CW-spheres encoded by polyspherical coordinates

Gábor Hetyei*
Department of Mathematics and Statistics
UNC Charlotte
Charlotte, NC 28223

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Abstract

We construct spherical CW-complexes whose face structure may be conveniently described using a system of polyspherical coordinates introduced by Vilenkin, Kuznetsov and Smorodinskii. We prove that these complexes may be constructed by repeated use of CW-suspension, free join, and edge subdivision. We show that all CW-spheres constructed this way have a non-negative cd-index and thus verify Stanley's famous conjecture. Among the particular examples we find a new class of partially ordered sets whose order complexes encode the derivative polynomials for secant of even degree. The geometric constructions presented here generalize CW-complexes whose flag numbers are suitable to encode systems of orthogonal polynomials.

Résumé

Nous construisons des sphères CW dont la structure de faces se décrit d'une manière convenable en utilisant des coordonnées polysphériques de Vilenkin, Kuznetsov, et Smorodinskii. Nous montrons que ces complexes peuvent être construits récursivement en utilisant des suspensions des complexes CW, des joins libres et des sous-division des arêtes. Nous démontrons ques tous nos sphères ont un indexe cd positif, en accord avec la fameuse conjecture de Stanley. Parmis les examples particulières nous retrouvons un nouveau class d'ensembles ordonnés dont le complex des chaînes croissants code les polynômes dérivés pour la fonction secant de degré pair. Nos constructions géométriques généralisent des complexes CW dont les nombres de drapeaux codent des systemes des polynômes orthogonaux.

Introduction

In a recent paper [13] the present author introduced sequences of CW-spheres whose ce-indices may be transformed into sequences of orthogonal polynomials by sending c into x and e into 1.

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These complexes were shown to be spherically shellable, and the resulting non-negativity of their cd-index implied by Stanley's [21, Theorem 2.2] induces a new proof for the fact that the true interval of orthogonality of any orthogonal polynomial system $\{Q_n(x)\}_{n=0}^{\infty}$, given by a recurrence formula $Q_n(x) = \nu_n \cdot x \cdot Q_{n-1}(x) - (\nu_n - 1) \cdot Q_{n-2}(x)$ where $\nu_i \geq 2$, is a subset of $[-1, 1]$.

The system of spherical coordinates used in that paper is the simplest example of a system of polyspherical coordinates introduced by Vilenkin, Kuznetsov and Smorodinskii [22] to encode the points of a unit sphere. Each such coordinate system may be described by a rooted binary tree. The spherical coordinate system used in [13] corresponds to the situation when the subtree of interior nodes (=the "small tree") is a rooted path.

In Section 2 we describe the faces of our complexes as intersections of certain lunes and hemispheres, and define our polyspherical complexes by explicitly listing their faces. The fact that our constructions yield CW-spheres may be shown by combining all results of Section 3 where we describe our polyspherical complexes recursively.

Our main result is in Section 4: every polyspherical complex we constructed has a non-negative cd-index. Unlike [13], it seems to be extremely hard to find a proof that uses spherical shelling, the main problem being with the spherical shellability of a CW-complex that arises as the free join of two CW-spheres. Fortunately, the dual version of a result of Ehrenborg and Fox [9] (based on the work of Ehrenborg and Readdy [10]) provides an immediate proof of the fact that the non-negativity of the cd-index is preserved by the free join operation. The same question for CW-suspension is trivial. Finally, edge-subdivision does not necessarily preserve the non-negativity of the cd-index (since it involves changing by a multiple of the cd-index of a proper interval in the associated poset). Fortunately, the face posets of our polyspherical complexes belong to a narrower class of Eulerian posets: not only their cd-index but the cd-index of every interval of the form $[x, \widehat{1}]$ turns out to be non-negative. This fact is easily shown by proving that all poset-operations used preserve this stronger positivity property.

