ON GROWTH OF SOLVABLE LIE SUPERALGEBRAS AND GENERATING FUNCTIONS

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Abstract. Finitely generated solvable Lie algebras have an intermediate growth between polynomial and exponential. Recently the second author suggested the scale to measure such an intermediate growth of Lie algebras. The growth was specified for solvable Lie algebras $F(\mathbf{A}^q, k)$ with a finite number of generators k, and which are free with respect to a fixed solubility length q. Later, an application of generating functions allowed to obtain more precise asymptotic. These results were obtained in generality of polynilpotent Lie algebras.

Now we consider the case of Lie superalgebras; we announce main results and describe methods. Our goal is to compute the growth for $F(\mathbf{A}^q, m, k)$, the free solvable Lie superalgebra of length q with m even and k odd generators. The proof is based upon a precise formula of the generating function for this algebra obtained earlier. The result is obtained in the generality of free polynilpotent Lie superalgebras.

We also consider almost solvable finitely generated Lie algebras and establish an upper bound on their growth in terms of our scale of subexponential functions.

1. Introduction

The ground field is denoted by K. Recall that a $\mathbb{Z}_2$-graded algebra $L = L_+ \oplus L_-$ is called a *Lie superalgebra* if it satisfies the following graded identities [27]. Let $\epsilon(L_+, L_+) = \epsilon(L_+, L_-) = \epsilon(L_-, L_+) = 1$, and $\epsilon(L_-, L_-) = -1$. We suppose that

$$[x, y] = -\epsilon(x, y)[y, x], \qquad x, y \in L_\pm \qquad \text{(anticommutativity);}$$
$$[x, [y, z]] = [[x, y], z] - \epsilon(y, z)[[x, z], y], \qquad x, y, z \in L_\pm \qquad \text{(Jacobi identity).}$$

(In case $\operatorname{char} K = 2$ some more additional assumptions should be imposed [2], also in case $\operatorname{char} K = 3$ one requires $[[y, y], y] = 0$ for all $y \in L_-$, this identity being satisfied in other characteristics). A *variety* of (Lie) (super)algebras is a class of all (Lie) (super)algebras that satisfy some set of (graded or non-graded) identical relations. Concerning varieties of Lie algebras we refer the reader to the monograph [1]. On Lie superalgebras and their varieties see also [2] and [16].

Let L be a Lie (super)algebra. Then the *lower central series* is defined by iteration $L^1 = L$, $L^{i+1} = [L, L^i]$, $i = 1, 2, \dots$. Now L is called *nilpotent* of class s provided that $L^{s+1} = \{0\}$. All Lie algebras nilpotent of class s form the variety denoted by $\mathbf{N}_s$. This notation we also shall use for the variety of nilpotent Lie superalgebras of class s. Recall that L is *polynilpotent* with a tuple $(s_q, \dots, s_2, s_1)$ iff there exists a chain of ideals $0 = L_{q+1} \subset L_q \subset \dots \subset L_2 \subset L_1 = L$ such that $L_i/L_{i+1} \in \mathbf{N}_{s_i}$, $i = 1, \dots, q$. All polynilpotent Lie (super)algebras with the fixed tuple $(s_q, \dots, s_2, s_1)$ form the variety denoted by $\mathbf{N}_{s_q} \cdots \mathbf{N}_{s_2}\mathbf{N}_{s_1}$. In the case $s_q = \dots = s_1 = 1$, one obtains as a particular case the variety $\mathbf{A}^q$ of *solvable* Lie (super)algebras of length q. Polynilpotent varieties of groups and Lie algebras as well as their interactions were studied by A.L. Shmelkin [28]. A basis for free polynilpotent Lie algebras was constructed by L.A. Bokut [4], for the case of free solvable Lie algebras, see also the monograph of C. Reutenauer [26].

