Some Symmetry and Unimodality Properties of the q, x, y-Hit Numbers

Fred Butler
Department of Mathematics
West Virginia University
P.O. Box 6310
Morgantown, WV 26506-6310
fbutler@math.wvu.edu

Abstract

We prove symmetry and in some cases symmetry and unimodality of polynomials related to the q, x, y-hit numbers introduced by Haglund. These results generalize theorems proven by Haglund for the q-hit numbers. We also apply one of these results to obtain a corollary concerning a generalization of the Eulerian numbers.

Résumé

Nous prouvons la symétrie et dans certains cas la symétrie et l'unimodalité de polynômes relatifs aux q, x, y nombres de contacts introduits par Haglund, généralisant ainsi certains théorèmes. Un de ces résultats nous permet d'obtenir un corollaire à propos d'une généralisation des nombres Eulériens.

1 Introduction

1.1 Preliminaries

We will use the notation SQ_n to denote the $n \times n$ square chess board. We will number the columns of SQ_n with 1 through n going from left to right across the bottom, and the rows of SQ_n with 1 through n going from bottom to top. We will label a square on SQ_n in column i row j with (i, j).

More generally, a *board* will be any subset of SQ_n for some $n \in \mathbb{N}$. A *Ferrers board* is a board with non-decreasing column heights from left to right, or more precisely a board of the form $\{(i, j) \in SQ_n | \ 1 \leq j \leq b_i, \ 1 \leq i \leq n\}$ where $b_1 \leq b_2 \leq \cdots \leq b_n$. We will denote the Ferrers board with

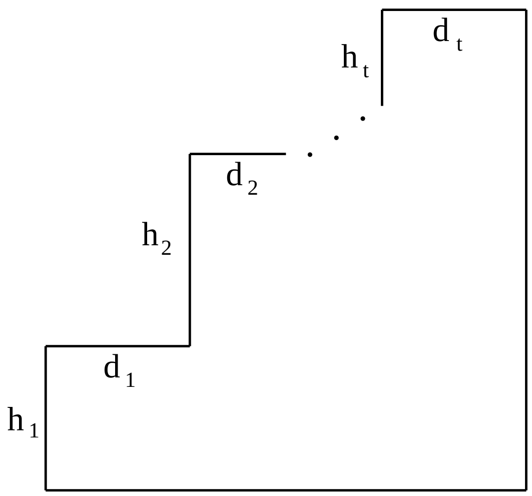

Figure 1: The Ferrers board $B(h_1, d_1; \ldots; h_t, d_t)$.

column heights $b_1, b_2, \ldots b_n$ by $B(b_1, \ldots, b_n)$. We will also specify a Ferrers board by its step heights and depths. The Ferrers board $B(h_1, d_1; \ldots; h_t, d_t)$ is shown in Figure 1. We will call $B = B(b_1, \ldots, b_n) = B(h_1, d_1; \ldots; h_t, d_t)$ a *regular Ferrers board* if $b_i \geq i$ for $1 \leq i \leq n$, or equivalently if $h_1 + \cdots + h_i \geq d_1 + \cdots + d_i$ for $1 \leq i \leq t$ as was defined in [9]. In this paper we will focus on regular Ferrers boards.

A *rook placement* on a board $B \subseteq SQ_n$ is a subset of squares of B such that no two of these squares lie in the same row or the same column. As the name suggests, these squares represent positions on an $n \times n$ chess board where non-attacking rooks can be placed. Let $r_k(B)$ denote the number of k rook placements on B, and let $h_{n,k}(B)$ denote the number of n rook placements on SQ_n such that exactly k rooks lie on B. These are known as the *kth rook number* and the *kth hit number*, respectively, of the board B. Classical rook theory is concerned with studying the relationships between these two numbers.

1.2 Cycle-counting q-rook theory

The cycle-counting q-rook numbers were first introduced in the unpublished work of Ehrenborg, Haglund, and Readdy [4], defined only for Ferrers boards. These rook numbers generalize both the q-rook numbers $R_k(q, B)$ of Garsia and Remmel [5], and the cycle-counting rook numbers $r_k(y, B)$ of Chung and Graham [2]. In order to describe them, we need to define the following three statistics.

The first statistic is denoted inv_B, a generalization of the number of inversions of a permutation. Given a placement P of rooks on a Ferrers board $B \subseteq SQ_n$, let each rook cancel all squares to the right in its row and below in its column. We can then define $inv_B(P)$ to be the number of squares of B which neither contain a rook from P nor are cancelled.

