DECIDING THE COHEN-MACAULAY PROPERTY FOR BIPARTITE GRAPHS

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Abstract. An algorithm is provided in order to decide whether a bipartite graph is Cohen-Macaulay. It works by appropriately deleting vertices from the given graph and by applying known properties on the obtained subgraphs.

1. Introduction

The Cohen-Macaulay property of a graph (CM for short) is worth investigating, since it comes from an algebraic concept and the combinatorial meaning is not so evident. In fact it is difficult to recognize a CM graph just by looking at it. So, it is interesting to find necessary and sufficient conditions for a graph in order to be Cohen Macaulay and it would be very useful to find a decision procedure.
Cohen-Macaulay graphs are investigated in several works, see for example [6], where one can find constructions of CM graphs and properties about bipartite CM graphs. The latter ones are characterized in [4]. It is also known that a chordal graph is CM if and only if it is unmixed (see [5]) and that the complement of a d-tree is CM (see [2]). Actually there is no decision procedure for CM graphs. In this paper we show an algorithm for checking Cohen-Macauly property of a bipartite graph. Such algorithm uses some results about CM graphs in [6] and it is based on the decision procedure for bipartite graphs and vertex covers in [1].

2. Cohen-Macaulay Graphs

Here we will introduce the concept of Cohen-Macaulay graph and all definitions and properties, that we will use as tools for studying such graphs.

DEFINITION 2.1. *The ascending chain condition, commonly abbreviated "A.C.C.," for a partially ordered set X requires that all increasing sequences in X become stationary.*

DEFINITION 2.2. *A ring is called* Noetherian *if it does not contain an infinite ascending chain of ideals.*

REMARK 2.1. *If R is Noetherian, it satisfies the ascending chain condition on ideals.*

PROPOSITION 2.1. *The following properties are equivalent.*

(1) *R satisfies the ascending chain condition on ideals.*
(2) *Every ideal of R is finitely generated.*
(3) *Every set of ideals contains a maximal element.*

Let M be a module over a ring R. We say that $x \in R$ is a *M-regular element* if it is not a zero-divisor on M.

DEFINITION 2.3. *A sequence* $\mathbf{x} = x_1, \ldots, x_n$ *of elements of R is called a* M-regular sequence *if*

- *(i)* x_i *is a* $M/(x_1, \ldots, x_{i-1})$ *M-regular element for* $i = 1, \ldots, n$;
- *(ii)* $M/\mathbf{x}M \neq 0$.

EXAMPLE 2.1. *The typical example of regular sequence is the sequence* $x_1, \ldots, x_n$ *of indeterminates in a polynomial ring* $R = S[x_1, \ldots, x_n]$.

Let R be a Noetherian ring and let M be a R-module. If $\mathbf{x} = x_1, \ldots, x_n$ is a M-sequence, then the sequence of ideals $(x_1) \subset (x_1, x_2) \subset \ldots \subset (x_1, \ldots, x_n)$ is strictly ascending. Therefore a M-sequence can be extended to a maximal sequence in the following way: a M-sequence $\mathbf{x}$ in an ideal I is *maximal* in I if $x_1, \ldots, x_{n+1}$ is not a M-sequence for any $x_{n+1} \in I$.

THEOREM 2.1. *(Rees)*
Let R be a Noetherian ring. Let M be a finite $R-$*module and let I be an ideal, such that* $IM \neq M$. *Then all maximal M-sequences in I have the same length n, that is called* grade *of I on M, and it is denoted by* $grade(I, M)$.

DEFINITION 2.4. *A ring R is called* local*, if it has a unique maximal ideal $\mathfrak{m}$ and it is denoted by $(R, \mathfrak{m})$.*

The notions of grade and Noetherian local ring imply the definition of depth of a local ring R.

DEFINITION 2.5. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let M be a finite R-module. Then the grade of $(R, \mathfrak{m})$ on M is called the* depth *of M and it is denoted by $depth M$.*

We introduce the notion of height of an ideal $\mathfrak{p}$ in R, in order to define the dimension of a commutative ring R. *height* $\mathfrak{p}$, is the supremum of the lengths t of strictly descending chains $\mathfrak{p} = \mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \ldots \supset \mathfrak{p}_t$ of prime ideals.