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Let L be a Lie superalgebra, then by $L = L_+ \oplus L_-$ we denote its decomposition into the even and odd components. Suppose that $\mathbf{M}$ is a variety of Lie superalgebras, then by $F(\mathbf{M}, X)$, $X = X_+ \cup X_-$ we denote its free algebra generated by X. Let $X_+ = \{x_i | i \in I_+\}$, $X_- = \{x_j | j \in I_-\}$. Recall that $F(\mathbf{M}, X)$ is the algebra generated by X and such that for all $H = H_+ \oplus H_- \in \mathbf{M}$ and any $y_i \in H_+$, $i \in I_+$; $y_j \in H_-$, $j \in I_-$, there exists a homomorphism $\phi : F(\mathbf{M}, X) \to H$ with $\phi(x_i) = y_i$, $i \in I_+ \cup I_-$. In case $|X_+| = m$, $|X_-| = k$ we also denote $F(\mathbf{M}, X) = F(\mathbf{M}, m, k)$.

One verifies that each polynilpotent Lie (super)algebra is solvable, i.e. belongs to some $\mathbf{A}^q$ for a sufficiently large q. So, by studying free polynilpotent Lie (super)algebras we study first, some solvable Lie (super)algebras. Second, this setting, as a particular case, includes the free solvable (super)algebras $F(\mathbf{A}^q, X) = F(\underbrace{\mathbf{N}_1 \cdots \mathbf{N}_1}_{q \text{ times}}, X)$.

Let A be a Lie (associative) algebra over a field K, generated by a finite set X. Denote by $A^{(X,n)}$ the subspace spanned by all monomials in X of length not exceeding n. Denote

$$\gamma_A(n) = \gamma_A(X, n) = \dim_K A^{(X,n)}, \qquad n \in \mathbb{N};$$
$$\lambda_A(n) = \gamma_A(n) - \gamma_A(n-1), \qquad n \in \mathbb{N}.$$

where $\dim_K$ stands for the dimension of a vector space over K. If A is an associative algebra with unity then we consider that this unity belongs to $A^{(X,n)}$, $n \geq 0$, and $\gamma_A(0) = \lambda_A(0) = 1$. On functions $f : \mathbb{N} \to \mathbb{R}^+$, where $\mathbb{R}^+ = \{\alpha \in \mathbb{R} \,|\, \alpha > 0\}$, we consider the partial order: $f(n) \overset{\text{a}}{\leq} g(n)$ iff there exists $N > 0$, such that $f(n) \leq g(n)$, $n \geq N$.

Consider two extreme examples of growth. Suppose that A is a free associative algebra (or a free Lie algebra) of finite rank. Then the growth function $\gamma_A(X, n)$ is an exponential. On the other hand, let $A = U(L)$ be the universal enveloping algebra of a finite dimensional Lie algebra L. Then the growth function $\gamma_A(X, n)$ is a polynomial of degree $k = \dim_K L$, and the degree k is extracted by computing of the Gelfand-Kirillov dimension [13].

But there are growths between these two extreme types of growth. The growth less than any exponent is called *subexponential.* If it is also greater than any polynomial growth, then it is called *intermediate.* For study of such growths the following series of dimensions has been suggested [17], [18]. Denote by iteration

$$\ln^{(1)} n = \ln n; \qquad \ln^{(q+1)} n = \ln(\ln^{(q)} n), \qquad q = 1, 2, \ldots.$$

Consider the series of functions $\Phi_\alpha^q(n)$, $q = 1, 2, 3, \ldots$ of the natural argument with the parameter $\alpha \in \mathbb{R}^+$:

$$\Phi_\alpha^1(n) = \alpha; \qquad q = 1;$$
$$\Phi_\alpha^2(n) = n^\alpha; \qquad q = 2;$$
$$\Phi_\alpha^3(n) = \exp(n^{\alpha/(\alpha+1)}); \qquad q = 3;$$
$$\Phi_\alpha^q(n) = \exp\left(\frac{n}{(\ln^{(q-3)} n)^{1/\alpha}}\right), \qquad q = 4, 5, \ldots.$$

Suppose that $f(n)$ is a positive valued function of a natural argument. We define the *(upper) dimension of level* q, $q = 1, 2, 3, \ldots$, and the *lower dimension of level* q by

$$\operatorname{Dim}^q f(n) = \inf\{\alpha \in \mathbb{R}^+ \mid f(n) \overset{\text{a}}{\leq} \Phi_\alpha^q(n)\},$$
$$\underline{\operatorname{Dim}}^q f(n) = \sup\{\alpha \in \mathbb{R}^+ \mid f(n) \overset{\text{a}}{\geq} \Phi_\alpha^q(n)\}.$$