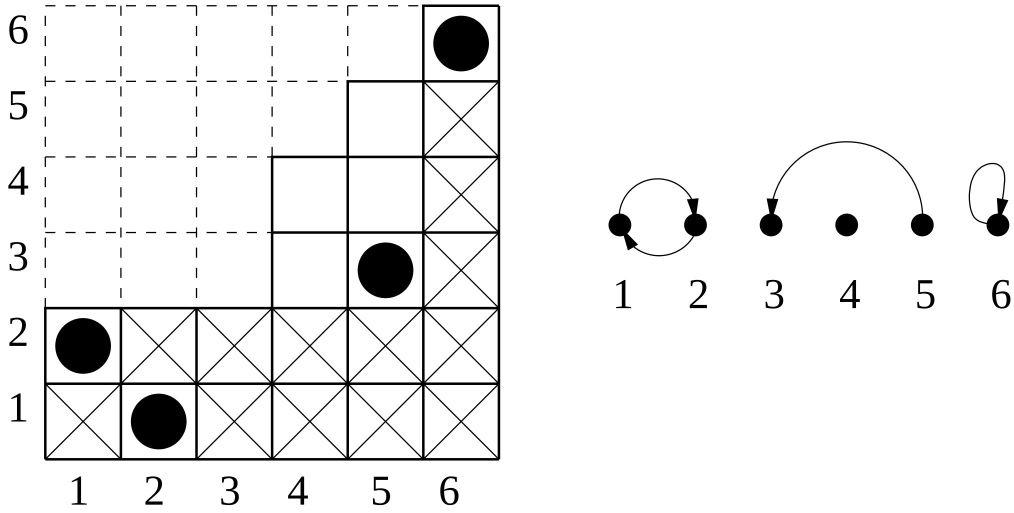

Figure 2: The placement P on B and the associated digraph G_P.

The second statistic is denoted *cyc*, and is a generalization of the number of cycles of a permutation. Given a rook placement P on a board $B \subseteq SQ_n$, it is possible to associate to P a simple directed graph G_P on n vertices. This fact was first noted in [6] (see also [2] and [3]). There is an edge from i to j in G_P if and only if there is a rook from P on the square (i, j). We can then define $cyc(P)$ to be the number of cycles in G_P.

The third statistic, denoted E, depends on the following fact. Given any placement P of j non-attacking rooks in columns 1 through $i-1$ of a Ferrers board B (where $j \leq i-1$), it is an easy exercise to see that if $b_i \geq i$ then there is exactly one square in column i where placement of a rook will complete a new cycle in the digraph G_P. If $b_i < i$ then there is no square where placing a rook will complete a new cycle. Note that a regular Ferrers board will have such a square in each of its columns (since $b_i \geq i$ for all $1 \leq i \leq n$). Now for i with $b_i \geq i$ we can define $s_i(P)$ to be the unique square which, considering only rooks from P in columns 1 through $i-1$ of P, completes a new cycle. Then let $E(P)$ be the number of i such that $b_i \geq i$ and there is no rook from P in column i on or above square $s_i(P)$.

For the rook placement P pictured in Figure 2, we see that $inv_B(P) = 4$, $cyc(P) = 2$, and $E(P) = 2$ (corresponding to $i = 4$ and $i = 5$). We will use the common notation of

$$[x] = \frac{1-q^x}{1-q}$$

to denote the *q-analog* of the real number x. Note that when $x = n \in \mathbb{N}$,

$$[n] = \frac{1-q^n}{1-q} = 1 + q + \cdots + q^{n-1}$$

is a polynomial in q. We use the notation $[n]!$ to denote the q-analog of $n!$, the product $[n][n-1]\cdots[2][1]$. Finally, for $n, k \in \mathbb{N}$ we denote by $\left[n \atop k \right]$ the

q-analog of the binomial coefficient $\binom{n}{k}$, equal to

$$\frac{[n]!}{[k]![n-k]!} = \frac{[n][n-1]\cdots[n-k+1]}{[k]!}$$

for $k \leq n$ and equal to 0 for $k > n$. The fact that $\begin{bmatrix} n \\ k \end{bmatrix}$ is also a polynomial in q is proven in [10]. More generally for $z \in \mathbb{C}$ we will write $\begin{bmatrix} z \\ k \end{bmatrix}$ for $[z][z-1]\cdots[z-k+1]/[k]!$.