Finally, in Section 5 we focus on a special class of polyspherical complexes, that has about the same "degree of freedom" as the ones studied in [13]: we require the underlying small trees to be strongly binary, and we forbid subdividing the intervals of angles associated to interior nodes. Using the dual version of the type B quasisymmetric functions defined by C.-O. Chow [5], we obtain an explicit formula for their flag f-vectors. We show that substituting x into c and 1 into e in the ce-index of a free join of quadrilaterals yields the derivative polynomials for secant of even degree. These polynomials appear in chain enumeration related to some generalization of the Tchebyshev posets introduced in [12] for the second time (the first connection was noted in [14, Section 9]). The appearance of this second, non-isomorphic connection suggests that the Tchebyshev polynomials of the first and second kind and the derivative polynomials for tangent and secant may be more intimately related at a combinatorial level than we ever thought.

1 Preliminaries

1.1 Graded and Eulerian posets, transformations, flag-enumeration

A partially ordered set P is *graded* if it has a unique minimum $\widehat{0}$, a unique maximum $\widehat{1}$, and a rank function ρ. If P has rank $n+1$ and $S \subseteq \{1, \ldots, n\}$, $f_S(P)$ is the number of saturated chains in $P_S = \{x \in P : \rho(x) \in S\} \cup \{\widehat{0}, \widehat{1}\}$. The vector $(f_S(P) : S \subseteq \{1, \ldots, n\})$ is the *flag f-vector* of P. It has several equivalent encodings. The connection between the flag f-vectors of P, Q, and *direct product* $P \times Q$ is most easily expressed by the *flag quasisymmetric function* $F(P)$ [8], given by

$$F(P) = \lim_{m \longrightarrow \infty} \sum_{\widehat{0}=x_0 \leq x_1 \leq \cdots \leq x_m = \widehat{1}} t_1^{\rho(x_1)-\rho(x_0)} t_2^{\rho(x_2)-\rho(x_1)} \cdots t_m^{\rho(x_m)-\rho(x_{m-1})}. \tag{1}$$

By Ehrenborg's result [8, Proposition 4.4], $F(P \times Q) = F(P) \times F(Q)$. A modified version of $P \times Q$ is the *diamond product* $P \diamond Q := (P \setminus \{\widehat{0}\}) \times (Q \setminus \{\widehat{0}\}) \cup \{\widehat{0}\}$. The analogue of [8, Proposition 4.4], recently discovered by Ehrenborg and Readdy [11], is

$$F_B(P \diamond Q) = F_B(P) \cdot F_B(Q), \tag{2}$$

where $F_B(Q)$ is the *type B quasisymmetric function*

$$F_B(P) = \lim_{m \longrightarrow \infty} \sum_{\widehat{0} < x_0 \leq x_1 \leq \cdots \leq x_m = 1} s^{\rho(x_0)-1} t_1^{\rho(x_1)-\rho(x_0)} t_2^{\rho(x_2)-\rho(x_1)} \cdots t_m^{\rho(x_m)-\rho(x_{m-1})}.$$

Another equivalent encoding of the flag f-vector is the *flag h-vector* $(h_S(P) : S \subseteq \{1, \ldots, n\})$ (see [21]), given by $h_S(P) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} f_T(P)$.

A graded poset is *Eulerian* if every interval $[x, y]$ of positive rank in it satisfies $\sum_{x \leq z \leq y} (-1)^{\rho(z)} = 0$. All linear relations holding for the flag f-vector of an arbitrary Eulerian poset of rank n were determined by Bayer and Billera [1]. These linear relations were rephrased by J. Fine as follows (see Bayer and Klapper [2]). For any $S \subseteq \{1, \ldots, n\}$ define the non-commutative monomial $u_S = u_1 \ldots u_n$ by setting

$$u_i = \begin{cases} b & \text{if } i \in S, \\ a & \text{if } i \notin S. \end{cases}$$

