The Bruhat ordering on the Coxeter group of type $\widetilde{C}_n$

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Abstract

This paper deals with two topics about the Bruhat ordering on the Coxeter groups.

The first topic is a general theory concerning with Coxeter graph automorphisms.A Coxeter graph automorphism σ of a Coxeter system (W,S) extends to a group automorphism on W, and gives rise to a subgroup W_σ of W consisting of all elements fixed by σ. Then we prove that the Bruhat ordering on W_σ is the restriction of the Bruhat ordering on W. We study a relation between the Bruhat ordering and a Coxeter graph automorphism.

The second topic is an application of the above theory. By applying this theory to a Coxeter group $W(\widetilde{A}_{2n-1})$ of type $\widetilde{A}_{2n-1}$ and a combinatorial description of the Bruhat ordering on it due to Björner–Brenti and Lascoux, we give a combinatorial description of the Bruhat ordering on $W(\widetilde{C}_n)$ of type $\widetilde{C}_n$.

1 Introduction

Let (W,S) be a Coxeter system, and ℓ be the length function on W. Let $T=\{wsw^{-1} \mid w \in W,\ s \in S\}$ be the set of reflections. The Bruhat ordering on W is a partial ordering on W defined as follows:

For $w, w' \in W$, we define $w \le w'$ if there exist elements $t_1, \cdots, t_r \in T$ such that

(1) $w' = t_r \cdots t_1 w$,

(2) $\ell(t_i \cdots t_1 w) \ge \ell(t_{i-1} \cdots t_1 w) \quad (1 \le \forall i \le r)$.

A Coxeter graph automorphism of (W,S) is a bijection from S to itself such that

$$m(\sigma(s),\sigma(s')) = m(s,s') \qquad (\forall s, s' \in S),$$

where $m(s,s')$ is the order of the product ss' for $s,s' \in S$. Such σ extends to an automorphism on W, which is called *a Coxeter graph automorphism of* (W,S).

In this paper, we are interested in the fixed-point subgroup

$$W_\sigma = \{w \in W \mid \sigma(w) = w\}$$

of W. R. Steinberg proved that W_σ is a Coxeter group. More precisely,

Theorem 1.1 (R. Steinberg [12])
Let (W,S) be a Coxeter system, and let σ be a Coxeter graph automorphism of (W,S). If we put

$$S_\sigma = \{w_X \mid X \text{ is a } \langle\sigma\rangle\text{-orbit in } S \text{ with } W_X = \langle X\rangle \text{ finite}\},$$

where w_X is the longest element in W_X, then (W_σ, S_σ) is a Coxeter system. □

From Theorem 1.1, W_σ has its own Bruhat ordering with respect to S_σ. The author clarifies the relation between the Bruhat ordering on W_σ and that on W. The following is the one of our main results:

Theorem 1.2 **(M. Nanba [10])**
In the setting of Theorem 1.1, the Bruhat ordering on (W_σ, S_σ) is the restriction of the Bruhat ordering on (W,S) to W_σ. That is, if $\leq$ is the Bruhat ordering on W with respect to S, and $\leq_\sigma$ is the Bruhat ordering on W_σ with respect to S_σ, then

$$w \leq w' \iff w \leq_\sigma w'$$

for all $w, w' \in W_\sigma$.

The outline of the proof of this theorem is explained in Section 2.2.

As application of this theorem, we give a combinatorial description of the Bruhat ordering on the Coxeter group $W(\widetilde{C}_n)$ of type $\widetilde{C}_n$, which is embedded into the Coxeter group $W(\widetilde{A}_{2n-1})$ of type $\widetilde{A}_{2n-1}$.

Let $W(\widetilde{A}_{N-1})$ be the group of affine permutations, i.e., an element π in $W(\widetilde{A}_{N-1})$ is a bijection from $\mathbb{Z}$ to $\mathbb{Z}$ such that

(1) $\pi(x) + N = \pi(x+N)$ for all $x \in \mathbb{Z}$,

(2) $\pi(1) + \pi(2) + \cdots + \pi(N) = \dfrac{1}{2}N(N+1)$.

