The Discrete Fundamental Group of the Order Complex of B_n

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Abstract

A-theory is a recently developed area of algebraic combinatorics that takes concepts from algebraic topology and transfers them to a combinatorial setting. It contains discrete analogues to continuity, homotopy, and fundamental group, defined on graphs and simplicial complexes. We provide a construction for a graph arising from the order complex of the direct product of two graded lattices. With this construction, we show that the rank of the abelianization of the discrete fundamental group of the order complex of the Boolean lattice, B_n, is $2^{n-3}(n^2-5n+8)-1$. This result recovers a formula from Björner and Welker's work on the computational complexity of the k-equal problem, a computer science application.

1 Introduction

An early appearance of a discrete homotopy theory can be found in the work of Atkin [1, 2] in the early 1970s. A physicist modeling social networks using simplicial complexes, Atkin developed Q-analysis, a discrete topological theory used to measure the combinatorial connectivity of a complex and identify combinatorial "holes" in the complexes. In 1972, Maurer [8] developed a similar concept of discrete deformation of paths in graphs while working on his dissertation, developing a characterization of basis graphs of matroids. In 1983, Malle [7] also defined a notion of equivalence of graph maps, as well as discrete fundamental group. These authors were apparently unaware of each other's work, but in fact the concepts they created are all equivalent. More recently, Laubenbacher and Kramer [6] became aware of Atkin's work while conducting research in social and communications networks. With Barcelo and Weaver [3], they pursued Atkin's ideas in Q-analysis and extended them to include graphs and discrete analogues to higher homotopy groups. They also named their work A-theory in his honor.

Let $\Gamma=(V,E)$ and $\Gamma'=(V',E')$ be *simple* graphs, with no loops or parallel edges. A *graph map* $f:\Gamma\to\Gamma'$ is a set map $V\to V'$ that preserves adjacency, that is, if $vw\in E$, then either $f(v)$ is adjacent to $f(w)$ in Γ', denoted by $f(v)\sim_{\Gamma'} f(w)$, or $f(v)=f(w)$. Let $v\in V$ and $v'\in V'$ be distinguished vertices. A *based graph map* is a graph map $f : (\Gamma, v) \to (\Gamma', v')$ such that $f(v) = v'$. The *box product* $\Gamma \Box \Gamma'$ of two graphs, Γ and Γ', is the graph with vertex set $V \times V'$ and an edge between (v, v') and (w, w') if either

1. $v = w$ and $v' \sim_{\Gamma'} w'$, or
2. $v' = w'$ and $v \sim_{\Gamma} w$.

Let I_m be the path on $m+1$ vertices, with vertices labeled from 0 to m. The boundary, ∂I_m, is the set of vertices 0 and m. This path has the same role as that of the unit interval in classical homotopy theory.

Definition 1.1. [3]

Let $\Gamma = (V, E)$ and $\Gamma' = (V', E')$ be simple graphs with distinguished vertices $v_0, v_1 \in V$ and $v_0', v_1' \in V'$. Let f and g be based graph maps $\Gamma \to \Gamma'$ such that $f(v_0) = g(v_0) = v_0'$ and $f(v_1) = g(v_1) = v_1'$. We say that f and g are G-homotopic relative to v_0' and v_1', denoted by $f \simeq_G g \quad rel(v_0', v_1')$ if there is an integer n and a graph map $F : \Gamma \Box I_n \to \Gamma'$ that discretely deforms f into g, specifically

1. $F(v, 0) = f(v) \qquad \forall\, v \in V$
2. $F(v, n) = g(v) \qquad \forall\, v \in V$
3. $F(v_0, j) = v_0' \qquad 0 \le j \le n$
4. $F(v_1, j) = v_1' \qquad 0 \le j \le n.$

If $v_0' = v_1'$, then we write $f \simeq_G g \quad rel(v_0')$, or simply $f \simeq_G g$ of the base vertex is clear.