DEFINITION 2.6. *Let $(R, \mathfrak{m})$ be a local ring. The* dimension *of R is the height of $\mathfrak{m}$ and it is denoted by $dim R$.*

In general $depth R \leq dim R$.
Finally we define a Cohen-Macaulay ring

DEFINITION 2.7. *Let R be a Noetherian local ring. A finite R-module $M \neq 0$ is a* Cohen-Macaulay *module if depth M=dim M. If R itself is a Cohen-Macaulay module, then it is called a* Cohen Macaulay *ring.*

DEFINITION 2.8. *A noetherian ring R is said to be a* Cohen-Macaulay *ring if $R_{\mathfrak{m}}$ is a* Cohen-Macaulay *ring for every maximal ideal $\mathfrak{m}$ of R.*

To every undirected graph G with the vertex set $V(G) = \{v_1, \ldots, v_n\}$ and the edge set $E(G) = \{e_1, \ldots, e_m\}$ it is possible to associate a monomial ideal $I(G)$, that is generated by all square free monomials $v_i v_j$, such that $\{v_i, v_j\} = e_h$ is an edge of G. Such an ideal is usually called the monomial *edge ideal*.

DEFINITION 2.9. *G is said* Cohen-Macaulay *(CM for short) with respect to the field K, if the quotient ring $K[v_1, \ldots, v_n]/I(G)$ is Cohen-Macaulay.*

DEFINITION 2.10. *A* vertex cover V' *of a graph* G *is a subset of vertices of* G*, such that at least one vertex of every edge of* G *is in* V'*. A vertex cover* V' *is said to be* minimal *if no subsets of* V' *is itself a vertex cover.*

Of course every graph has vertex covers (it is enough to take the whole set of vertices). By using the following proposition of Villarreal it is possible to find them by looking at the primary decomposition of the monomial edge ideal.

PROPOSITION 2.2. *(See* [6]*, chapter 6 proposition 1.16)*
Let $K[v] = K[v_1, ..., v_n]$ *be a polynomial ring over a field* K *and let* G *be an undirected graph. If* P *is the ideal of* $K[v]$ *generated by* $A = \{v_{i1}, \ldots, v_{ir}\}$*, then* P *is a minimal prime over the edge ideal* $I(G)$ *if and only if* A *is a minimal vertex cover of* G*.*

As a corollary of the previous proposition we obtain a way to compute the height of an edge ideal.

COROLLARY 2.1. *(See* [6]*, chapter 6 corollary 1.18)*
If G *is a graph and* $I(G)$ *its monomial edge ideal, then the height of* $I(G)$ *is equal to the vertex covering number* $\alpha_0(G)$*, that is the smallest number of vertices in a minimal vertex cover.*

DEFINITION 2.11. *A graph is said* unmixed *if all minimal vertex covers have the same cardinality.*

REMARK 2.2. *A Cohen-Macaulay graph is unmixed. (See, for instance,* [4]*)*

Finally it is useful to introduce the definition of bipartite graph.

DEFINITION 2.12. *A graph* G *is* bipartite*, if its vertices can be divided in two sets, such that no edge connects vertices in the same set. Here we will call these two sets* partition sets*. Equivalently* G *is* bipartite *iff all cycles in* G *are even.*

2.1. **Construction of Cohen-Macaulay Graphs.** The main part of the results in this subsection can be found in [6], chapter 6 section 2.

The degree of a vertex v, $deg(v)$, is the number of edges incident in v and the set of neighbors of v, $N(v)$ is the set of vertices connected with v.
Of course $\mid deg(v) \mid = \mid N(v) \mid, \forall v \in V(G)$.

First construction

Let G be a graph on the vertex set $V = \{v_1, \ldots, v_r, z, w\}$ with $deg(w) = 1$, $N(w) = \{z\}$, $deg(z) = k + 1$, $N(z) = \{w, v_1, \ldots, v_k\}$.
Let G_1 be the graph obtained by deleting the vertices w and z in G, and let F_1 be the graph obtained by deleting the vertices $\{v_1, \ldots, v_k\}$ in G_1 .[1]
Then the following propositions hold:

(1) If G is CM, then both G_1 and F_1 are CM
(2) If G_1 and F_1 are CM and $\{v_1, \ldots, v_k\}$ form a part of a minimal vertex cover for G, then G is CM
(3) If G_1 is CM and $\{v_1, \ldots, v_k\}$ is a minimal vertex cover for G_1, then G is CM
(4) Every bipartite CM graph has a vertex of degree 1.