Suppose that A is a finitely generated algebra. We define the *q-dimension* and the *lower q-dimension*, $q = 1, 2, 3, \ldots$, of A by

$$\operatorname{Dim}^q A = \operatorname{Dim}^q \gamma_A(n), \quad \underline{\operatorname{Dim}}^q A = \underline{\operatorname{Dim}}^q \gamma_A(n).$$

Roughly speaking, the condition $\operatorname{Dim}^q A = \underline{\operatorname{Dim}}^q A = \alpha$ means that the growth function $\gamma_A(n)$ behaves like $\Phi_\alpha^q(n)$. These q-dimensions do not depend on a generating set X [18].

Remark that 1-dimension coincides with the dimension of the vector space A over K. Dimensions of level 2 are exactly the upper and lower Gelfand-Kirillov dimensions [7], [13], and [15]. Dimensions of level 3 correspond to the superdimensions of [5] up to normalization (see [18]). If L is a Lie (super)algebra, then by $U(L)$ we denote its universal enveloping algebra. The following two theorems are the crucial facts about this scale of functions and dimensions.

Theorem 1.1 ([17], [18]). *Let L be a finitely generated Lie algebra with $\operatorname{Dim}^q L = \alpha > 0$, $q = 1, 2, \ldots$. Also for $q \geq 3$ suppose that $\underline{\operatorname{Dim}}^q L = \alpha$ and for $q = 2$ suppose that $\underline{\operatorname{Dim}}^2 \lambda_L(n) = \alpha - 1$, and $\alpha \geq 1$. Then*

$$\underline{\operatorname{Dim}}^{q+1} U(L) = \operatorname{Dim}^{q+1} U(L) = \alpha.$$

A. Lichtman proved that the growth of finitely generated solvable Lie algebras is subexponential [14]. The following result specifies the growth of such algebras in terms of our scale of functions.

Theorem 1.2. [18] *Let $L = F(\mathbf{N}_{s_q} \cdots \mathbf{N}_{s_2}\mathbf{N}_{s_1}, k)$ be the free polynilpotent Lie algebra of rank k, $k \geq 2$, $q \geq 2$. Then*

$$\underline{\operatorname{Dim}}^q L = \operatorname{Dim}^q L = s_2 \dim_K F(\mathbf{N}_{s_1}, k).$$

As a particular case, we have the following.

Corollary 1.1. [17] *Let $L = F(\mathbf{A}^q, k)$ be the free solvable Lie algebra of length q and rank k, $k \geq 2$, $q \geq 1$. Then $\underline{\operatorname{Dim}}^q L = \operatorname{Dim}^q L = k$.*

More precise asymptotic for free polynilpotent Lie algebras $L = F(\mathbf{N}_{s_q} \cdots \mathbf{N}_{s_1}, k)$ is found in [19]. As an application, this result gave also an answer to the question of M. I. Kargapolov [12] to describe the lower central series ranks for free polynilpotent finitely generated groups. Earlier exact recursive formulae were found by G.P. Egorychev [6]. Another answer to this problem was given by the second author by describing the asymptotic behaviour of these ranks [19]. In general, the approach [19] heavily relies on the use of generating functions and study of their growth.

Denote by $\zeta(*)$, $\Gamma(*)$, $\mu(*)$ the Riemann zeta-function, the Gamma-function, and the Möbius function, respectively. By $\delta_{i,j}$ denote the Kronecker symbol.

2. Main result: Growth of solvable Lie superalgebras

Now, our goal is to study the growth of solvable Lie *super*algebras. Let us formulate the first result in this direction. Its says that $F(\mathbf{N}_{s_q} \cdots \mathbf{N}_{s_1}, m, k)$ lies, as a rule, also on the level q.