While G-homotopy is defined for graph maps in general, for the remainder of this discussion, we will limit our investigation to graph maps defined on the discrete interval I_m. If a based graph map $f : (I_m, \partial I_m) \to (\Gamma, v)$ sends ∂I_m to the base vertex v in Γ, then the image of f is a *string loop* in Γ, or simply a *loop*, based at v. Furthermore, we can "stretch" a graph map $f : (I_m, \partial I_m) \to (\Gamma, v)$ to define it on a larger discrete interval by sending vertices with labels $> m$ to v. We can view these graph maps as being defined on $\mathbb{Z}$, with only finitely many images not equal to v, so we may drop the subscript m.

$$f : (I_m, \partial I_m) \to (\Gamma, v) \simeq_G \tilde{f} : (I_p, \partial I_p) \to (\Gamma, v), \quad p > m$$

Multiplication of graph maps f and g is equivalent to the concatenation of the loops corresponding to the maps. Furthermore, G-homotopy is an equivalence relation on the set of based graph maps from the discrete interval I to Γ. Barcelo, Kramer, Laubenbacher, and Weaver [3] showed that these equivalence classes, with multiplication, form a group, denoted by $A_1^G(\Gamma, v)$, and referred to simply as the A_1-group of Γ. As in classical topology, if Γ is connected, the discrete fundamental group $A_1^G(\Gamma, v)$ is independent of the choice of base vertex.

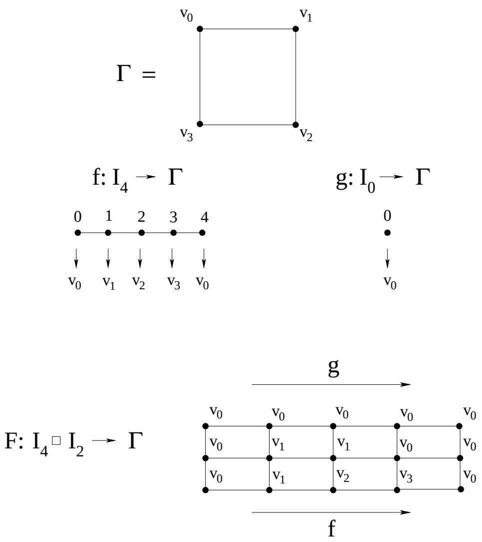

Figure 1: A G-homotopy from f to g.

Figure 1 shows an example of two G-homotopic graph maps; the image of f is the 4-cycle Γ, and the image of g is the single vertex v_0. The vertices of the graph (grid) $I_4 \Box I_2$ are labeled with the image of a G-homotopy from f to g, where g has been stretched so that it is also defined on I_4. The G-homotopy F is itself a graph map which must preserve adjacency, thus each edge in the grid corresponds to an edge or a single vertex in Γ.

Furthermore, is is straightforward to show that any based graph map from the discrete interval to the 4-cycle is G-homotopic to the constant map g, so the A_1 group of the 4-cycle, and similarly the 3-cycle, is trivial. Barcelo et al. [3] also showed that $A_1^G(\Gamma, v) \simeq \pi_1(\Gamma, v)/N$, where $\pi_1(\Gamma, v)$ is the classical fundamental group of Γ when considered as a 1-dimensional simplicial complex and N is the normal subgroup generated by 3-and 4-cycles. Thus, computing the A_1 group of a graph is equivalent to attaching 2-cells to the 3- and 4-cycles of the graph and computing the classical fundamental group of the resulting topological space.

2 Constructing the Graph for the Boolean Lattice

There is an equivalent definition of discrete homotopy for simplicial complexes that we will use in order to compute the discrete fundamental group of the order complex of B_n, the Boolean lattice. This definition includes a graded version of the discrete fundamental group, related to the dimension of the intersection of simplices in a simplicial complex. This complete definition can be found in [3], however, here we will only be concerned with the highest of these groups. In general, to compute the discrete fundamental group of a simplicial complex Δ, we first construct a graph $\Gamma(\Delta)$ and then we compute $A_1^G(\Gamma(\Delta))$.