Then the polynomial $\Psi_{ab}(P) = \sum_S h_S u_S$ in non-commuting variables a and b, called the *ab-index of* P, is a polynomial $\Phi_{cd}(P)$ of $c = a + b$ and $d = ab + ba$, called the *cd-index* of P. It was noted by Stanley [21] that the existence of the *cd*-index is equivalent to saying that the *ab*-index rewritten as a polynomial of $c = a + b$ and $e = a - b$ involves only even powers of e. Stanley's conjecture [21, Conjecture 2.1] states that the *cd*-index of any Gorenstein* poset has non-negative coefficients. The *cd*-index form is convenient to calculate the flag f-vector of the *join* $P * Q := (P \setminus \{\widehat{1}_P\}) \cup (Q \setminus \{\widehat{0}_Q\})$ of two Eulerian posets. (We place the elements of Q above the elements of P.) By [21, Lemma 1.1] we have $\Phi_{cd}(P * Q) = \Phi_{cd}(P)\Phi_{cd}(Q)$.

1.2 The Tchebyshev transform of a graded poset

We define the *Tchebyshev transform* $T(P)$ of a graded poset P as follows: we adjoin a new minimum element $\widehat{-1} < \widehat{0}$ to P and set $T(P) = \{(x,y) \ : \ x < y, \ x,y \in \{\widehat{-1}\} \cup P\} \cup \{\widehat{1}_{T(P)}\}$. Here $\widehat{1}_{T(P)}$ is the maximum element of $T(P)$, and we set $(x_1,y_1) < (x_2,y_2)$ if either $y_1 < x_2$ or $x_1 = x_2$ and $y_1 < y_2$. It was shown in [12] that this operation yields a graded poset and preserves the Eulerian property. It was observed in [14, Section 9] that substituting x into c and 1 into e in the ce index of the Tchebyshev transform of the Boolean algebra of rank n yields the polynomial $(\sqrt{-1})^n \cdot Q_n(x \cdot \sqrt{-1})$ where $Q_n(x)$ is the *n-th derivative polynomial for secant* given by $\frac{d^n}{du^n}\sec(u) = Q_n(\tan u)\sec u$. Further information on these polynomials may be found in Hoffman's papers [15] and [16]. The study of these polynomials goes back to Krichnamachary and Rao [18] and Knuth and Buckholtz [17]. Further results on the Tchebyshev transform are in [11], [12], and [14].

1.3 Free join and suspension of CW-spheres

Given a regular CW-complex Ω, we obtain a graded poset $P_1(\Omega)$ by adjoining a maximum element $\widehat{1}$ to the face poset of Ω. Such posets are called *CW-posets* and were characterized by Björner [3]. $P_1(\Omega)$ is Eulerian if Ω is a CW-sphere. We use two basic operations on CW-complexes: *free join* and *CW-suspension*. The free join $X * Y$ of the topological spaces X and Y is $X \times Y \times [0,1]/\equiv$, where the only nontrivial equivalence classes of $\equiv$ are $\{(x,y_0,0) \ : \ x \in X\}$ and $\{(x_0,y,1) \ : \ y \in Y\}$. The free join $\Omega * \Omega'$ of two CW-complexes Ω and Ω' may also be given as a CW-complex (see May [19, Chapter 10, Section 2]). This operation satisfies

$$P_1(\Omega * \Omega') = P_1(\Omega) \diamond^* P_1(\Omega') \tag{3}$$

where $P \diamond^* Q = (P^* \diamond Q^*)^*$. The *suspension* of a topological space X is usually defined as $\mathrm{Susp}(X) = X * \mathbb{S}^0$ (see, e.g., Dong [7] or Readdy [20]), because this operation assigns a simplicial complex to a simplicial complex. For CW-complexes there is also a more economical way to create a face structure $\mathrm{Susp}(\Omega)$ on the suspension of the topological space underlying Ω. This is noted by Dold [6, Chapter V, Example 3.10]. Using that construction we obtain

$$P_1(\mathrm{Susp}(\Omega)) \cong B_2 * P_1(\Omega). \tag{4}$$

Here and later B_m is the Boolean algebra of rank m.

