the recurrence

$$f_{j+2}(q;x) = 2f_{j+1}(q;x) - (1-q^{j+2}x)^2 f_j(q;x) \tag{12}$$

with the initial conditions $f_0(q;x) = 1$ and $f_1(q;x) = 1 + 2qx - q^2x^2$. This completely defines the polynomials $f_k(q;x)$.

Then we have that

$$a_j(q;x,y) = \frac{q^{j+1}x^j f_j(q;x)}{(1-qx)^2(1-q^2x)^2\cdots(1-q^jx)^2(1-q^{j+1}xy)}$$

and finally

$$a(q;x,y,z) = \sum_{n,i,j\geq 0} a_{n,i,j}(q)x^ny^iz^j = \sum_{k\geq 0} \frac{q^{k+1}x^kz^kf_k(q;x)}{(1-qx)^2(1-q^2x)^2\cdots(1-q^kx)^2(1-q^{k+1}xy)} \tag{13}$$

From this series we can deduce several other generating series. First of all for, $q = 1$, equation (12) becomes

$$f_{j+2}(1;x) = 2f_{j+1}(1;x) - (1-x)^2 f_j(1;x)$$

with $f_0(1;x) = 1$ and $f_1(1;x) = 1 + 2x - x^2$. Then the generating series for these polynomials is

$$f(1;x,t) = \frac{1-(1-x)^2t}{1-2t+(1-x)^2t^2}. \tag{14}$$

Hence, for $q = 1$ the series (13) becomes

$$a(1;x,y,z) = \frac{1}{1-xy} f\left(1;x,\frac{xz}{(1-x)^2}\right) = \frac{(1-x)^2(1-xz)}{(1-xy)(1-2x+x^2-2xz+x^2z^2)}.$$

In particular, for $y = z = 1$, we obtain another time the generating series

$$a(1;x,1,1) = \frac{(1-x)^2}{1-4x+2x^2}$$

for the number of L-convex polyominoes according to semi-perimeter.

Finally, let a_n be the number of all L-convex polyominoes with area n. The first terms of this sequence are: 1, 1, 2, 6, 15, 35, 76, 156, 310, 590, 1098, 1984, 3515, 6094, 10398, 17434, 28837, 47038, 75820. This sequence is not in the *Encyclopedia of Integer Sequences* [11]. From (13) follows the main proposition:

Proposition 5 *The generating series of the sequence* a_n *is*

$$a(q) = \sum_{n\geq 0} a_n\, q^n = 1 + \sum_{k\geq 0} \frac{q^{k+1}f_k(q;1)}{(1-q)^2(1-q^2)^2\cdots(1-q^k)^2(1-q^{k+1})}.$$

This series is very similar to the generating series [13]

$$s(q) = \sum_{n\geq 0} s_n\, q^n = 1 + \sum_{k\geq 0} \frac{q^{k+1}}{(1-q)^2(1-q^2)^2\cdots(1-q^k)^2(1-q^{k+1})}.$$

for the numbers s_n that count stacks with area n (sequence A001523 in [11]). They only differ for the presence of the polynomials $f_k(q;1)$. It could be interesting a deeper investigation about the (combinatorial and formal) structure of such polynomials.

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