As in [4], we now define the *kth cycle-counting q-rook number* of a Ferrers board B by the equation

$$R_k(y,q,B) = \sum_{P \ k \text{ rooks on } B} [y]^{cyc(P)} q^{inv_B(P)+(y-1)E(P)}. \tag{1}$$

Letting $y = 1$ in (1) yields the q-rook numbers of [5], and letting $q \to 1$ gives the cycle-counting rook numbers of [2]. The $R_k(y,q,B)$ satisfy the useful equation

$$\sum_{k=0}^{n} R_{n-k}(y,q,B)[z][z-1]\cdots[z-k+1] =$$
$$\prod_{i \text{ with } b_i \geq i} [z+b_i-i+y] \prod_{i \text{ with } b_i < i} [z+b_i-i+1], \tag{2}$$

a version of the well-known factorization theorems proven for the $r_k(B)$ [7], $R_k(q,B)$ [5], and $r_k(y,B)$ [2].

Haglund [9] further extended this model by defining the *q,x,y-hit numbers* algebraically by the equation

$$\sum_{k=0}^{n} A_{n,k}(x,y,q,B)z^k = \tag{3}$$
$$\sum_{k=0}^{n} R_{n-k}(y,q,B)[x][x+1]\cdots[x+k-1]z^k \prod_{i=k+1}^{n}(1-zq^{x+i-1}),$$

which generalize the $a_{n,k}(x,y,B)$ also discussed in [9]. The case $x = y$ is studied in [1], where a combinatorial interpretation for $A_{n,k}(y,y,q,B)$ is given. In addition to generalizing the q-hit numbers of Garsia and Remmel [5], the $A_{n,k}(y,y,q,B)$ also generalize the cycle-counting hit numbers in the model of Chung and Graham [2].

In Section 2 we prove symmetry and unimodality of $A_{n,k}(a,b,q,B)$ for $a,b \in \mathbb{N}$. We then apply this theorem to prove a symmetry and unimodality property of the *cycle-counting q-Eulerian numbers* introduced in [1]. In Section 3, we prove symmetry of the polynomial $A_{n,k}(a,b,q,B)/\prod_{i=1}^{t}[d_i]!$ for any regular Ferrers board $B = B(h_1,d_1;\ldots;h_t,d_t)$. Finally in Section 4, we prove unimodality of $A_{n,k}(a,b,q,B)/\prod_{i=1}^{t}[d_i]!$ for a certain class of regular Ferrers boards.

2 Symmetry and Unimodality of $A_{n,k}(a, b, q, B)$

If $B = B(h_1, d_1; \ldots ; h_t, d_t) \subseteq SQ_n$ is a Ferrers board, let us denote by $B - h_p - d_p$ the Ferrers board $B(h_1, d_1; \ldots ; h_p - 1, d_p - 1; \ldots h_t, d_t) \subseteq SQ_{n-1}$, obtained from B by decreasing the pth step by 1. We will call the number of squares in the board B the $Area(B)$.

Suppose

$$f(q) = \sum_{i=M}^{N} a_i q^i,$$

is a polynomial in q with $a_M, a_N \neq 0$. We call $M + N$ the *virtual degree* of f. We will say the polynomial $f(q)$ is $zsu(d)$ if either

1. $f(q)$ is identically zero, or

2. $f(q)$ is in $\mathbb{N}[q]$, symmetric, and unimodal with virtual degree d.

Note that for $s \in \mathbb{N}$, q^s is $zsu(2s)$ and $[s]$ is $zsu(s - 1)$. We have the following lemmas. The proof of Lemma 2.1 is trivial, and a proof of Lemma 2.2 can be found in [11].

Lemma 2.1. *If f and g are polynomials which are both $zsu(d)$, then $f + g$ is $zsu(d)$.*

Lemma 2.2. *If f is $zsu(d)$ and g is $zsu(e)$, then fg is $zsu(d + e)$.*

Combining Lemmas 2.1 and 2.2 with (2) and (3), we can easily prove the following.

Lemma 2.3. *Let $a, b \in \mathbb{N}$. For any regular Ferrers board $B \subseteq SQ_n$, $A_{n,0}(a, b, q, B)$ is $zsu(Area(B) + n(b - 1) - \binom{n+1}{2})$.*

Lemma 2.4. *Let $B = B(h_1, d_1; \ldots ; h_t, d_t) \subseteq SQ_n$ be a regular Ferrers board, $B - h_t - d_t \subseteq SQ_{n-1}$ as described earlier. Then*

$$A_{n,k}(x, y, q, B) = [k + y + d_t - 1]A_{n-1,k}(x, y, q, B - h_t - d_t)+$$

$$q^{k+y+d_t-2}[n + x - y - d_t - k + 1]A_{n-1,k-1}(x, y, q, B - h_t - d_t)$$

for any $1 \leq k \leq n$.