1.4 Polyspherical coordinates

Our polyspherical coordinates $(\theta_1,\ldots,\theta_{n-1})$ parameterize the *unit* $(n-1)$*-sphere* defined by $\sum_{i=1}^n x_i^2 = 1$. Recall that a *binary tree* is a planar rooted tree such that each internal node has at most two children. It is *strongly binary* if each internal node has exactly two children. Let us fix strongly binary tree with

n leaves, and label its leaves with the rectangular coordinates $x_1, \ldots, x_n$, as shown in Fig. 1. Associate to each internal node an angle θ_i ($i = 1, \ldots, n-1$). We call such a labeled tree a *large tree*. For the

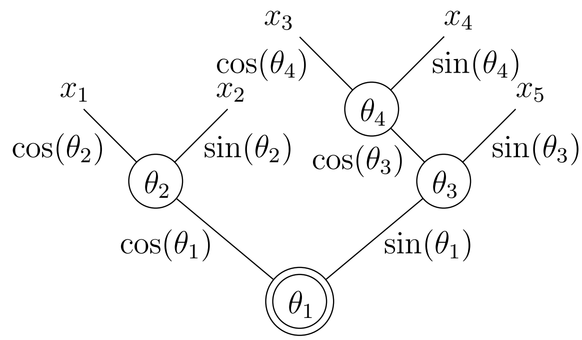

Figure 1: Large tree of polyspherical coordinates for a 4-dimensional sphere

internal node labeled θ_i, we label the edge to its left child with $\cos(\theta_i)$ and the edge to its right child with $\sin(\theta_i)$. Define the value of each x_j as the product of the labels on the edges along the unique path connecting the root with x_j. For example, for the labeled tree in Fig. 1 we set

$$\begin{aligned} x_1 &= \cos(\theta_1) \cdot \cos(\theta_2) \\ x_2 &= \cos(\theta_1) \cdot \sin(\theta_2) \\ x_3 &= \sin(\theta_1) \cdot \cos(\theta_3) \cdot \cos(\theta_4) \\ x_4 &= \sin(\theta_1) \cdot \cos(\theta_3) \cdot \sin(\theta_4) \\ x_5 &= \sin(\theta_1) \cdot \sin(\theta_3) \end{aligned}$$

Using the rule to express the x_i's as products of edge labels, every point of the unit sphere is already determined by the subtree of internal nodes and the labels θ_j associated to them. We call the rooted tree (T, r), consisting of these internal nodes only, a *small tree*. According to [22, (13)], our parameterization provides a single covering of the unit sphere if we set the following restrictions:

(S1) If the node associated to θ_i is a leaf in the small tree, we require $\theta_i \in [0, 2\pi]$.
(S2) If the node of θ_i has only a right child, we require $\theta_i \in [0, \pi]$.
(S3) If the node of θ_i has only a left child, we require $\theta_i \in [-\pi/2, \pi/2]$.
(S4) If the node θ_i has two children, we require $\theta_i \in [0, \pi/2]$.

The representation by polyspherical coordinates may be made unique by factoring by the following equivalence relation.

Definition 1.1 *We say that $(\theta_1, \ldots, \theta_{n-1})$ and $(\theta'_1, \ldots, \theta'_{n-1})$ are equivalent, if for each j such that $\theta_j \neq \theta'_j$ at least one of the following holds:*

(i) θ_j (and θ'_j) is the label of a leaf in the small tree and $\theta_j, \theta'_j \in \{0, 2\pi\}$,
(ii) θ_j (and θ'_j) is the right descendant of a node whose labels in both vectors satisfy $\theta_i = \theta'_i \in \{0, \pi\}$,
(iii) θ_j (and θ'_j) is the left descendant of a node whose labels in both vectors satisfy $\theta_i = \theta'_i \in \{-\pi/2, \pi/2\}$.

Thus, for example, the labels of the right descendants of a node labeled 0 or π are irrelevant. Each equivalence class may be represented by a *simplified code*, where the irrelevant coordinates are replaced with a dash.