This group $W(\widetilde{A}_{N-1})$ is known to be *a Coxeter group of type* $\widetilde{A}_{N-1}$. ([8].) Note that an affine permutation π is determined by its values $\pi(1), \pi(2), \cdots, \pi(N)$. Moreover, it is obvious from (2) that an affine permutations π has the following property:
For $i, j \in \mathbb{Z}$,

$$\pi(i) \equiv \pi(j) \pmod N \iff i \equiv j \pmod N. \tag{1.1}$$

In order to describe the Bruhat ordering on $W(\widetilde{A}_{N-1})$, we need investigate the Bruhat ordering on the following subset:

$$W(\widetilde{A}_{N-1})^{J_0} = \{\pi \in W(\widetilde{A}_{N-1}) \mid \pi(1) < \pi(2) < \cdots < \pi(N)\} \ \subset W(\widetilde{A}_{N-1}).$$

This subset $W(\widetilde{A}_{N-1})^{J_0}$ is the set of distinguished coset representatives with respect to a maximal parabolic subgroup. Moreover, the elements in $W(\widetilde{A}_{N-1})^{J_0}$ are in one-to-one correspondence with N-cores. In particular, this encoding enables us to describe combinatorially the Bruhat ordering on $W(\widetilde{A}_{N-1})^{J_0}$. (See Theorem 4.5.)

Let $N = 2n$ be an even integer and put

$$W(\widetilde{C}_n) = \{w \in W(\widetilde{A}_{2n-1}) \mid w(i) + w(2n+1-i) = 2n+1 \ (\forall i \in \mathbb{Z})\}.$$

This subgroup $W(\widetilde{C}_n)$ of $W(\widetilde{A}_{2n-1})$ is the fixed-point subgroup under a certain Coxeter graph automorphism, and it it is the Coxeter group of type $\widetilde{C}_n$. (See Theorem 3.3.) Hence we can apply the above Theorem 1.2 to obtain a combinatorial description of the Bruhat ordering on it.
If we put

$$W(\widetilde{C}_n)^{I_0} = \{w \in W(\widetilde{C}_n) \mid w(1) < w(2) < \cdots < w(2n)\} = W(\widetilde{C}_n) \cap W(\widetilde{A}_{2n-1})^{J_0},$$

then $W(\widetilde{C}_n)^{I_0}$ is the set of a distinguished coset representatives of $W(\widetilde{C}_n)$ with respect to a maximal parabolic subgroup and has also the encoding with $2n$-core. Another main result is the following theorem. (See Theorem 5.3.)

Theorem 1.3 *The elements of $W(\widetilde{C}_n)^{I_0}$ are in one-to-one correspondence with symmetric $2n$-cores. Moreover, if w and $v \in W(\widetilde{C}_n)$ correspond to symmetric $2n$-cores λ and μ respectively, then $w \leq v$ if and only if $\lambda \subseteq \mu$.* □

This paper is organized as follows.

Section 2: In this section, we review some facts about Coxeter groups and the outline of the proof of Theorem 1.2.

Section 3: We consider two Coxeter groups: $W(\widetilde{A}_{N-1})$ of type $\widetilde{A}_{N-1}$ and $W(\widetilde{C}_n)$ of type $\widetilde{C}_n$.

Section 4: We review Lascoux's encoding of the Bruhat ordering on $W(\widetilde{A}_{N-1})^{J_0}$.

Section 5: In Section 5, we give the combinatorial description of the Bruhat ordering on $W(\widetilde{C}_n)^{I_0}$ by using Lascoux's description of that on $W(\widetilde{A}_{2n-1})$ and Theorem 1.2.

Section 6: By Theorem 1.3 and a certain Coxeter graph automorphism ω, we find the combinatorial description of the Bruhat ordering on the whole group $W(\widetilde{C}_n)$.

2 General theory

In this section, we review some key facts on Coxeter groups and give an outline of the proof of Theorem 1.1 and 1.2. Throughout this section, let (W, S) be a Coxeter system and σ a Coxeter graph automorphism, otherwise stated.

2.1 Key facts

Since a Coxeter graph automorphism σ is a bijection an S, we have

$$\ell(\sigma(w)) = \ell(w) \qquad (\text{for all } w \in W), \tag{2.1}$$
$$\sigma(T) = T. \tag{2.2}$$

Hence it follows from the definition of Bruhat ordering that σ is order-preserving automorphism of W.