Theorem 2.1. *Let $L = F(\mathbf{N}_{s_q} \cdots \mathbf{N}_{s_2}\mathbf{N}_{s_1}, m, k)$ be the free polynilpotent Lie superalgebra, where $m + k \geq 2$ and $q \geq 2$. Then*

(1) $\underline{\operatorname{Dim}}^q L = \operatorname{Dim}^q L = s_2 \dim F_+(\mathbf{N}_{s_1}, m, k)$.
(2) $\dim F_+(\mathbf{N}_{s_1}, m, k) = 0$ *is equivalent to* $s_1 = 1$ *and* $m = 0$. *In this case we also suppose that* $q \geq 3$, *then* $\underline{\operatorname{Dim}}^{q-1} L = \operatorname{Dim}^{q-1} L = s_3 \dim F_+(\mathbf{N}_{s_2}\mathbf{A}, 0, k)$.

As a particular case, we obtain.

Corollary 2.1. *Let $L = F(\mathbf{A}^q, m, k)$ be the free solvable Lie superalgebra of length q, where $m + k \geq 2$, $q \geq 2$. Then*

(1) $\underline{\operatorname{Dim}}^q L = \operatorname{Dim}^q L = m$;
(2) *If* $m = 0$ *and* $q \geq 3$ *then* $\underline{\operatorname{Dim}}^{q-1} L = \operatorname{Dim}^{q-1} L = 1 + (k-1)2^{k-1}$.

We shall refer to the second case as the *degenerate case*. Now we formulate our main result, that immediately implies Theorem 2.1. Complete proofs of main results will appear in [10].

Theorem 2.2. *Consider the polynilpotent variety $\mathbf{V}=\mathbf{N}_{s_q}\cdots\mathbf{N}_{s_1}$, $q\ge 2$, of Lie superalgebras. Suppose that $L=F(\mathbf{V},m,k)$ is its free superalgebra, generated by set $X=X_+\cup X_-$, $X_+=\{x_1,\dots,x_m\}$, $X_-=\{x_{m+1},\dots,x_{m+k}\}$. Then*

(1) *If $s_1>1$ or $m>0$ then*

$$\gamma_L(X,n)=\begin{cases}\dfrac{A+o(1)}{N!}n^N, & q=2;\\ \exp\left((C+o(1))n^{\frac{N}{N+1}}\right), & q=3;\\ \exp\left((B^{1/N}+o(1))\dfrac{n}{(\ln^{(q-3)}n)^{1/N}}\right), & q\ge 4;\end{cases}$$

where the constants are

$$N=s_2\dim F_+(\mathbf{N}_{s_1},m,k),\qquad A=\frac{1}{s_2}\left((m+k-1)\frac{2^{\dim F_-(\mathbf{N}_{s_1},m,k)}}{\prod_{j=1}^{s_1}j^{\psi_+(j)}}\right)^{s_2},$$

$$B=s_3A\zeta(N+1)\left(1-\frac{1-\delta_{k,0}}{2^{N+1}}\right),\qquad C=\left(1+\frac{1}{N}\right)(BN)^{\frac{1}{1+N}};$$

and $\psi_+(j)=\dim F_{j,+}(\mathbf{N}_{s_1},m,k)$, $1\le j\le s_1$, are dimensions of the even parts of the homogeneous components.

(2) *If $s_1=1$, $m=0$, and additionally $q\ge 3$ then*

$$\gamma_L(X,n)=\begin{cases}\dfrac{A+o(1)}{N!}n^N, & q=3;\\ \exp\left((C+o(1))n^{\frac{N}{N+1}}\right), & q=4;\\ \exp\left((B^{1/N}+o(1))\dfrac{n}{(\ln^{(q-4)}n)^{1/N}}\right), & q\ge 5;\end{cases}$$

where the constants in this case are

$$N=s_3\dim F_+(\mathbf{N}_{s_2}\mathbf{A},0,k),\qquad A=\frac{1}{s_3}\left((k-1)\frac{2^{\dim F_-(\mathbf{N}_{s_2}\mathbf{A},0,k)}}{\prod_{2|j}j^{\phi(j)}}\right)^{s_3},$$

$$B=s_4A\zeta(N+1)\left(1-\frac{1}{2^{N+1}}\right),\qquad C=\left(1+\frac{1}{N}\right)(BN)^{\frac{1}{1+N}};$$

and $\phi(j)=\dim F_j(\mathbf{N}_{s_2}\mathbf{A},0,k)$ are dimensions of the homogeneous components of the finite dimensional algebra $F(\mathbf{N}_{s_2}\mathbf{A},0,k)$.