Second construction

Let G be a graph on the vertex set $V = \{v_1, \ldots, v_n, z\}$ with $deg(z) \geq 2$, $N(z) = \{v_1, \ldots, v_k\}$, and $deg(v_i) \geq 2$ for all $i = 1, \ldots, k$.
Let G_1 be the graph obtained by deleting z in G and let F_1 be the graph obtained from by deleting $v_1, \ldots, v_k$ in G_1.
Let I be the edge ideal of G_1.
Then the following propositions hold:

(1') If G is CM, then F_1 is CM
(2') Suppose that $\{v_1, \ldots, v_k\}$ do not form a part of a minimal vertex cover for G_1 and $height(I, v_1, \ldots, v_k)$=$height(I) + 1$. If F_1 and G_1 are CM, then G is CM
(3') If G_1 is CM and $\{v_1, \ldots, v_{k-1}\}$ is a minimal vertex cover for G_1, then G is CM

[1]When a vertex is removed from a graph, then all edges incident in the vertex are also removed.

(4') For every CM graph two vertices of degree 1 cannot have a vertex in common.

2.2. **Characterization of bipartite CM graphs.** The main results in this subsection can be found in [4].

DEFINITION 2.13. *A* simple *graph is an unweighted, undirected graph and with no self-loops.*

Let G be a simple finite bipartite graph and let us suppose G unmixed. Let us call W and W' the two bipartition sets.

REMARK 2.3. *If G is a bipartite graph without isolated vertices, then the partition sets are minimal vertex covers of G. In fact if W and W' are the two partition sets, then W (resp W') is a vertex cover, because every edge has a vertex in W (resp W'). Moreover W (resp.W') is minimal, because there are no edges connecting two vertices in W' (resp. W).*

So, if G is bipartite and unmixed, then W and W' have the same cardinality n.
Now $(W \backslash U) \cup N(U)$ is a vertex cover of G for every subset U of W. In fact every edge incident in a vertex of U is covered by a vertex in $N(U)$ and every vertex not incident in a vertex of U is covered by a vertex in $W \backslash U$.
So $|(W \backslash U) \cup N(U)| \geq |W|$ and then $|U| \leq |N(U)|$. By marriage theorem every vertex in W is connected with a vertex in W'. This means we can relabel the names of the vertices in the following way: $W = \{x_1, \ldots, x_n\}$ and $W' = \{y_1, \ldots, y_n\}$, such that **(a)** $\{x_i, y_i\}$ is an edge of G for all $1 \leq i \leq n$.

LEMMA 2.1. *(see* [4]*, lemma 3.3)*
*With the above notation, let us suppose that G is a simple bipartite Cohen-Macaulay graph. Then G satisfies condition **(a)** and, furthermore, it satisfies also the condition **(b)** if $\{x_i, y_j\}$ is an edge of G, then $i \leq j$.*

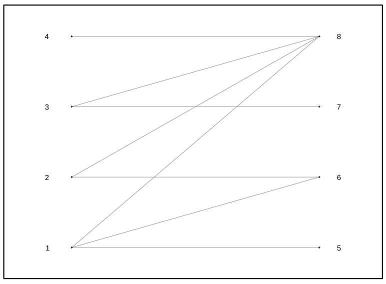
THEOREM 2.2. *(see [4], theorem 3.4)*
Let G be a simple bipartite graph without loops on the vertex set $W \bigcup W'$, with $W = \{x_1, \ldots, x_n\}$, $W' = \{y_1, \ldots, y_n\}$ such that

(a): *$\{x_i, y_i\}$ is an edge for all $1 \leqslant i \leqslant n$;*
(b): *if $\{x_i, y_j\}$ is an edge then $i \leqslant j$;*

then G is CM iff

(c): *whenever $\{x_i, y_j\}$ and $\{x_j, y_k\}$ are edges, then $\{x_i, y_k\}$ is an edge.*

The previous theorem allows to know how a bipartite CM graph looks like. See the picture below. This fact is not trivial, because it is not clear just by looking only at the definition.



3. A Decision Algorithm for CM Graphs

Actually there is no algorithm for checking whether a graph is CM. Here we found a decision procedure for graphs, when the graph is bipartite. The main strategy is given by removing vertices from the initial graph and by checking some properties for the corresponding subgraphs. So if the graph is CM, then at the end of the algorithm we will find either a vertex or an edge, that are trivially Cohen-Macaulay. In order to write the algorithm we need the following results. (See [1]).