Proposition 2.1 *Let (W, S) be a Coxeter system, and σ be a Coxeter graph automorphism of (W, S). Let $\leq$ be the Bruhat ordering on W, then for $w, w' \in W$,*

$$w \leq w' \iff \sigma(w) \leq \sigma(w').$$

□

For a subset $J \subseteq S$, let $W_J = \langle J \rangle$ be the parabolic subgroup and W^J be the set of *distinguished (left) coset representatives*. For each $w \in W$, we can find a unique pair of elements $w^J \in W^J$ and $w_J \in W_J$ such that $w = w^J w_J$ and $\ell(w) = \ell(w^J) + \ell(w_J)$. (See J. E. Humphreys [5]) In this decomposition, $w^J \in W^J$ is called the *W^J-part* of w. The equation (2.1) can also leads the following proposition:

Proposition 2.2 *For a Coxeter system (W, S), let σ be a Coxeter graph automorphism of (W, S). For all $J \subseteq S$ and $w \in W$, $\sigma(w^J) = (\sigma(w))^{\sigma(J)}$.* □

We define the Bruhat ordering on W^J by restricting the Bruhat ordering on W to W^J. The following theorem can be deduced immediately from [4]:

Theorem 2.3 (See V. V. Deodhar [4])
Let (W, S) be a Coxeter system with the identity element e and J be a subset of S. Let $\leq$ be a relation on W^J. Then the following are equivalent:

(1) *The relation $\leq$ is the Bruhat ordering on W^J.*

(2) *The relation $\leq$ satisfies (a) and (b) as follows:*

  (a) *$w \leq e$ if and only if $w = e$.*

  (b) *The following is satisfied :*

Property $Z^J(s, w_1, w_2)$: For $w_1, w_2 \in W^J$ and $s \in S$ such that $\ell(sw_2) \leq \ell(w_2)$ and $\ell((sw_1)^J) \leq \ell(w_1)$, one has $w_1 \leq w_2 \Leftrightarrow (sw_1)^J \leq w_2 \Leftrightarrow (sw_1)^J \leq (sw_2)^J$.

□

In particular, if $J = \emptyset$, then the Property $Z^J(s, w_1, w_2)$ is the same as the Property $Z(s, w_1, w_2)$ which appeared in [4].

2.2 Outline of the proof of Theorem 1.1 and 1.2

R. Steinberg proved Theorem 1.1 by using a root system. Here we give another proof to Theorem 1.1 based on the Exchange Condition:

Theorem 2.4 (See N. Bourbaki [3].) *Let W be a group, and S be a subset of W such that S generates W and $m(s, s) = 1$ for all $s \in S$. Then the pair (W, S) is a Coxeter system if and only if (W, S) satisfies the Exchange Condition (EC) :*

(EC) Let $w \in W$ and $w = s_1 \cdots s_q$ be an arbitrary expression. If $\ell(sw) \leq \ell(w)$ for $s \in S$, then there exists j with $1 \leq j \leq q$ such that $sw = s_1 \cdots s_{j-1}s_{j+1} \cdots s_q$. □

The following lemma is obtained from the Exchange Condition and (2.1):

Lemma 2.5 *For a Coxeter system (W, S) with a Coxeter group automorphism σ, we have*

(1) *Let $X \subset S$ be a $\langle\sigma\rangle$-orbit. Then, for all $w \in W_\sigma$,*

$$\exists s \in X \text{ s.t. } \ell(sw) \leq \ell(w) \iff \ell(s'w) \leq \ell(w) \ (\forall s' \in X),$$
$$\exists s \in X \text{ s.t. } \ell(sw) \geq \ell(w) \iff \ell(s'w) \geq \ell(w) \ (\forall s' \in X)$$

(2) *For each $w \in W_\sigma$, there exist $w_{X_1}, w_{X_2}, \cdots, w_{X_r} \in S_\sigma$ such that $w = w_{X_1}w_{X_2}\cdots w_{X_r}$ and $\ell(w) = \ell(w_{X_1}) + \ell(w_{X_2}) + \cdots + \ell(w_{X_r})$.* □