The simplicial complex we will consider here is the order complex of B_n. The i-faces of $\Delta(B_n)$ correspond to the i-chains of B_n. When we construct the graph, $\Gamma((\Delta(B_n))$, or simply $\Gamma(B_n)$, associated with the highest of the discrete fundamental groups, the vertices of the graph correspond to the maximal faces of $\Delta(B_n)$. These maximal faces are the maximal chains in B_n after the $\hat{0}$ and $\hat{1}$ are removed, or equivalently, permutations in S_n. Two vertices in $\Gamma(B_n)$ are adjacent if the two chains in B_n differ in precisely one element. In this case, the associated permutations in S_n differ by multiplication on the right by a simple transposition $(i, i+1)$, for some $1 \leq i \leq n-1$.

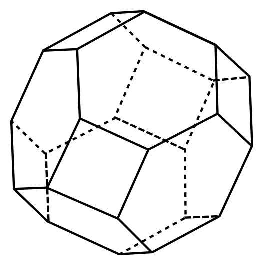

Figure 2: The 1-skeleton of the permutahedron P_3.

We note that $\Gamma(B_n)$ is the 1-skeleton of the permutahedron P_{n-1} [11], and we see the graph for $n = 4$ in Figure 2. We can see that if we attach 2-cells to the 4-cycles in $\Gamma(B_4)$, we are left with 6-cycles. Thus, in order to compute the rank $A_1^G(\Gamma(B_n))^{ab}$, the abelianization of the A_1-group, we are looking for a way to define and count equivalence classes of based graph maps defined on the discrete interval whose images are 6-cycles in $\Gamma(B_n)$. Unfortunately, it is not easy to see a relationship just by looking at the permutahedron.

The breakthrough that allows us to understand the G-homotopy relation on $\Gamma(B_n)$ is the simple observation that B_n is isomorphic to the direct product $B_{n-1} \times B_1$. We will use this observation to define a method for constructing $\Gamma(B_n)$ that will make it easier for us to compute the rank of $A_1^G(\Gamma(B_n))^{ab}$. The graph $\Gamma(B_n)$ is not isomorphic to $\Gamma(B_{n-1})\Box\Gamma(B_1)$ because a maximal chain in B_n corresponds to a shuffle of the edges of a maximal chain in B_{n-1} with the single edge from B_1. These edges can be shuffled in more than one way, so the product of the graphs for the smaller lattices will not have enough vertices.

To solve this problem, we introduce the *shuffle graph*. The vertices of the shuffle graph $\Gamma^{n-1,1}_{shuffle}$ correspond to shuffles of a maximal chain in B_{n-1} with the single edge from B_1. Two vertices are adjacent if the shuffles differ by a switch of two consecutive edges, one from B_{n-1} and the other from B_1. We note that $\Gamma(B_1)$ is a single vertex and $\Gamma^{n-1,1}_{shuffle}$ is a path on n vertices. We use this shuffle graph in the construction of another graph, $\widetilde{\Gamma}(B_n)$:

$$\widetilde{\boldsymbol{\Gamma}}(\mathbf{B_n}) = \boldsymbol{\Gamma}(\mathbf{B_{n-1}})\Box\boldsymbol{\Gamma}(\mathbf{B_1})\Box\boldsymbol{\Gamma}^{\mathbf{n-1,1}}_{\mathbf{shuffle}}$$

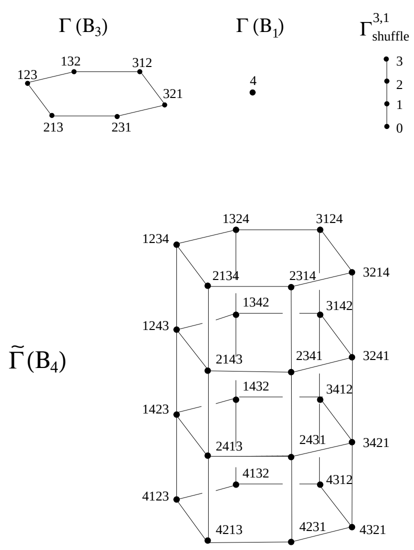

Figure 3: The intermediate graph $\widetilde{\Gamma}(B_4)$.