Proof. Let $p = t$ in Lemma 5.7 of [9]. □

The following is now a simple corollary of the above lemmas.

Corollary 2.5. *Let $B = B(h_1, d_1; \ldots ; h_t, d_t) \subseteq SQ_n$ be a regular Ferrers board, $a, b \in \mathbb{N}$. If $n+a+1 \geq b+d_t+k$, then $A_{n,k}(a, b, q, B)$ is $zsu(Area(B)+ n(b + k - 1) + k(a - 1) - \binom{n+1}{2})$ for $0 \leq k \leq n$.*

Proof. The proof is by induction on $Area(B)$. When $Area(B) = 1$ the only regular Ferrers board is the 1×1 square SQ_1. A quick calculation from the definition shows that $A_{1,0}(x, y, q, SQ_1) = [y]$, $A_{1,1}(x, y, q, SQ_1) = q^y[x - y]$, and $A_{1,k}(x, y, q, SQ_1) = 0$ for $k > 1$. Thus $A_{1,0}(a, b, q, B) = [b]$ and $A_{1,1}(a, b, q, B) = q^b[a - b]$ are $zsu(b - 1)$ and $zsu(a + b - 1)$ respectively, and the result holds for the case $Area(B) = 1$.

Now assume the result holds for all regular Ferrers boards with $Area < A$, and let B be such a board with $Area(B) = A$. We know by Lemma 2.3 that the result holds for $A_{n,0}(a, b, q, B)$, so assume $k > 0$. Then by Lemma 2.4 with $x = a$ and $y = b$, we have that

$$\begin{aligned} A_{n,k}(a, b, q, B) = [k + b + d_t - 1]A_{n-1,k}(a, b, q, B - h_t - d_t) + \\ q^{k+b+d_t-2}[n + a - b - d_t - k + 1]A_{n-1,k-1}(a, b, q, B - h_t - d_t). \end{aligned} \tag{4}$$

Now we know that $[k + b + d_t - 1]$ is $zsu(k + b + d_t - 2)$, and by the induction hypothesis, $A_{n-1,k}(a, b, q, B - h_t - d_t)$ is $zsu(Area(B - h_t - d_t) + (n - 1)(b + k - 1) + k(a - 1) - \binom{n}{2})$. Note here that $Area(B - h_t - d_t) = Area(B) - n - d_t + 1$. Then by Lemma 2.2, the first term on the right side of (4) is $zsu(Area(B) + n(b + k - 1) + k(a - 1) - \binom{n+1}{2})$.

For the second term on the right side of (4), we know that q^{k+b+d_t-2} is $zsu(2k + 2b + 2d_t - 4)$, $[n + a - b - d_t - k + 1]$ is $zsu(n + a - b - d_t - k)$ (since we have assumed $n + a + 1 \geq b + d_t + k$), and by the induction hypothesis $A_{n-1,k-1}(a, b, q, B - h_t - d_t)$ is $zsu(Area(B - h_t - d_t) + (n - 1)(b + k - 2) + (k - 1)(a - 1) - \binom{n}{2})$. Finally, applying Lemma 2.2 one last time we get that the second term on the right side of (4) is $zsu(Area(B) + n(b + k - 1) + k(a - 1) - \binom{n+1}{2})$, and Lemma 2.1 gives us the result for $A_{n,k}(a, b, q, B)$ as well. $\square$

A cycle-counting version of the Eulerian numbers is given in [1], defined by the equation

$$\tilde{E}_{n,k}(y, q) = \sum_{\sigma \in S_n,\ des(\sigma) = k-1} [y]^{\ell rmin(\sigma)} q^{(n - \ell rmin(\sigma))(y-1) + maj(\sigma)}. \tag{5}$$

Here $\ell rmin(\sigma)$ denotes the number of *left-to-right minima* of the permutation σ, computed by the following algorithm. For $\sigma = \sigma_1\sigma_2 \cdots \sigma_n \in S_n$, if $\sigma_{j_1} = 1$ then let y_1 be the cycle $(\sigma_1 \cdots \sigma_{j_1})$. If α is the smallest integer not contained in y_1, and $\sigma_{j_2} = \alpha$, let y_2 be the cycle $(\sigma_{j_1+1} \cdots \sigma_{j_2})$, etc. If the result of the above procedure is the product of cycles $y_1y_2 \cdots y_p$, we will let $p = \ell rmin(\sigma)$.