The second part of this lemma shows that S_σ generates W_σ. Therefore we can define the length function ℓ_σ on W_σ with respect to S_σ. We have the following proposition from the Exchange Condition:

Proposition 2.6 *Suppose that $w \in W_\sigma$ has an expression $w = w_{X_1}w_{X_2}\cdots w_{X_r}$ as a product of $w_{X_1}, w_{X_2}, \cdots, w_{X_r} \in S_\sigma$. Let X be a subset $\langle\sigma\rangle$-orbit with W_X finite. If $\ell(sw) \leq \ell(w)$ for an element $s \in X$, there exists an integer k with $1 \leq k \leq r$ such that*

$$w_X w = w_{X_1} \cdots w_{X_{k-1}} w_{X_{k+1}} \cdots w_{X_r}.$$

. □

Corollary 2.7 *For $w_X \in S_\sigma$ and $w \in W_\sigma$,*

(1) *$w = w_{X_1}w_{X_2}\cdots w_{X_r}$ is reduced if and only if $\ell(w) = \ell(w_{X_1}) + \ell_\sigma(w_{X_2}) + \cdots + \ell_\sigma(w_{X_r})$.*

(2) *$\ell_\sigma(w_X w) \leq \ell_\sigma(w) \iff \exists s \in X$ s.t. $\ell(sw) \leq \ell(w)$. Then $\ell(w_X w) = \ell(w) - \ell(w_X)$.*

(3) *$\ell_\sigma(w_X w) \geq \ell_\sigma(w) \iff \exists s \in X$ s.t. $\ell(sw) \geq \ell(w)$. Then $\ell(w_X w) = \ell(w) + \ell(w_X)$.* □

Now we are in position to give proofs of Theorem 1.1 and 1.2.

Proof of Theorem 1.1. It follows from Proposition 2.6 and Corollary 2.7 that (W_σ, S_σ) satisfies the Exchange Condition. □

Proof of Theorem 1.2. It is enough to show that the restriction to W_σ of the Bruhat order $\leq$ on W satisfies the condition (a) and (b) in Theorem 2.3 (2) for $J = \emptyset$.

It is obvious that the restriction $\leq$ satisfies Theorem 2.3 (2) (a).

To prove the condition (b), take elements $w_X \in S_\sigma$ and $w_1, w_2 \in W_\sigma$ satisfying $\ell_\sigma(w_X w_1) \leq \ell_\sigma(w_1)$ and $\ell_\sigma(w_X w_2) \leq \ell_\sigma(w_2)$. Let $w_X = s_k \cdots s_1$ be a reduced expression. Then $s_1, s_2, \cdots, s_k \in X$. Since

$\ell_\sigma(w_X w_1) \leq \ell_\sigma(w_1)$, we see that $\ell(w_X w_1) = \ell(w_1) - \ell(w_X)$ by Corollary 2.7 (2). Thus $\ell(s_i \cdots s_1 w_1) \leq \ell(s_{i-1} \cdots s_1 w_1)$ for $1 \leq i \leq k$. Similarly, $\ell(s_i \cdots s_1 w_2) \leq \ell(s_{i-1} \cdots s_1 w_2)$ for $1 \leq i \leq k$. In particular, since X is a $\langle\sigma\rangle$-orbit, $\ell(s_1 w_2) \leq \ell(w_2)$ and Lemma 2.5 imply that $\ell(s_i w_2) \leq \ell(w_2)$ for $1 \leq i \leq k$. Now, using induction, we show that, for $1 \leq i \leq k$, the following condition $(**_i)$ holds:

$$w_1 \leq w_2 \quad \Leftrightarrow \quad s_i \cdots s_1 w_1 \leq w_2 \quad \Leftrightarrow \quad s_i \cdots s_1 w_1 \leq s_i \cdots s_1 w_2 \tag{$**_i$}$$

If $i = 1$, then $\ell(s_1 w_1) \leq \ell(w_1)$, $\ell(s_1 w_2) \leq \ell(w_2)$ and Property $Z(s_1, w_1, w_2)$ imply that the condition $(**_1)$ holds.