The vertices in $\widetilde{\Gamma}(B_n)$, the box product of three graphs, are ordered triples. The first coordinate is a permutation in S_{n-1} corresponding to a maximal chain in B_{n-1}. The second coordinate is the element n, corresponding to the single edge in B_1. The third coordinate is an integer i, $0 \le i \le n-1$, and it uniquely defines the shuffle of the two chains by indicating how many edges of the chain from B_{n-1} are below the edge from B_1 in the resulting chain. Two vertices in $\widetilde{\Gamma}(B_n)$ are adjacent if either the chains in B_{n-1} are the same and the shuffles are adjacent in $\Gamma^{n-1,1}_{shuffle}$, or the chains differ in one element and the shuffles are the same.

The graph $\widetilde{\Gamma}(B_n)$ now has the right number of vertices, but there are too many edges because some edges are incident to pairs of chains in B_n that differ in more than one element. This problem, however, is easily solved by removing a well-defined set of edges from $\widetilde{\Gamma}(B_n)$ to obtain the desired graph $\Gamma(B_n)$ [9]; we can use $\Gamma(B_4)$ to illustrate the following properties of $\Gamma(B_n)$.

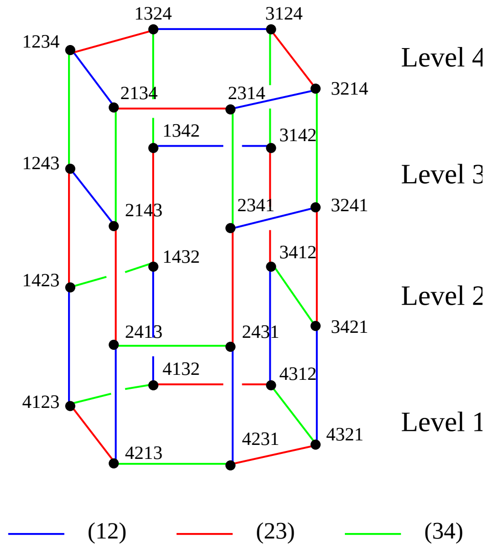

Figure 4: The final graph $\Gamma(B_4)$.

1. **Vertices.** We label the vertices with permutations in S_n, written in single line notation.

2. **Edges.** Each edge corresponds to a simple transposition. The graph is bipartite and $(n-1)$-regular, with each vertex incident to precisely one edge for each of the $n-1$ simple transpositions in S_n.

3. **Bipartite.** The graph is bipartite, with vertices partitioned into even and odd permutations, thus all cycles in the graph are of even length.

4. **Cycles.** The set of transpositions labeling the edges of a cycle correspond to a representation of the identity in S_n. Each 4-cycle in the graph correspond to a pair of disjoint transpositions such as (12) and (34). Each 6-cycle corresponds to a pair of consecutive simple transpositions, such as (12) and (23). All other cycles of length ≥ 8 can be expressed as the concatenation of 4- and 6-cycles. Therefore, we can limit our investigation to 6-cycles which are not the concatenation of 4-cycles. We want to count the G-homotopy equivalence classes of 6-cycles in B_n, which form a basis for $A_1^G(\Gamma(B_n))^{ab}$.

5. **Levels.** The may view the graph as having n levels; each level was initially a copy of $\Gamma(B_{n-1})$ before we removed edges from $\widetilde{\Gamma}(B_n)$. All vertices in a single level resulted from the use of the same shuffle, so all permutations in level i have the element n as entry i when the permutations are written in single line notation. We can classify each edge in the graph as *horizontal* if it is incident to two vertices within the same level of $\Gamma(B_n)$, or *vertical* if it is incident to vertices in consecutive levels of the graph. We note that all vertical edges between two consecutive levels correspond to the same transposition. We can similarly define horizontal and vertical 6-cycles. All vertices in a horizontal cycle are in the same level. A vertical 6-cycle contains two vertices in each of three consecutive levels. We identify each vertical 6-cycle with the middle of the three levels. For example, 1243-2143-2413-4213-4123-1423 is a vertical 6-cycle at level 2 in $\Gamma(B_4)$, and its edges correspond to the transpositions (12) and (23).