DEFINITION 3.1. *Let G=$(V(G), E(G))$ be a finite undirected graph. The binomial* extended edge ideal of *G is the ideal $I(G, E(G))$= ($e_h - v_i v_j$: $e_h = \{v_i, v_j\}$ is in $E(G)$).*
The ideal $I(G)_{E(G)} = I(G, E(G)) \cap K[e_1, \ldots, e_m]$ is the binomial edge ideal of *G.*

REMARK 3.1. *$I(G)_{E(G)}$ is the toric ideal of the incidence matrix $IM(G) = (a_{ih})_{i=1,\dots,n,h=1,\dots,m}$ of G defined by $a_{ih} = 1$ if $v_i \in e_h$ and $a_{ih} = 0$ if $v_i \notin e_h$ for every $v_i \in V(E)$ and $e_h \in E(G)$.*

DEFINITION 3.2. *Let G=$(V(G), E(G))$ be a finite undirected graph. The ideal $I(G, V(G))$=($v_i - \prod e_h$: v_i belongs to the edge e_h in G) is the binomial* extended vertex ideal of G. $I_{V(G)}$=$I(G, V(G)) \cap K[v_1, \dots, v_n]$. *is the binomial* vertex ideal of G.

$I_{V(G)}$ is the toric ideal of the transpose of the incidence matrix $IM(G)$ of G.

THEOREM 3.1. *Let G=$(V(G), E(G)$ be a finite undirected graph with $V(G)$= $\{v_1, \dots, v_n\}$ and $E(G)$=$\{e_1, \dots, e_m\}$. The odd cycle C=$(e_{i_1} = \{v_{i_1}, v_{i_2}\}, e_{i_2} = \{v_{i_2}, v_{i_3}\}, \dots, e_{i_{2q-2}} = \{v_{i_{2q-2}}, v_{i_{2q-1}}\}, e_{i_{2q-1}} = \{v_{i_{2q-1}}, v_{i_1}\})$ is in G iff the binomial $f_C = \prod_{k=1,\dots,q-1} e_{i_{2k}} v_{i_1}^2 - \prod_{k=1,\dots,q} e_{i_{2k-1}} \in I(G, E(G))$.
Let σ be a lexicographic term ordering on the set of the power products in $\{e_1, \dots, e_m, v_1, \dots, v_n\}$ with $v_i > e_j$ for all i and j and $v_{i_{2q-1}} >_\sigma v_{i_{2q-2}} >_\sigma \dots >_\sigma v_{i_1}$. If C is minimal, then the binomial f_C is in the Gröbner basis of $I(G, E(G))$ with respect to σ.*

THEOREM 3.2. *Let G=$(V(G), E(G))$ be a simple undirected connected graph without isolated vertices. If $I_{V(G)}$ contains an irreducible polynomial p of the form p=$\prod_{j \in J} v_j - \prod_{k \in K} v_k$, then G is bipartite and the partition sets are V'=$\{v_j : j \in J\}$ and V''=$\{v_k : k \in K\}$.*

3.1. **The Decision Algorithm.** Now we can show our decision procedure for bipartite CM graphs. First of all we have to check if our graph is bipartite. This can be done in the following way:

- We observe that a graph is CM if and only if every connected components is CM, and an isolated vertex is trivially CM; so we can apply the following step to the graph, that it is obtained by deleting the isolated vertices of G;

- Given a graph G, check if $\mid V(G) \mid$ is even. If $\mid V(G) \mid$ is odd, then G cannot be a bipartite unmixed graph and then it is not CM.
- Given a graph G, by using the package networks of Maple we can get the incidence matrix I_G of G;
- Given the matrix I_G, we can get the ideal $I(G, E(G))$ in the ring $K[e_1, \ldots, e_m, v_1, \ldots, v_n]$;
- Given the ideal $I(G, E(G))$, by using the package Groebner of Maple we can get a Gröbner basis of it with respect to a lexicographic term ordering σ with $v_i >_\sigma e_j$ for all i and j and then we can get the ideal $I(G)_{E(G)}$;
- By using the theorem 3.1 and the property of Gröbner bases about the decidability of the membership problem for polynomials, if G has no odd cycle, then G is bipartite;
- If G is not bipartite, we cannot conclude anything about the Cohen-Macaulay property.

Once we know that G is bipartite we can start the following algorithm

- We can get the transpose of the incidence matrix I_G^T of G;
- Given I_G^T we can get the ideal $I(G, V(G))$ in the ring $K[v_1, \ldots, v_n, e_1, \ldots, e_m]$;
- We can find the two partition sets by computing a Gröbner basis of the ideal $I(G, V(G))$ with respect to a lexicographic term ordering τ with $e_j >_\tau v_i$ for all i and j and then we can get the ideal $I(G)_{V(G)}$. The monomials appearing in the binomial of the basis represent the two sets, as in theorem 3.2;
- If the partition sets have different cardinality, then we can conclude that G is not CM. (In fact, the graph is not unmixed by remark as above);
- Given the bipartite graph G we can get the monomial edge ideal $I(G)$ and then we can find its minimal primes, that represent the minimal vertex covers of G, according to proposition 2.2
- Given the minimal vertex covers, we can decide if G is unmixed by looking at the cardinality of its minimal vertex covers;