Suppose that $i > 1$ and the condition $(**_{i-1})$ holds. Since $\ell(s_i s_{i-1} \cdots s_1 w_1) \leq \ell(s_{i-1} \cdots s_1 w_1)$ and $\ell(s_i w_2) \leq \ell(w_2)$, Property $Z(s_i, s_{i-1} \cdots s_1 w_1, w_2)$ implies that $s_{i-1} \cdots s_1 w_1 \leq w_2$ is equivalent to $s_i \cdots s_1 w_1 \leq w_2$. Thus $w_1 \leq w_2$ is equivalent to $s_i \cdots s_1 w_1 \leq w_2$. And since $\ell(s_i \cdots s_1 w_1) \leq \ell(s_{i-1} \cdots s_1 w_1)$ and $\ell(s_i \cdots s_1 w_2) \leq \ell(s_{i-1} \cdots s_1 w_2)$, Property $Z(s_i, s_{i-1} \cdots s_1 w_1, s_{i-1} \cdots s_1 w_2)$ implies that $s_{i-1} \cdots s_1 w_1 \leq s_{i-1} \cdots s_1 w_2$ is equivalent to $s_i \cdots s_1 w_1 \leq s_i \cdots s_1 w_2$. Thus $w_1 \leq w_2$ is equivalent to $s_i \cdots s_1 w_1 \leq s_i \cdots s_1 w_2$. Thus the condition $(**_i)$ holds.

It is follows from the condition $(**_k)$ that the restriction satisfies the condition (b) in Theorem 2.3 (2) for $J = \emptyset$.

□

3 A Coxeter group of type $\widetilde{C}_n$

3.1 Affine permutation of Type A_{N-1}

Let $W(\widetilde{A}_{N-1})$ be the group of affine permutations introduced in Section 1. Recall that an affine permutation π is determined by its values $\pi(1), \pi(2), \cdots, \pi(n)$. Therefore we denote $\pi \in W(\widetilde{A}_{N-1})$ by writing $\pi = [\pi(1), \cdots \pi(N)]$, and this notation of affine permutations is called the *window*.

For each integer i with $0 \leq i \leq N-1$, let s_i be the affine permutation defined by

$$s_i(j) = \begin{cases} j+1 & \text{if } j \equiv i \pmod N, \\ j-1 & \text{if } j \equiv i+1 \pmod N, \\ j & \text{otherwise.} \end{cases}$$

Put $S(\widetilde{A}_{N-1}) = \{s_0, s_1, \cdots, s_{N-1}\}$. Then G. Lusztig proved, in [8], that $(W(\widetilde{A}_{N-1}), S(\widetilde{A}_{N-1}))$ is a Coxeter system of type $\widetilde{A}_{N-1}$ with the Coxeter graph in Figure 1.

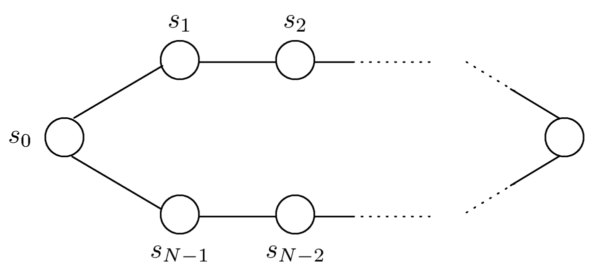

Figure 1: The Coxeter graph of type $\widetilde{A}_{N-1}$.

In the case of $(W(\widetilde{A}_{N-1}), S(\widetilde{A}_{N-1}))$, the length function ℓ is given by [2]:

$$\ell(\pi) = \sum_{1 \leq i < j \leq N} \left| \left\lfloor \frac{\pi(j) - \pi(i)}{N} \right\rfloor \right|,$$

where, for a rational number x, $\lfloor x \rfloor$ is the largest integer not exceeding x. Therefore *the descent set* $D(\pi)$ of π is given by the following:

$$D(\pi) \stackrel{\text{def}}{=} \{s \in S(\widetilde{A}_{N-1}) \mid \ell(\pi s) \leq \ell(\pi)\} = \{s_i \in S(\widetilde{A}_{N-1}) \mid \pi(i) > \pi(i+1)\} \tag{3.1}$$