2 The polyspherical complex $C((T,r);m_1,\ldots,m_{n-1})$

Definition 2.1 *Consider a code $(\sigma_1,\ldots,\sigma_{n-1})$ associated to a small tree, such that each σ_j is either a real number, or an interval $[\alpha,\beta]$, or the $*$ sign. We call such a code a* standard lune code *if it satisfies the following conditions:*

(i) Exactly one σ_i is an interval $[\alpha,\beta]$. The node of this σ_i is the root *of the standard lune code.*
(ii) The interval $[\alpha,\beta]$ is a proper *subset of the interval $I(\sigma_i)$ that occurs in the restrictions (S1)–(S4) applied to the node of σ_i. Moreover, $0<\beta-\alpha<\pi$.*
(iii) All descendants of the node of σ_i are labeled with a real number, subject to the restrictions (S1)–(S4).
(iv) All other nodes are labeled with $*$.

We use a standard lune code to denote the set of all polyspherical vectors $(\theta_1,\ldots,\theta_{n-1})$ satisfying $\theta_i\in[\alpha,\beta]$, and $\theta_j=\sigma_j$ whenever $j\neq i$ and $\sigma_j\neq *$. A *standard hemisphere code* is defined analogously, the only difference is setting $\alpha=\beta$. The equivalence relation for polyspherical coordinates may be extended to codes of standard lunes and hemispheres, so we may obtain a simplified code for them, and think of them as subsets of the unit sphere.

Proposition 2.2 *A standard lune code with $d-1$ star signs encodes a d-dimensional closed region, whose boundary is the union of two spheres obtained by replacing the interval $[\alpha,\beta]$ with α or β respectively (thus obtaining standard hemispheres).*

Taking intersections of standard lunes motivates the introduction of canonical regions.

Definition 2.3 *A* canonical region code *is a vector $(\sigma_1,\ldots,\sigma_{n-1})$ such that each σ_i is either a real number, or an interval $[\alpha,\beta]$, or the symbol $*$, or the symbol $-$, subject to the following conditions:*

(i) If σ_i is a real number or an interval, then it is an element or proper subset of $I(\sigma_i)$.
(ii) If $\sigma_i\in\{0,\pi\}$ then every right descendant of the node labeled σ_i is labeled $-$.
(iii) If $\sigma_i\in\{-\pi/2,\pi/2\}$ then every left descendant of the node labeled σ_i is labeled $-$.
(iv) σ_i is $-$ then every descendant of the node labeled σ_i is labeled $-$.
(v) If σ_i is $-$ then the parent of the node labeled σ_i is either labeled with $-$, or it is labeled with an element of $\{0,\pi\}$ and the node of σ_i is the right child, or it is an element of $\{-\pi/2,\pi/2\}$ and the node of σ_i is the left child.

(vi) If $\sigma_i \neq *$ and some descendant of its node is $*$ or an interval, then the node of σ_i has two children in the small tree, and σ_i is a real number belonging to $\{0, \pi/2\}$. (Thus either its left or its right subtree will have all of its nodes labeled with $-$.)

Again we obtain equivalence classes of polyspherical vectors; thus we may think of *canonical regions* as subsets of the unit sphere.

Proposition 2.4 *Every canonical region, except for the entire unit sphere, may be written as an intersection of finitely many standard hemispheres and standard lunes. Conversely, the intersection of any two canonical regions may be written as a union of canonical regions.*

Definition 2.5 *We call a rooted tree (T, r) whose nodes are labeled with positive integers such that the label on each leaf is at least two a* loopless complex code.

Given a loopless complex code, consider the family $C((T, r); m_1, \ldots, m_{n-1})$ consisting of the empty set, and of all canonical regions whose code is subject to the following conditions:

(C1) No leaf is labeled with $*$.

(C2) Each node labeled with a real number has either an ancestor whose label is an interval, or an ancestor labeled with $\sigma_i = *$ for which the corresponding m_i is equal to 1.