3 Equivalence Classes

In this section, we describe how to distinguish between different G-homotopy equivalence classes so that we may count them. The proof of both Lemma 3.1 and Theorem 3.2 rely heavily on checking many possible cases of the labelings of G-homotopy grids, the precise details of which we will not go into here, but we give a brief outline of each proof to capture the flavor of the argument. The complete details of the proofs are contained in [9].

Let C_1 and C_2 be two distinct 6-cycles in $\Gamma(B_n)$, and suppose that the edges of C_1 are associated with the transpositions $(i-1, i)$ and $(i, i+1)$ for some i, $2 \leq i \leq n-1$. If $C_1 \simeq_G C_2$, then, as in our example in Figure 1, we must be able to construct a G-homotopy grid so that the image of the first row of the grid is C_1 and the image of the last row is C_2. Recall that a G-homotopy is itself a graph map and must preserve adjacency. When we consider the various changes that we can make from row to row in the grid that will preserve adjacency, we find that they will also preserve the parity of the number edges in each row that are associated with $(i-1,i)$ and $(i,i+1)$. In particular, the last row must also contain an odd number of edges associated with each of $(i-1,i)$ and $(i,i+1)$, and consequently the edges of C_2 are also associated with this same pair of simple transpositions. This leads us to our initial description of equivalence classes of 6-cycles.

Lemma 3.1. *Let C_1 and C_2 be two distinct 6-cycles in $\Gamma(B_n)$. If $C_1 \simeq_G C_2$, then they are associated with the same pair of transpositions, $(i-1,i)$ and $(i,i+1)$, for some i, $2 \leq i \leq n-1$.*

Association with the same pair of transpositions is a necessary condition for G-homotopy of 6-cycles, but it turns out not to be sufficient. In order to guarantee that two 6-cycles, C_1 and C_2, are G-homotopic, they must also differ by a sequence of simple transpositions, $\tau_1\tau_2...\tau_k$, where each τ_j is disjoint from $(i-1,i)$ and $(i,i+1)$. That is, if we multiply each of the six permutations in C_1 by the same sequence $\tau_1\tau_2\cdots\tau_k$, the result is precisely the six permutations in C_2. To indicate this relationship, we write $C_2 = C_1\tau_1\tau_2\cdots\tau_k$.

Theorem 3.2. *Let C_1 and C_2 be two distinct 6-cycles in $\Gamma(B_n)$. Then $C_1 \simeq_G C_2$ iff there exists an integer $k \geq 1$ such that $C_2 = C_1\tau_1\ldots\tau_k$ where C_1 and C_2 are both associated with $(i-1,i)$ and $(i,i+1)$ for some i, $2 \leq i \leq n-1$, and the τ_j are simple transpositions in S_n that are disjoint from $(i-1,i)$ and $(i,i+1)$.*

Proof sketch. The first part of the proof is constructive: assuming $C_2 = C_1\tau_1\ldots\tau_k$, we construct a G-homotopy from C_1 to C_2 whose image is a sequence of 6-cycles connected by 4-cycles. Figure 5 is the image of a such a G-homotopy from C_1 to $C_2 = C_1\tau_1\tau_2\tau_3$.

In the second part of the proof we assume $C_1 \simeq_G C_2$, which means there is a path P such that $C_1PC_2^{-1}P^{-1} \simeq_G \sigma$, where σ is a permutation in C_1. The edges of P correspond to simple transpositions, and the product of these transpositions is a permutation in S_n. Therefore we must be able to construct another valid G-homotopy grid, this time with the first row corresponding to $C_1PC_2^{-1}P^{-1}$ and the last row corresponding to the single vertex σ. By again performing a check of all possible cases, we can show that

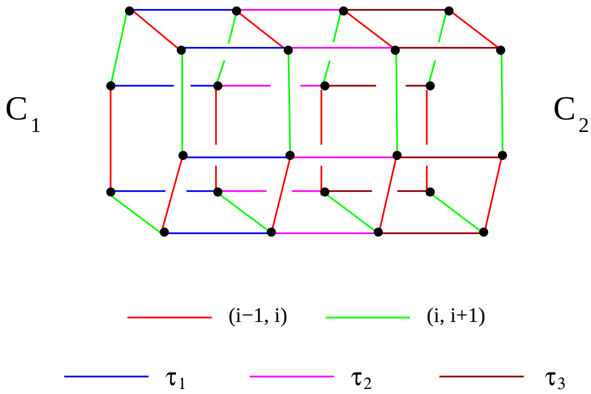

Figure 5: A G-homotopy from C_1 to C_2.

this permutation can be written using only transpositions that are disjoint from the pair $(i-1, i)$ and $(i, i+1)$ associated to C_1 and C_2.