It was proven in [1] that

$$\tilde{E}_{n,k}(y, q) = A_{n,k-1}(y, y, q, \mathbb{T}_n), \tag{6}$$

where $\mathbb{T}_n = B(1, 2, \ldots, n)$ denotes the triangular Ferrers board. In light of (6) and Corollary 2.5, the following can be easily proven.

Corollary 2.6. *For $m \in \mathbb{N}$, the polynomial*

$$\sum_{\sigma \in S_n,\ des(\sigma) = k-1} [m]^{\ell rmin(\sigma)} q^{(n - \ell rmin(\sigma))(m-1) + maj(\sigma)}$$

is symmetric and unimodal with virtual degree $n(m + k - 2) + (k - 1)(m - 1)$.

3 Symmetry of $A_{n,k}(a,b,q,B)/\prod_{i=1}^{t}[d_i]!$

In this section we prove a more general symmetry result for all regular Ferrers boards, namely the symmetry of the polynomial

$$\frac{A_{n,k}(a,b,q,B)}{\prod_{i=1}^{t}[d_i]!},$$

where $B = B(h_1, d_1; \ldots; h_t, d_t)$. Throughout the rest of the paper we will use the notation H_i for the partial sum $h_1 + \cdots + h_i$, and D_i for $d_1 + \cdots + d_i$. We have the following lemmas.

Lemma 3.1. *Let $B = B(b_1, \ldots, b_n) = B(h_1, d_1; \ldots; h_t, d_t)$ be a regular Ferrers board, $j \in \mathbb{N}$. Then*

$$\frac{\prod_{i=1}^{n}[j+b_i-i+y]}{\prod_{i=1}^{t}[d_i]!} = \prod_{i=1}^{t}\begin{bmatrix} j+H_i-D_{i-1}+y-1 \\ d_i \end{bmatrix}.$$

Proof. We see that

$$\prod_{i=1}^{n}[j+b_i-i+y] =$$

$$\prod_{i=1}^{t}[j+H_i-D_{i-1}+y-1][(j+H_i-D_{i-1}+y-1)-1]\cdots[(j+H_i-D_{i-1}+y-1)-d_i+1].$$

Thus

$$\frac{\prod_{i=1}^{n}[j+b_i-i+y]}{\prod_{i=1}^{t}[d_i]!} =$$

$$\prod_{i=1}^{t}\frac{[j+H_i-D_{i-1}+y-1]\cdots[(j+H_i-D_{i-1}+y-1)-d_i+1]}{[d_i]!},$$

which is

$$\prod_{i=1}^{t}\begin{bmatrix} j+H_i-D_{i-1}+y-1 \\ d_i \end{bmatrix}$$

by definition. □

Lemma 3.2. *Let $B = B(b_1, \ldots, b_n) = B(h_1, d_1; \ldots; h_t, d_t) \subseteq SQ_n$ be a regular Ferrers board. Then $A_{n,k}(x,y,q,B)/\prod_{i=1}^{t}[d_i]! =$*

$$\sum_{j=0}^{k}\begin{bmatrix} n+x \\ k-j \end{bmatrix}\begin{bmatrix} x+j-1 \\ j \end{bmatrix}(-1)^{k-j}q^{\binom{k-j}{2}}\prod_{i=1}^{t}\begin{bmatrix} j+H_i-D_{i-1}+y-1 \\ d_i \end{bmatrix}.$$

Proof. By Lemma 5.1 of [9], we have

$$A_{n,k}(x,y,q,B) = \sum_{j=0}^{k} \begin{bmatrix} n+x \\ k-j \end{bmatrix} \begin{bmatrix} x+j-1 \\ j \end{bmatrix} (-1)^{k-j} q^{\binom{k-j}{2}} \prod_{i=1}^{n} [j+b_i-i+y].$$

The lemma now follows trivially from Lemma 3.1. □

We can now prove the following.