We draw the readers attention that these asymptotics hold for the growth with respect to the standard generating set X only. One can also easily derive the corollary for the particular case of free solvable Lie superalgebras $L=F(\mathbf{A}^q,m,k)$, $q\ge 2$.

Our approach heavily relies on application of generating functions. Suppose that an algebra A is generated by a finite set X and is homogeneous with respect to the degree in X. So, we have $A=\mathop{\oplus}\limits_{n=0}^{\infty}A_n$ and $\dim A_n=\lambda_A(X,n)$. In this case we define the Hilbert-Poincaré series

$$\mathcal{H}_X(A,t)=\sum_{n=0}^{\infty}\dim A_n t^n.$$

We introduce some more series in the next section. The following result plays an important role in our proof. It is also of independent interest.

Theorem 2.3. *Consider the polynilpotent variety $\mathbf{V}=\mathbf{N}_{s_q}\cdots\mathbf{N}_{s_1}$, $q\ge 2$, of Lie superalgebras and $L=F(\mathbf{V},m,k)$, $m+k\ge 2$. Then the Hilbert-Poincaré series with respect to the standard generating set X has the following growth, while $t\to 1-o$:*

(1) *If $s_1 > 1$ or $m > 0$ then*

$$\mathcal{H}_X(L,t) = \begin{cases} \dfrac{A+o(1)}{(1-t)^N}, & q=2; \\ \exp^{(q-2)}\left(\dfrac{B+o(1)}{(1-t)^N}\right), & q\ge 3. \end{cases}$$

(2) *If $s_1 = 1$, $m = 0$, and additionally $g \ge 3$ then*

$$\mathcal{H}_X(L,t) = \begin{cases} \dfrac{A+o(1)}{(1-t)^N}, & q=3; \\ \exp^{(q-3)}\left(\dfrac{B+o(1)}{(1-t)^N}\right), & q\ge 4. \end{cases}$$

where the constants N, A, and B are the same as in the respective cases of Theorem 2.2.

3. Generating functions for solvable Lie superalgebras

As it is seen from the previous theorem, the generating functions and their growth play an important role in our arguments. The goal of this section is to formulate a precise formula for generating functions for free solvable (more generally, polynilpotent) Lie superalgebras (Theorem 3.1). The proof of our main result is based on this formula. So, we solve a problem of enumerative combinatorics [8, 29, 30].

Let $X = \{x_i | i \in I\}$, $I = I_+ \cup I_-$ be an at most countable generating set for a superalgebra $A = A_+ \oplus A_-$. We assume that A is multihomogeneous with respect to X. For example, this is the case when A is the relatively free algebra of some variety of superalgebras and $X = X_+ \cup X_-$ is the free generating set. Denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We define the grading $A = \bigoplus_{\alpha\in\mathbb{N}_0^I} A_\alpha$ induced by setting $x_i \in A_{\alpha_i}$, where $\alpha_i = (\dots,0,1,0,\dots) \in \mathbb{N}_0^I$ with 1 on the ith place. Also we define a homomorphism $\varepsilon : \mathbb{N}_0^I \to \{\pm 1\}$ by $\varepsilon(\alpha_i) = \pm 1$, $i \in I_\pm$. The sequence $\alpha = \sum_{i\in I} m_i\alpha_i \in \mathbb{N}_0^I$ has only finitely many nonzero entrees m_i, and we denote $\mathbf{t}^\alpha = \prod_{i\in I} t_i^{m_i}$. We consider the formal power series ring $\mathbb{Q}[[\mathbf{t}]] = \mathbb{Q}[[t_i | i \in I]]$. Suppose that $W = \bigoplus_{\alpha\in\mathbb{N}_0^I} W_\alpha \subset A$ is a homogeneous subalgebra, then we define its *Hilbert-Poincaré series*

$$\mathcal{H}_X(W,\mathbf{t}) = \mathcal{H}_X(W, t_i | i\in I) = \sum_{\alpha\in\mathbb{N}_0^I} \dim(W_\alpha)\,\mathbf{t}^\alpha \in \mathbb{Q}[[\mathbf{t}]].$$