(C3) If σ_i is a real number and $I(\sigma_i) = [\gamma, \delta]$, then we have

$$\sigma_i = \gamma + t \cdot \frac{\delta - \gamma}{m_i} \quad \text{for some} \quad t \in \{0, 1, \ldots, m_i\}.$$

(C4) If σ_i is an interval and $I(\sigma_i) = [\gamma, \delta]$, then we have

$$[\alpha, \beta] = \left[\gamma + t \cdot \frac{\delta - \gamma}{m_i}, \gamma + (t+1) \cdot \frac{\delta - \gamma}{m_i}\right] \quad \text{for some} \quad t \in \{0, 1, \ldots, m_i - 1\}.$$

Theorem 2.6 *Given a loopless code associated to a small tree (T, r), $C((T, r); m_1, \ldots, m_{n-1})$ is a CW-complex, homeomorphic to an $(n-1)$-sphere.*

3 A recursive description of the polyspherical complexes

Any binary tree (T, r) having at least two nodes may be reconstructed from knowing the children r_1 and r_2 of the root and the subtrees T_i of descendants of r_i. (Only at least one of the r_i's needs to exist.) In this section we describe the structure of a polyspherical complex in terms of the polyspherical complexes associated to the subtrees of the children of the root in its small tree. The arising operations assign the face poset of a CW-sphere to face posets of CW-spheres. Thus the aggregate of the statements in this section provides proof of Theorem 2.6.

Consider a family of canonical regions $C((T,r);m_1,\ldots,m_{n-1})$. Assume w.l.o.g. that m_1 is associated to the root. We introduce $P_1(C((T,r);m_1,\ldots,m_{n-1}))$ to denote the partially ordered set obtained by taking the elements of $C((T,r);m_1,\ldots,m_{n-1})$, ordered by inclusion, and add a new unique maximum element $\widehat{1}$, which we associate to the canonical region $(*,\ldots,*)$.

Lemma 3.1 *If $m_1 = 1$ and the root r has only a right child r' then $P_1(C((T,r);1,m_2,\ldots,m_{n-1})) = B_2 * P_1(C((T',r');m_2,\ldots,m_{n-1}))$.*

Proposition 3.2 *Assume that Ω is an $(n-2)$-dimensional CW-sphere, whose faces subdivide the unit sphere $\{(x_2,\ldots,x_n) : x_2^2+\cdots+x_{n-1}^2 = 1\}$. Then the CW-suspension $\Omega' = \mathrm{Susp}(\Omega)$ may be realized as a CW-complex subdividing the unit sphere $\{(x_1,\ldots,x_n) : x_1^2+\cdots+x_{n-1}^2 = 1\}$. As a consequence we have $P_1(\Omega') = B_2 * P_1(\Omega)$.*

To handle the case $m_1 > 1$, we introduce *m-fold edge subdivisions*. Given an edge e in a CW complex, connecting the vertices u and v, we introduce $m-1$ new vertices $u_1,u_2,\ldots,u_{m-1}$, and set $u_0 := u$ and $u_m := v$. We remove e and introduce m new edges $e_1,\ldots,e_m$ such that e_i connects u_{i-1} and u_i for $i = 1,2,\ldots,m$ and a face f contains any of the e_i's in the new CW-complex if and only if it contains e in the original complex. We make the analogous definition for graded posets as well. An m-fold edge subdivision does not change the homeomorphy type of the CW-complex and, for posets, it preserves the Eulerian property. We denote by $E_m(P)$ the poset obtained by applying m-fold edge subdivision to all rank 2elements of the graded poset P.

Theorem 3.3 *Assume that the root r in the small tree associated to $C((T,r);m_1,m_2,\ldots,m_{n-1})$ has only a right child r'. Then $P_1(C((T,r);m_1,m_2,\ldots,m_{n-1})) \cong E_{m_1}(B_2 * P_1(C((T',r');m_2,\ldots,m_{n-1})))$. Here T' is the subtree of descendants of r'.*

The case when the root of the small tree has only a left child is completely analogous. Consider finally the case when the root r has two children: a left child r_1 and a right child r_2, with subtrees of descendants T_1 and T_2. W.l.o.g. we may assume that $m_2,\ldots,m_k$ belong to the nodes in T_1 and $m_{k+1},\ldots,m_{n-1}$ belong to the nodes of T_2. Again we discuss first the case $m_1 = 1$ separately.