Theorem 3.3. *Let $B = B(h_1, d_1; \dots; h_t, d_t)$ be a regular Ferrers board (so $H_i \geq D_i$ for $1 \leq i \leq t$). Let $a, b \in \mathbb{N}$ with $a \geq b \geq 1$, and set*

$$L_k^{a,b}(B) = Area(B) + n(b-1) + k(n+a-1) - \sum_{i=1}^{t} d_i D_i.$$

Then $A_{n,k}(a,b,q,B)/\prod_{i=1}^{t}[d_i]!$ is either zero or symmetric with virtual degree $L_k^{a,b}(B)$.

Proof. By Lemma 3.2, $A_{n,k}(a,b,q,B)/\prod_{i=1}^{t}[d_i]! =$

$$\sum_{j=0}^{k} \begin{bmatrix} n+a \\ k-j \end{bmatrix} \begin{bmatrix} a+j-1 \\ j \end{bmatrix} (-1)^{k-j} q^{\binom{k-j}{2}} \prod_{i=1}^{t} \begin{bmatrix} j+H_i-D_{i-1}+b-1 \\ d_i \end{bmatrix},$$

which is a polynomial in q (the first two q-binomial coefficients in each summand are clearly polynomials, and the third is since $H_i \geq D_i \geq D_{i-1}$ and $b \geq 1$). Using the fact that $\begin{bmatrix} r \\ s \end{bmatrix}$ is $zsu(s(r-s))$ (see [8, 12] for a proof) and Lemma 2.2, each term on the right side above has virtual degree $(k-j)(n+a-k+j)+j(a-1)+(k-j)(k-j-1)+\sum_{i=1}^{t} d_i(j+H_i-D_i+b-1)$, which is exactly $L_k^{a,b}(B)$. Since the sign alternates, we can only conclude that $A_{n,k}(a,b,q,B)/\prod_{i=1}^{t}[d_i]!$ is symmetric with virtual degree $L_k^{a,b}(B)$. □

4 Unimodality of $A_{n,k}(a,b,q,B)/\prod_{i=1}^{t}[d_i]!$

In this section we give some sufficient conditions on the regular Ferrers board B for the polynomial of the previous section to also be unimodal. Let us first define some more notation.

Suppose we have integers $h_1, \dots, h_t$, $d_1, \dots, d_t$, and $e_1, \dots, e_t$ with $d_i \in \mathbb{P}$, $h_i \in \mathbb{N}$, and $0 \leq e_i \leq d_i$. We will denote the vector $(e_1, e_2, \dots, e_t)$ by $\vec{e}$. We will continue to denote the partial sum $h_1 + \cdots + h_i$ by H_i, $d_1 + \cdots + d_i$ by D_i, and we will also let $E_i = e_1 + \cdots + e_i$. We make the convention that $H_0 = D_0 = E_0 = 0$. For fixed $h_1, \dots, h_t$ and $d_1, \dots, d_t$ we can define

$$P(\vec{e},x,y) = \prod_{i=1}^{t} \begin{bmatrix} H_i - D_{i-1} + E_{i-1} + y - 1 \\ d_i - e_i \end{bmatrix} \begin{bmatrix} D_i + D_{i-1} - H_i - E_{i-1} + x - y \\ e_i \end{bmatrix}$$

and prove the following lemmas.

Lemma 4.1. *Let $B = B(h_1, d_1; \ldots h_{t-1}, d_{t-1}; h_t, d_t) \subseteq SQ_n$ be a regular Ferrers board, $B' = B(h_1, d_1; \ldots; h_{t-1}, d_{t-1}) \subseteq SQ_{H_{t-1}}$. Then*

$$A_{n,k}(x, y, q, B) = [d_t]! \sum_{s=k-d_t}^{k} A_{H_{t-1},s}(x, y, q, B') \begin{bmatrix} y + d_t + s - 1 \\ d_t - k + s \end{bmatrix}$$

$$\times \begin{bmatrix} n - y - d_t + x - s \\ k - s \end{bmatrix} q^{(k-s)(y+k-1)}.$$

Proof. Let $p = t$ in Corollary 5.10 of [9] and note that because B is a regular Ferrers board, $H_t = D_t = n$. □

Lemma 4.2. *Let $B = B(h_1, d_1; \ldots; h_t, d_t)$ be a regular Ferrers board. Then*

$$A_{n,k}(x, y, q, B) =$$

$$\prod_{i=1}^{t} [d_i]! \sum_{e_1+\cdots+e_t=k,\ 0 \le e_i \le d_i} P(\vec{e}, x, y) \prod_{i=1}^{t} q^{e_i(H_i - D_i + E_i + y - 1)}. \quad (7)$$