□

Theorem 3.2 stems from the definition of a G-homotopy from C_1 to C_2, and the limitations on the types of changes we are able to make from row to row of the G-homotopy grid. We can combine this theorem with our new understanding of the structure of $\Gamma(B_n)$ to make the following observations about equivalence classes:

1. **Horizontal and vertical 6-cycles are in different equivalence classes.** All permutations in a single horizontal 6-cycle have the element n in the same position when they are written in single-line notation because they result from the use of the same shuffle. In a vertical 6-cycle, the element n will be in different positions in the permutations, depending on which of three consecutive levels each permutation is in. Consequently a horizontal 6-cycle cannot be G-homotopic to a vertical 6-cycles, so we may count horizontal and vertical equivalence classes separately.

2. **Counting Horizontal Equivalence Classes.** We can count the horizontal equivalence classes in level n, which remains a copy of $\Gamma(B_{n-1})$ even after we removed edges in the construction of $\Gamma(B_n)$. We can show that a horizontal 6-cycle in another level of $\Gamma(B_n)$ is G-homotopic to the concatenation of a 6-cycle in level n with vertical 4- and 6-cycles,

and will therefore be contained in the product of a horizontal equivalence class counted at level n with vertical equivalence classes described below.

3. **Vertical 6-cycles at different levels of Γ_{B_n} are in different equivalence classes.** This is a direct consequence of Lemma 3.1 and the observation we made in Section 2 that a vertical 6-cycle at level i, $2 \leq i \leq n-1$ is associated with transpositions $(i-1,i)$ and $(i,i+1)$.

4. **There are $\binom{n-1}{i}\binom{i}{2}$ equivalence classes at level i of Γ_{B_n}, $2 \leq i \leq n-1$.** We can count the number of vertical equivalence classes at level i by using the order of the subgroup of S_n generated by transpositions disjoint from $(i-1,i)$ and $(i,i+1)$ to determine the number of 6-cycles in each equivalence class.

Using standard techniques [10] to count the equivalence classes described above gives us a total of $2^{n-3}(n^2-5n+8)-1$ classes. Each cycle of length ≥ 8 is contained in the product of one or more of these classes, thus the collection of equivalence classes forms a basis for $A_1^G(\Gamma(B_n))^{ab}$.

4 Related Questions

In the beginning of Section 2, we noted that computing the A_1 group of $\Gamma(B_n)$ is equivalent to attaching 2-cells to the 4-cycles of the graph and computing the classical fundamental group of the resulting 2-dimensional topological space. Eric Babson noted in 2001 that attaching 2-cells to the 4-cycles in $\Gamma(B_n)$ results in a topological space that is homotopy equivalent to the complement (in $\mathbb{R}^n$) of the 3-equal hyperplane arrangement. The k-equal hyperplane arrangements have been extensively studied, and it turns out that these two problems are in fact related (for more details see [4]). Our computation of the rank of $A_1^G(\Gamma(B_n))^{ab}$ recovers a formula of Björner and Welker [5] for the dimensions of the homology groups of the 3-equal arrangements.

The definition of the shuffle graph $\Gamma^{n-1,1}_{shuffle}$ can be generalized to a graph $\Gamma^{k,l}_{shuffle}$ and used to construct the graph associated with the $\Delta(L_1 \times L_2)$, the order complex of the direct product of finite ranked lattices of rank k and l, respectively. The arguments in the computation of the rank of $A_1^G(\Gamma(B_n))^{ab}$ depended heavily on the structure of S_n in determining what changes are permissible from row to row in a valid G-homotopy grid. Nevertheless, using the construction described in Section 2 to build $\Gamma(\Delta(L))$ from smaller graphs may prove useful in obtaining results for lattices other than B_n.

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