Let $J_0 = S(\widetilde{A}_{N-1}) \setminus \{s_0\}$. By (3.1), $W(\widetilde{A}_{N-1})^{J_0}$ is a subset of $W(\widetilde{A}_{N-1})$ that consists of the elements π satisfying

$$\pi(1) < \pi(2) < \cdots < \pi(N). \tag{3.2}$$

For each $w \in W(\widetilde{A}_{N-1})_{J_0}$, we have

$$\{w(1), w(2), \cdots, w(N)\} = \{1, 2, \cdots, N\}.$$

Thus, given $\pi = [\pi(1), \cdots, \pi(N)] \in W(\widetilde{A}_{N-1})$, the window of πw can be written by permuting $\pi(1), \pi(2), \cdots, \pi(N)$. From (3.2), this implies that the window of π^{J_0} is given as follows:

Lemma 3.1 *For $\pi = [\pi(1), \cdots, \pi(N)] \in W(\widetilde{A}_{N-1})$, the window of π^{J_0} is given by sorting $\pi(1), \cdots, \pi(N)$ in increasing order.* □

3.2 $W(\widetilde{C}_n)$ as a subgroup of $W(\widetilde{A}_{2n-1})$

Let $N = 2n$ and we take a Coxeter group of $W(\widetilde{A}_{2n-1})$.

Let σ be a Coxeter graph automorphism of $(W(\widetilde{A}_{2n-1}), S(\widetilde{A}_{2n-1}))$ given by

$$\sigma(s_i) = \begin{cases} s_0 & i = 0, \\ s_{2n-i} & 1 \leq i \leq 2n-1. \end{cases} \tag{3.3}$$

This action of σ on $S(\widetilde{A}_{2n-1})$ is described as Figure 2.

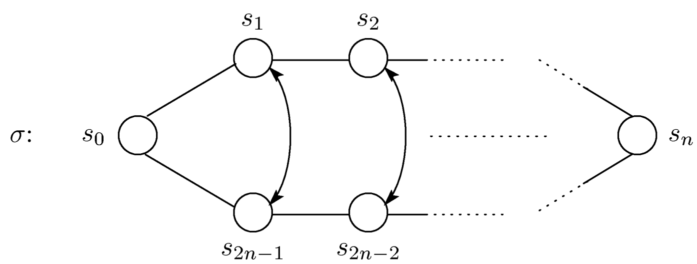

Figure 2: The action of σ on $S(\widetilde{A}_{2n-1})$.

Then σ extends to an automorphism on $W(\widetilde{A}_{2n-1})$ as follows:

Proposition 3.2 *For each $\pi \in W(\widetilde{A}_{2n-1})$, this Coxeter graph automorphism σ defined as* (3.3) *maps $\pi = [\pi(1), \pi(2), \cdots, \pi(2n+1)]$ to the following element:*

$$\sigma(\pi) = [2n+1-\pi(2n), 2n+1-\pi(2n-1), \cdots, 2n+1-\pi(1)]. \tag{3.4}$$

Proof. We can prove this proposition by using induction on $r = \ell(\pi)$. □

For this automorphism σ, Proposition 3.2 implies that

$$\begin{aligned} W(\widetilde{A}_{2n-1})_\sigma &= \{\pi \in W(\widetilde{A}_{2n-1}) \mid \sigma(\pi) = \pi\} \\ &= \{\pi \in W(\widetilde{A}_{2n-1}) | \ \pi(i) + \pi(2n+1-i) = 2n+1 \ (\forall i \in \mathbb{Z})\} \\ &= W(\widetilde{C}_n). \end{aligned}$$

We set $S(\widetilde{C}_n) = \{w_{X_0}, w_{X_1}, w_{X_2}, \cdots, w_{X_{n-1}}, w_{X_n}\}$, where

$$w_{X_i} = \begin{cases} s_0 & i = 0, \\ s_i s_{2n-i} & 1 \leq i \leq n-1, \\ s_n & i = n. \end{cases}$$

Then Theorem 1.1 and Theorem 1.2 imply that we obtain the following result:

Theorem 3.3 *$(W(\widetilde{C}_n), S(\widetilde{C}_n))$ is a Coxeter system of type $\widetilde{C}_n$. And the Bruhat ordering on $W(\widetilde{C}_n)$ is the restriction of the Bruhat ordering on $W(\widetilde{A}_{2n-1})$ to $W(\widetilde{C}_n)$.*

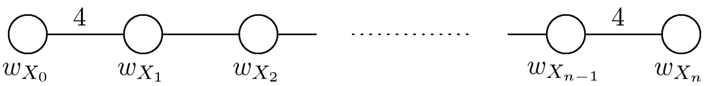

Figure 3: The Coxeter graph of type $\widetilde{C}_n$.