Proof. By induction on t. When $t = 1$ we have that $d_1 = n$, and Lemma 4.1 gives us

$$A_{n,k}(x, y, q, B) = [d_1]! \sum_{s=k-n}^{k} A_{0,s}(x, y, q, \emptyset) \begin{bmatrix} y + n + s - 1 \\ d_1 - k + s \end{bmatrix} \quad (8)$$

$$\times \begin{bmatrix} n - y - n + x - s \\ k - s \end{bmatrix} \times q^{(k-s)(y+k-1)}.$$

In this case we have that $H_1 = D_1 = d_1 = n$ and $D_0 = H_0 = 0$, so we get that the $s = 0$ term in (8) is equal to

$$[d_1]! \begin{bmatrix} H_1 - D_0 + y - 1 \\ d_1 - k \end{bmatrix} \begin{bmatrix} D_1 + D_0 - H_1 + x - y \\ k \end{bmatrix} \times q^{k(H_1 - D_1 + k + y - 1)}. \quad (9)$$

Note that by definition

$$A_{0,s}(x, y, q, \emptyset) = \delta_{s,0},$$

so the only nonzero summand in (8) occurs when $s = 0$ and hence (9) is actually equal to (8). Finally if we recall that $E_1 = e_1$ and $E_0 = 0$, we can rewrite (9) as

$$[d_1]! \sum_{e_1=k,\ 0 \le e_1 \le d_1} \begin{bmatrix} H_1 - D_0 + E_0 + y - 1 \\ d_1 - e_1 \end{bmatrix}$$

$$\times \begin{bmatrix} D_1 + D_0 - H_1 - E_0 + x - y \\ e_1 \end{bmatrix} \times q^{e_1(H_1 - D_1 + E_1 + y - 1)},$$

which is exactly of the form of (7).

For $t > 1$, Lemma 4.1 gives that

$$A_{n,k}(x,y,q,B) = [d_t]! \sum_{E_{t-1}=E_t-d_t}^{E_t} A_{H_{t-1},E_{t-1}}(x,y,q,B') \begin{bmatrix} y+d_t+E_{t-1}-1 \\ d_t-e_t \end{bmatrix}$$

$$\times \begin{bmatrix} n-y-d_t+x-E_{t-1} \\ e_t \end{bmatrix} \times q^{e_t(y+E_t-1)}. \tag{10}$$

Here we are letting $E_{t-1} = s$ and defining $e_t = k-s$ and $E_t = E_{t-1}+e_t = k$. Since B is regular $H_t = D_t = n$, so $H_t - D_{t-1} = D_t - D_{t-1} = d_t$ and (10) can be rewritten as

$$A_{n,k}(x,y,q,B) = [d_t]! \sum_{e_t=0}^{d_t} A_{H_{t-1},E_{t-1}}(x,y,q,B') \begin{bmatrix} H_t-D_{t-1}+E_{t-1}+y-1 \\ d_t-e_t \end{bmatrix}$$

$$\times \begin{bmatrix} D_t+D_{t-1}-H_t-E_{t-1}+x-y \\ e_t \end{bmatrix} \times q^{e_t(H_t-D_t+E_t+y-1)}.$$

By the inductive hypothesis, the above is equal to

$$[d_t]! \sum_{e_t=0}^{d_t} \left\{ \prod_{i=1}^{t-1} [d_i]! \sum_{e_1+\cdots+e_{t-1}=E_{t-1},\ 0\le e_i\le d_i} \prod_{i=1}^{t-1} \begin{bmatrix} H_i-D_{i-1}+E_{i-1}+y-1 \\ d_i-e_i \end{bmatrix} \right.$$

$$\left. \times \begin{bmatrix} D_i+D_{i-1}-H_i-E_{i-1}+x-y \\ e_i \end{bmatrix} q^{e_i(H_i-D_i+E_i+y-1)} \right\}$$

$$\times \begin{bmatrix} H_t-D_{t-1}+E_{t-1}+y-1 \\ d_t-e_t \end{bmatrix} \begin{bmatrix} D_t+D_{t-1}-H_t-E_{t-1}+x-y \\ e_t \end{bmatrix} q^{e_t(H_t-D_t+E_t+y-1)}$$

which is

$$\prod_{i=1}^{t} [d_i]! \sum_{e_1+\cdots+e_t=k,\ 0\le e_i\le d_t} P(\vec{e},x,y) \prod_{i+1}^{t} q^{e_i(H_i-D_i+E_i+y-1)}$$