We use the following operators acting on the formal power series ring $\mathbb{Q}[[\mathbf{t}]]$. These operators were introduced in [22], [23]:

$$\phi^{[-]}(\mathbf{t}) = \phi\Big|_{t_i=\varepsilon(t_i)t_i,\ i\in I};$$

$$\phi^{[m]}(\mathbf{t}) = \phi\Big|_{t_i=\varepsilon(t_i)^{m+1}t_i^m,\ i\in I}, \quad m\in\mathbb{N};$$

$$\mathcal{E}(\phi)(\mathbf{t}) = \exp\left(\sum_{m=1}^\infty \frac{1}{m}\phi^{[m]}(\mathbf{t})\right) = \prod_{\alpha\in\mathbb{N}_0^I}(1-\varepsilon(\alpha)\mathbf{t}^\alpha)^{-\varepsilon(\alpha)b_\alpha}, \qquad (1)$$

$$\text{where}\quad \phi(\mathbf{t}) = \sum_{\alpha\in\mathbb{N}_0^I} b_\alpha\mathbf{t}^\alpha \in \mathbb{Q}[[\mathbf{t}]]_0;$$

and $\mathbb{Q}[[\mathbf{t}]]_0$ denotes series with zero constant term. For the equality (1) see [22]. For the right hand side of (1) to be defined we also suppose that $b_\alpha \in \mathbb{Z}$, such expressions enumerate universal enveloping algebras [31]. The importance of $\mathcal{E}$ is explained by the following.

Lemma 3.1 ([22]). *Let L be a Lie superalgebra generated by an at most countable generating set $X = X_+ \cup X_-$ and multihomogeneous with respect to it. Then the series for the universal enveloping algebra equals $\mathcal{H}_X(U(L),\mathbf{t}) = \mathcal{E}(\mathcal{H}_X(L,\mathbf{t}))$.*

The proof of our main result is based on the following explicit formula of the generating function for free polynilpotent Lie superalgebras [23]. In a particular case of the free metabelian Lie superalgebra $F(\mathbf{A}^2, m, k)$ this series was found in [3], see also [2].

Theorem 3.1 ([23]). *Let $L = F(\mathbf{N}_{s_q} \cdots \mathbf{N}_{s_2}\mathbf{N}_{s_1}, X)$ be the free polynilpotent Lie superalgebra generated by an at most countable generating set $X = \{x_i | i \in I\}$, where $I = I_+ \cup I_-$ and $X_+ = \{x_i | i \in I_+\}$, $X_- = \{x_i | i \in I_-\}$ are the even and odd generators, respectively. We define functions $g_i(\mathbf{t})$, $f_i(\mathbf{t}) \in \mathbb{Q}[[\mathbf{t}]]$, $i = 0, \dots, q$ by $g_0(\mathbf{t}) = 0$, $f_0(\mathbf{t}) = \sum_{i\in I} t_i$, and*

$$g_i(\mathbf{t}) = g_{i-1}(\mathbf{t}) + \sum_{m=1}^{s_i} \frac{1}{m} \sum_{a|m} \mu(a) \left(f_{i-1}^{[a]}(\mathbf{t}) \right)^{m/a}, \quad 1 \le i \le q;$$

$$f_i(\mathbf{t}) = 1 + \left(\sum_{i\in I} t_i - 1 \right) \cdot \mathcal{E}(g_i(\mathbf{t})), \quad 1 \le i \le q.$$

Then $\mathcal{H}_X(L, \mathbf{t}) = g_q(\mathbf{t})$.