Proof. It remains to show that type of $(W(\widetilde{C}_n), S(\widetilde{C}_n))$ is $\widetilde{C}_n$ and it is easy. □

Let $I_0 = S(\widetilde{C}_n) \setminus \{w_{X_0}\}$. Then Lemma 2.5 leads to

$$\begin{aligned} w \in W(\widetilde{C}_n)^{I_0} &\iff \ell_\sigma(ww_{X_i}) \geq \ell_\sigma(w) \quad \text{for all } i = 1, 2, \cdots, n. \\ &\iff \ell(ws_j) \geq \ell(w) \text{ for all } j = 1, 2, \cdots, 2n+1. \\ &\iff w \in W(\widetilde{A}_{2n-1})^{J_0} \end{aligned}$$

for $w \in W(\widetilde{C}_n)$. Thus we have the following lemma:

Lemma 3.4

$$W(\widetilde{C}_n)^{I_0} = W(\widetilde{C}_n) \cap W(\widetilde{A}_{2n-1})^{J_0}.$$

That is,

$$W(\widetilde{C}_n)^{I_0} = \{w \in W(\widetilde{C}_n) \mid w(1) < w(2) < \cdots < w(2n)\}.$$

□

4 The Bruhat ordering on $W(\widetilde{A}_{N-1})^{J_0}$

By Theorem 3.3 and Lemma 3.4, the Bruhat ordering on $W(\widetilde{C}_n)^{I_0}$ is closely related to the Bruhat ordering on $W(\widetilde{A}_{2n-1})^{J_0}$ in a certain sense. A. Lascoux, in [7], described the Bruhat ordering on $W(\widetilde{A}_{N-1})^{J_0}$ by N-core encoding. In this section, we review Lascoux's encoding of $W(\widetilde{A}_{N-1})^{J_0}$ and give a description of the Bruhat ordering on it.

By Lemma 3.1, $W(\widetilde{A}_{N-1})^{J_0}$ consists of elements $\pi \in W(\widetilde{A}_{N-1})$ satisfying

$$\pi(1) < \pi(2) < \cdots < \pi(N).$$

Let λ be a Young diagram. To each box (i, j) in λ, we associate an integer p (called color) defined by

$$p \equiv j - i \pmod N \qquad 0 \leq p \leq N-1.$$

We introduce the notation of addable or removable i-corner of a Young diagram as follows.

(a) A box in λ is called a *removable corner* if we get another Young diagram by deleting it from λ. If a removable corner of λ has the color i, we call it a *removable i-corner* of λ.

(b) A box in λ is called an *addable corner* if we get another Young diagram by adding it to λ. If an addable corner of λ has the color i, we call it an *addable i-corner.*

Example 4.1 *If $N = 5$ and $\lambda = (4, 3, 1, 1, 1)$, then the coloring for λ and removable/addable corners are given by* Figure 4*.*

If a diagram λ has no hook whose length is a multiple of N, then λ is called an *N-core*. For example, $\lambda = (4, 3, 1, 1, 1)$ is a 5-core. From the definition, λ is an N-core if and only if λ satisfies the condition $(C_{N\text{-core}})$.

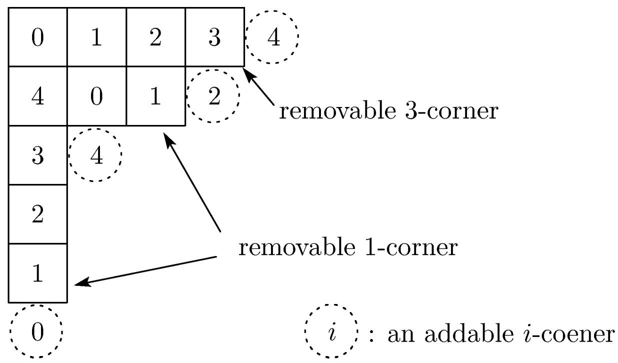

Figure 4: The coloring for $\lambda = (4,3,1,1,1)$ ($N = 5$), removable corners and addable corners.