- If G is not unmixed, then we can conclude that it is not CM;
- If G is unmixed, we can look at the degrees of its vertices; if there is not a vertex of degree one, then we can conclude that G is not CM, according to condition 4 in section 2.1;
- If G has a vertex of degree one, then we can consider the two partition sets being the sets W and W', with $W = \{x_1, \dots, x_n\}$, $W' = \{y_1, \dots, y_n\}$. W and W' have the same cardinality, and G is unmixed. We can order the vertices in such a way as the vertex of degree one is y_1 (it belongs just to the edge x_1, y_1 by condition **(b)** in theorem 2.1). $G_1 = G \setminus \{x_1, y_1\}$. If the set of neighbors of x_1 is the entire set $\{y_2, \dots, y_n\}$, that is a minimal vertex cover for G_1, then by condition 3 G is CM if and only if G_1 is CM. So we can apply the algorithm to G_1;
- If the set of neighbors of x_1 is a proper subset of $\{y_2, \dots, y_n\}$, [2], then by condition 2 in 2.1 G is CM if and only if both G_1 and F_1 are CM. So we start with F_1 (the smallest subgraph) as input of the algorithm. If F_1 is not CM, we can conclude that G is not CM. Else we put G_1 as input of the algorithm. If G_1 is not CM, then G is not CM;
- If in the previous steps we did not conclude that G is not CM, then we can conclude that G is CM.

Finiteness and correctness

Note that the algorithm finishes, because at each recursive step the input is a subgraph of the given one. So, in the worst case we apply the algorithm until the input is either an isolated vertex or an edge, that are trivially Cohen-Macaulay. Moreover of course by deleting vertices from a bipartite graph we get a graph, that is still bipartite, since there are no new edges.

[2]It is clear that every set of x_i's or y_i's is at least a part of a minimal vertex cover for G

4. Example

Our algorithm is implemented in Maple 9.5 by using the packages *networks*, for graphs, and *Groebner*, for Gröbner bases, and it is called *isbipCM*. Here we will show an example.

EXAMPLE 4.1. *Let us consider the following graph G with eight vertices*

$> with(networks):$

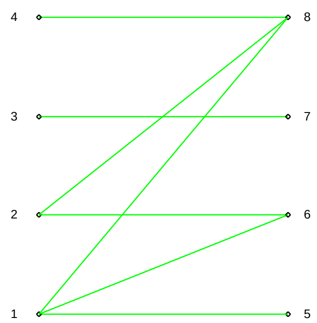
$> with(Groebner):$

$> G := void(8):$

$> addedge([\{1,5\},\{2,6\},\{3,7\},\{4,8\},\{1,6\},\{1,8\},\{2,8\}],G);$

$$e1, e2, e3, e4, e5, e6, e7$$

$> draw(Linear([1,2,3,4],[5,6,7,8]),G);$



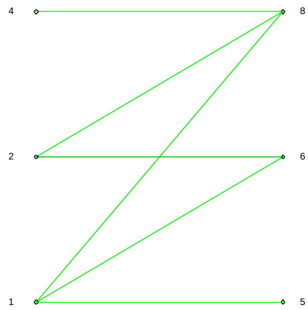
$> isbipCM(G);$

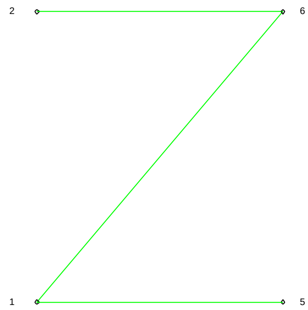
$$true$$

At each step the algorithm chooses a vertex of degree 1 and it works on the obtained subgraphs, according to the construction of CM graphs with the following choices:

$w = 3$, $z = 7$, $N(z)\backslash\{w\} = \{\}$, G_1 is the graph on the left of the following picture and F_1 is the same.

and then $w = 4$, $z = 8$, $N(z) \backslash \{w\} = \{1, 2\}$, G_1 is the graph on the right and since $\{1, 2\}$ is the entire set of bipartition of G_1 we do not need to construct F_1.





At the third step $w = 5$, $z = 1$, $N(z) \backslash \{w\} = \{6\}$, G_1 is just the edge $\{5, 6\}$ and we do not need to construct F_1.

An edge is trivially Cohen-Macaulay and the algorithm returns *true*.

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