Now assume that the generating set is finite. Let $I = I_+ \cup I_-$, $I_+ = \{1, \dots, m\}$, $I_- = \{m+1, m+2, \dots, m+k\}$, $1 < m+k < \infty$. We have the generating set $X = X_+ \cup X_-$, where $X_+ = \{x_1, \dots, x_m\}$, and $X_- = \{x_{m+1}, \dots, x_{m+k}\}$. Let $W \subset A$ be a multihomogeneous subspace. Then one has components for the gradation by the multidegree W_α, degree W_n, and "superdegree" W_{ij}, where the last space consists of elements of degree i with respect to X_+, and degree j with respect to X_-. So, we obtain the following different Hilbert–Poincaré series

$$\mathcal{H}_X(W, \mathbf{t}) = \mathcal{H}(W, t_1, \dots, t_{m+k}) = \sum_{\alpha \in \mathbb{N}_0^{m+k}} \dim W_\alpha \, t_1^{\alpha_1} \cdots t_{m+k}^{\alpha_{m+k}};$$

$$\mathcal{H}_X(W, x, y) = \sum_{i,j=0}^{\infty} \dim W_{ij} \, x^i y^j;$$

$$\mathcal{H}_X(W, t) = \sum_{n=0}^{\infty} \dim W_n \, t^n = \mathcal{H}(W, x, y)\Big|_{x=y=t}.$$

In [22] we used variables t_+, t_-, now in order to simplify formulas, we use symbols x, y and hope that they are not mixed up with the elements of the generating set X. We can now apply Theorem 3.1 and obtain $\mathcal{H}(L, x, y)$ just by the setting $t_i = x$ for $i \in I_+$, and $t_i = y$ for $i \in I_-$ in the formula $\mathcal{H}(L, \mathbf{t})$. The above operators look in this case as

$$\phi^{[-]}(x, y) = \phi(x, -y);$$

$$\phi^{[m]}(x, y) = \phi(x^m, (-1)^{m+1} y^m);$$

$$\mathcal{E}(\phi(x, y)) = \exp\left(\sum_{m=1}^{\infty} \frac{1}{m} \phi\big(x^m, (-1)^{m+1} y^m\big) \right);$$

$$\mathcal{E}\left(\sum_{i,j\ge 0;\ i+j>0} b_{ij} x^i y^j \right) = \prod_{i,j} \big(1 - (-1)^j x^i y^j\big)^{(-1)^{j+1} b_{ij}}.$$

These and other formulas proved to be useful earlier and allowed to obtain dimension formulas for free Lie superalgebras and study invariants of finite groups acting on free Lie superalgebras [20, 21, 22], see also similar formulas in [9].

Having the explicit formula for $\mathcal{H}(L, x, y)$, we obtain $\mathcal{H}(L, t) = \mathcal{H}(L, x, y)|_{x=t, y=t}$ and we proceed further similar to the case of Lie algebras [19] and using different asymptotic methods [24, 25].

In order to prove main results we study the growth for universal enveloping algebras of Lie superalgebras [10]. Also, some special bases for free Lie superalgebras are constructed [10].

4. Growth of almost solvable Lie algebras

We extend our results on the growth of solvable Lie algebras to *almost solvable* Lie algebras. A Lie algebra is called almost solvable if it has a solvable subalgebra of finite codimension.

Theorem 4.1 ([11]). *Let L be a finitely generated Lie algebra such that there exists a subalgebra $H \subseteq L$ of finite codimension and such that H is solvable of length q. Then L is of subexponential growth and*

$$\mathrm{Dim}^{q+1} L \leq \dim_K(L/H).$$

By Corollary 1.1 this upper bound is exact. Indeed, we consider $L = F(\mathbf{A}^{q+1}, k)$ and its commutator subalgebra $H = L^2$. Then $\dim_K L/H = k$, H is solvable of step q, and by Corollary 1.1, we have $\mathrm{Dim}^{q+1} L = k$. On the other hand, there are no lower bounds because L can be finite dimensional.

We say that a set $X = \cup_{m=1}^{\infty} X_m$, where $|X_m| < \infty$, $m \in \mathbb{N}$, is *graded*. We extend our definitions for algebras generated by such sets, where we assume that the elements of X_m have *weight* m in computations of all growth functions and series. We define $\mathcal{H}(X,t) = \sum_{n=1}^{\infty} |X_n|\, t^n$. The next result is a particular case of Theorem 3.1.