Lemma 3.4 *Under the conditions listed above, we have*

$$P_1(C((T,r);1,m_2,\ldots,m_{n-1})) \cong P_1(C((T_1,r_1);m_1,\ldots,m_k)) \diamond^* P_1(C((T_2,r_2);m_{k+1},\ldots,m_{n-1})).$$

Proposition 3.5 *Assume that the $(k-1)$-dimensional CW-sphere Ω_1 subdivides $\{(x_1,\ldots,x_k) : x_1^2+\cdots+x_k^2 = 1\}$ and the $(n-k-1)$-dimensional CW-sphere Ω_2 subdivides $\{(x_{k+1},\ldots,x_n) : x_{k+1}^2+\cdots+x_n^2 = 1\}$. Then $\Omega_1 * \Omega_2$ may be realized as a CW-sphere Ω, subdividing $\{(x_1,\ldots,x_n) : x_1^2+\cdots+x_{n-1}^2 = 1\}$. As a consequence we have $P_1(\Omega) = P_1(\Omega_1) \diamond^* P_1(\Omega_2)$.*

Finally, to state the analogue of Theorem 3.3, we need the following analogue of the operator E_m.

Definition 3.6 *The m-fold edge-subdivided dual diamond product $P \diamond_m^* Q$ is obtained from $P \diamond^* Q$ by applying m-fold edge subdivision to all elements of the form $(p, q) \in P \diamond^* Q$ such that both p and q have rank* 1.

Theorem 3.7 *Making the same assumptions as in Lemma 3.4 except for allowing $m_1 > 1$, we have*

$$P_1(C((T, r); m_1, m_2, \ldots, m_{n-1})) \cong P_1(C((T_1, r_1); m_2, \ldots, m_k)) \diamond_{m_1}^* P_1(C((T_2, r_2); m_{k+1}, \ldots, m_{n-1})).$$

When we prove Theorem 2.6 by induction, the basis is the case when the tree T has only one vertex, and $m_1 \geq 2$. The resulting complex $C(\{r\}, r; m_1)$ is a circle, subdivided into m_1 arcs.

4 Non-negativity of the cd-index

Theorem 4.1 *The cd-index of any poset $P_1(C((T, r); m_1, m_2, \ldots, m_{n-1}))$ associated to a loopless complex code is non-negative.*

This may be shown using the recursive description of our polyspherical complexes given in Section 3.

Definition 4.2 *We say that an Eulerian poset P is* upwards cd-positive *if, for every $x \in P \setminus \{\widehat{1}\}$, the interval $[x, \widehat{1}]$ has a non-negative cd-index.*

It is easy to see that $C(\{r\}, r; m_1)$ is upwards cd-positive for all $m_1 > 1$.

Proposition 4.3 *If P is an upwards cd-positive Eulerian poset, then so is $B_2 * P$.*

The following is an easy consequence of [9, Proposition 7.4].

Proposition 4.4 *If the Eulerian posets P and Q are upwards cd-positive, then so is $P \diamond^* Q$.*

Finally, since any m-fold subdivision may be obtained by iterated 2-fold subdivisions, it is sufficient to show the following.

Proposition 4.5 *Assume that P is an upwards cd-positive Eulerian poset of rank $n + 1$, and that $e \in P$ is an element of rank two. Let Q be the Eulerian poset obtained from P by applying* 2*-fold edge-subdivision to e. Then Q is also upwards cd-positive.*

5 Strongly binary small trees

It is hard to convert the *cd*-index to a type B quasisymmetric function; thus finding an explicit general formula for the flag f-vector of an arbitrary polyspherical complex seems difficult. The situation becomes easier when we may restrict ourselves to using only one of these two encodings of the flag f-vector. The special case when the small tree is a path is the subject of [13]. From now on, we assume the "other extreme" that all underlying small trees are strongly binary. To reduce the complexity of the question to a level similar to [13], we we also require that the number m_i associated to any interior node in the small tree has to be 1. Then the shape of the tree becomes irrelevant, because of the following:

Lemma 5.1 *Consider a loopless code* $((T,r);m_1,m_2,\ldots,m_{2n-1}))$ *such that the underlying small tree* (T,r)*, is strongly binary. Assume that* $m_{n+1},\ldots,m_{2n-1}$ *are associated to the interior nodes and that these numbers are equal to* 1*. Then we have*

$$P_1(C((T,r);m_1,m_2,\ldots,m_{2n-1}))\cong P_1(C(\{r\},r;m_1))\diamond^* P_1(C(\{r\},r;m_2))\diamond^*\cdots\diamond^* P_1(C(\{r\},r;m_n)).$$

The computation of the flag f-vector of such a poset is possible using *dual type B quasisymmetric functions*

$$F_B^*(P)=\sum_{\widehat{0}\le x_1\le\cdots\le x_m<1} s^{\rho(\widehat{1})-\rho(x_m)-1}\cdot t_1^{\rho(x_1)-\rho(x_0)}t_2^{\rho(x_2)-\rho(x_1)}\cdots t_m^{\rho(x_m)-\rho(x_{m-1})}.$$

Dually to (2) we have the identity $F_B^*(P\diamond^* Q)=F_B^*(P)\cdot F_B^*(Q)$. Direct substitution into the definitions yields

$$F_B^*(P_1(C(\{r\},r;m)))=m\cdot\left(\sum_i t_i+\frac{s}{2}\right)^2-\left(\frac{m}{4}-1\right)\cdot s^2. \tag{5}$$

Corollary 5.2 *Under the assumptions of Lemma 5.1 we have*

$$F_B^*(P_1(C((T,r);m_1,m_2,\ldots,m_{2n-1})))=\prod_{j=1}^{n}\left(m_j\cdot\left(\sum_i t_i+\frac{s}{2}\right)^2-\left(\frac{m_j}{4}-1\right)\cdot s^2\right).$$

Using this Corollary it is possible to calculate the f-vectors of the order complexes, and it will be worthwhile to explore the sequences of polynomials arising, in analogy to [13]. To conclude, consider the special case $m_1=\ldots=m_n=4$. Then all terms $(m_j/4-1)s^2$ vanish from all factors:

Proposition 5.3 *The* n*-th dual diamond power* $\mathcal{Q}_n$ *of* $P_1(C(\{r\},r;4))$ *satisfies*

$$F_B^*(\mathcal{Q}_n)=\left(2\cdot\sum_i t_i+s\right)^{2n}=\sum_{k=0}^{2k}\binom{2n}{k}s^{2n-k}\left(2\cdot\sum_i t_i\right)^k.$$

From this we may deduce

$$f_{\{s_1\dots,s_k\}}(\mathcal{Q}_n) = \binom{2n}{s_k} \cdot 2^{s_k} \cdot \binom{s_k}{s_1, s_2 - s_1, \dots, s_k - s_{k-1}} \qquad (6)$$

which is also equal to $f_{\{s_1\dots,s_k\}}(T(B_{2n})$.

Corollary 5.4 *Substituting x into c and 1 into e in the ce-index of $\mathcal{Q}_n$ yields $(-1)^n Q_{2n}(x \cdot \sqrt{-1})$, where $Q_{2n}(x)$ the $2n$-th derivative polynomial for secant.*

Remark 5.5 Among all posets of the form $P_1(C(\{r\}, r; m))$, only $P_1(C(\{r\}, r; 4))$ is the Tchebyshev transform of a poset of rank 2. Thus we may also use a result of Ehrenborg and Readdy [11] stating that for any pair of graded posets (P, Q), $T(P \times Q)$ has the same flag f-vector as $T(P) \diamond^* T(Q)$.

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Final note

To observe the 12 page limit, we omitted all proofs. A preprint with the title "Polyspherical complexes" is available at `http://www.math.uncc.edu/~ghetyei` .

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