as desired. □

Lemma 4.3. *Let $B = B(h_1,d_1;\ldots;h_t,d_t)$ be a regular Ferrers board, $a,b \in \mathbb{N}$ with $a \ge b \ge 1$. Let e_i, d_i, h_i, E_i, D_i, and H_i be as in the definition of $P(\vec{e},x,y)$. Assume that B is such that $d_{i-1}+d_i \ge h_i$ for $1 \le i \le t$ (where $d_0 := 0$). If any of the numerators of the q-binomial coefficients in*

$$P(\vec{e},a,b) = \prod_{i=1}^{t} \begin{bmatrix} H_i-D_{i-1}+E_{i-1}+b-1 \\ d_i-e_i \end{bmatrix} \begin{bmatrix} D_i+D_{i-1}-H_i-E_{i-1}+a-b \\ e_i \end{bmatrix}$$

are negative, then $P(\vec{e},a,b) = 0$.

Proof. First note that $H_i - D_{i-1} + E_{i-1} + b - 1 \geq 0$ for $1 \leq i \leq t$, since $H_i \geq D_i \geq D_{i-1}$ and $b \geq 1$, so none of the numerators in the first q-binomial coefficient of the product are ever negative.

Now suppose that $D_k + D_{k-1} - H_k - E_{k-1} + a - b < 0$ for some k with $0 \leq k \leq t$. Note $D_1 + D_0 - H_1 - E_0 + a - b = d_1 - h_1 + a - b$, and since we assumed $d_{i-1} + d_i \geq h_i$ (and in particular $d_1 \geq h_1$) and $a \geq b$, we have that $d_1 - h_1 + a - b \geq 0$. Thus we see that such a k must be greater than 2.

Now choose j such that $D_i + D_{i-1} - H_i - E_{i-1} + a - b \geq 0$ for $1 \leq i < j$, but $D_j + D_{j-1} - H_j - E_{j-1} + a - b < 0$ (such a j exists because of the remarks in the previous paragraph). Then $D_j + D_{j-1} - H_j - E_{j-1} + a - b < 0$ implies $D_j + D_{j-1} - H_j - E_{j-2} + a - b < e_{j-1}$, which is equivalent to $d_j + d_{j-1} - h_j + D_{j-1} + D_{j-2} - H_{j-1} - E_{j-2} + a - b < e_{j-1}$, which implies $D_{j-1} + D_{j-2} - H_{j-1} - E_{j-2} + a - b < e_{j-1}$ (since $d_j + d_{j-1} \geq h_j$). Hence

$$\begin{bmatrix} D_{j-1} + D_{j-2} - H_{j-1} - E_{j-2} + a - b \\ e_{j-1} \end{bmatrix} = 0$$

since the numerator is non-negative by definition of j but less than the denominator, so the product $P(\vec{e}, a, b) = 0$ as well. □

Theorem 4.4. *Let $B = B(h_1, d_1; \ldots; h_t, d_t)$ be a regular Ferrers board such that $d_{i-1} + d_i \geq h_i$ for $1 \leq i \leq t$. Let $a, b \in \mathbb{N}$ with $a \geq b \geq 1$, and set*

$$L_k^{a,b}(B) = Area(B) + n(b-1) + k(n + a - 1) - \sum_{i=1}^{t} d_i D_i$$

as before. Then $A_{n,k}(a, b, q, B) / \prod_{i=1}^{t} [d_i]!$ is $zsu(L_k^{a,b}(B))$.

Proof. We apply Lemma 4.2, which says that

$$\frac{A_{n,k}(a, b, q, B)}{\prod_{i=1}^{t} [d_i]!} = \sum_{e_1 + \cdots + e_t = k,\ 0 \leq e_i \leq d_i} P(\vec{e}, a, b) \prod_{i=1}^{t} q^{e_i (H_i - D_i + E_i + b - 1)},$$

and all of the terms on the right hand side above are in $\mathbb{N}[q]$ by Lemma 4.3. Each term is $zsu(\sum_{i=1}^{t} \{(d_i - e_i)(H_i - D_i + E_i + b - 1) + e_i(D_i + D_{i-1} - H_i - E_i + a - b) + 2e_i(H_i - D_i + E_i + b - 1)\})$, which a simple calculation shows is the same $zsu(L_k^{a,b}(B))$. Thus by Lemma 2.1, $A_{n,k}(a, b, q, B) / \prod_{i=1}^{t} [d_i]!$ is $zsu(L_k^{a,b}(B))$ as well. □

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