Theorem 4.2 ([23]). *Let $L = F(\mathbf{A}^q, Y)$ be the free solvable Lie algebra of length q, generated by a graded set Y. Denote $g_1(t) = \mathcal{H}(Y,t)$ and $g_i(t) = g_{i-1}(t) + 1 + (\mathcal{H}(Y,t) - 1)\mathcal{E}(g_{i-1}(t))$ for $i = 2, \dots, q$. Then $\mathcal{H}_Y(L,t) = g_q(t)$.*

We show how series are used to prove Theorem 4.1. Let L be generated by $Z = \{z_1, \dots, z_k\}$. By assumption, $\dim_K L/H = N < \infty$ and $H^{(q)} = 0$. Let $F = F(X)$ be the free Lie algebra generated by $X = \{x_1, \dots, x_k\}$. We have an epimorphism $\phi : F \to L$, $\phi(x_i) = z_i$ for $i = 1, \dots, k$. Denote $D = \phi^{-1}(H)$ and $G = \mathrm{Ker}\, \phi$. Then $\dim_K F/D = N$, $F/G \cong L$ and $D^{(q)} \subseteq G$.

We consider the filtration $F^1 \subseteq F^2 \subseteq \cdots$, given by the degree in X. For any $V \subseteq F$ let $\mathrm{gr}\, V = \oplus_{n=1}^{\infty} \mathrm{gr}_n V \subseteq F$ denote the associated graded space. The subalgebra $\mathrm{gr}\, D \subseteq F$ is free by Shirshov-Witt theorem [1]. Let $\bar{Y} = \cup_{n=1}^{\infty} \bar{Y}_n$ be a homogeneous generating set for $\mathrm{gr}\, D$, where $\bar{Y}_n \subset \mathrm{gr}_n F$. Let $\mathcal{H}_X(\bar{Y}, t) = \sum_{n=1}^{\infty} |\bar{Y}_n| t^n$, then we have an analogue of Schreier's formula [23]:

$$\mathcal{H}_X(\bar{Y}, t) - 1 = (kt - 1)\, \mathcal{E}(\mathcal{H}_X(F/\mathrm{gr}\, D, t)), \quad \text{where} \tag{2}$$

$\mathcal{H}_X(F/\mathrm{gr}\, D, t) = \sum_{n=1}^{\infty} c_n t^n$ is a polynomial, because $\sum_n c_n = \dim F/\mathrm{gr}\, D = \dim F/D = N$. We consider the chain of subspaces $F \supseteq D \supseteq G \supseteq D^{(q)}$. We observe that $\mathrm{gr}\, D/D^{(q)} \cong F(\mathbf{A}^q, \bar{Y})$. Suppose that $\mathcal{H}(F(\mathbf{A}^q, \bar{Y}), t) = \sum_{n=1}^{\infty} d_n t^n$, denote $\bar{d}_n = d_1 + \cdots + d_n$ for $n \in \mathbb{N}$. We get the following bound on the growth of L: $\gamma_L(n) \leq N + \bar{d}_n$, $n \in \mathbb{N}$.

We apply Theorem 4.2 and (2) to algebra $F(\mathbf{A}^q, \bar{Y})$.

$$g_1(t) = \mathcal{H}_X(\bar{Y}, t) = 1 + (kt - 1) \prod_{n \geq 1} \frac{1}{(1 - t^n)^{c_n}} = \frac{A + o(1)}{(1-t)^N}, \quad t \to 1 - o;$$
$$g_i(t) \leq g_{i-1}(t) + g_1(t) \cdot \mathcal{E}(g_{i-1}(t)), \quad 0 \leq t < 1; \qquad i = 2, \dots, q.$$

We apply facts on the growth of functions analytic in the unit circle [19] and obtain the following asymptotics

$$g_p(t) = \exp^{(p-1)}\left(\frac{A\zeta(N+1) + o(1)}{(1-t)^N} \right), \quad t \to 1 - o; \quad p = 2, \dots, q.$$

By Theorem 4.2, $\mathcal{H}(F(\mathbf{A}^q, \bar{Y}), t) = g_q(t)$. We use this asympotics along with properties of functions analytic in the unit circle [19] and conclude that $\mathrm{Dim}^{q+1} L \leq N$.

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