<u>$(C_{N\text{-}\mathbf{core}})$</u> For each integer i with $0 \le i \le N-1$, λ can not have both removable i-corners and addable i-corners.

That is,

(1) If λ has a removable i-corner, then λ can not have any addable i-corners.

(2) If λ has an addable i-corner, then λ can not have any removable i-corners.

Let $\mathcal{C}[N]$ be the set of all N-cores. We define the action of $W(\widetilde{A}_{N-1})$ to $\mathcal{C}[N]$ as follows:

First, for an N-core λ and $s_i \in S(\widetilde{A}_{N-1})$, we define a new diagram $s_i \cdot \lambda$ as follows:

- If λ has removable i-corners, then $s_i \cdot \lambda$ is the diagram obtained by removing all of them from λ.
- If λ has addable i-corners, then $s_i \cdot \lambda$ is the diagram obtained by adding boxes to the corners of λ.
- If λ doesn't have either removable or addable i-corners, then $s_i \cdot \lambda = \lambda$.

Then it is obvious from the condition $C_{N\text{-core}}$ that $s_i \cdot \lambda$ is an N-core if λ is an N-core. Moreover, we can easily deduce that

$$s_i \cdot (s_i \cdot \lambda) = \lambda \qquad (0 \le \forall i \le N-1), \tag{4.1}$$

$$s_i \cdot (s_j \cdot \lambda) = s_j \cdot (s_i \cdot \lambda), \qquad \text{if } |i-j| \ge 2, \tag{4.2}$$

and

$$s_i \cdot (s_{i+1} \cdot (s_i \cdot \lambda)) = s_{i+1} \cdot (s_i \cdot (s_{i+1} \cdot \lambda)) \qquad (0 \le \forall i \le N-1), \tag{4.3}$$

where we set $s_N = s_0$.

For general $\pi \in W(\widetilde{A}_{N-1})$, we take an expression $\pi = s_{i_1} s_{i_2} \cdots s_{i_r}$ with respect to $S(\widetilde{A}_{N-1})$, and define

$$\pi \cdot \lambda = s_{i_1} \cdot (s_{i_2} \cdot (\cdots (s_{i_r} \cdot \lambda) \cdots)).$$

This action of $W(\widetilde{A}_{N-1})$ on $\mathcal{C}[N]$ is well-defined from (4.1), (4.2) and (4.3).

Example 4.2 *For $N = 5$ and 5-core $\lambda = (4, 3, 1, 1, 1)$,*

$$\begin{array}{cccc} 0 & 1 & 2 & 3 \\ 4 & 0 & 1 & \\ 3 & & & \\ 2 & & & \\ 1 & & & \end{array} \xrightarrow{s_4} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \\ 4 & 0 & 1 & & \\ 3 & 4 & & & \\ 2 & & & & \\ 1 & & & & \end{array} \xrightarrow{s_3} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \\ 4 & 0 & 1 & & \\ 3 & 4 & & & \\ 2 & 3 & & & \\ 1 & & & & \end{array} \xrightarrow{s_1} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \\ 4 & 0 & & & \\ 3 & 4 & & & \\ 2 & 3 & & & \end{array}$$

Therefore, for $\pi = s_1 s_3 s_4 = [2, 1, 4, 5, 3]$, we have $\pi \cdot \lambda = (5, 2, 2, 2)$.

This action induces the map $\mathcal{C}_0 : W(\widetilde{A}_{N-1}) \to \mathcal{C}[N]$ as follows:

$$\mathcal{C}_0(\pi) = \pi \cdot \emptyset \in \mathcal{C}[N]$$

where the diagram $\emptyset$ is the empty diagram, i.e., the partition of 0. By the definition of $\mathcal{C}_0$, it is obvious that the map $\mathcal{C}_0$ is compatible with the action of $W(\widetilde{A}_{N-1})$. That is, for $\tau, \pi \in W(\widetilde{A}_{N-1})$,

$$\tau \cdot \mathcal{C}_0(\pi) = \mathcal{C}_0(\tau\pi). \tag{4.4}$$

Then the equation (4.4) leads to the following properties:

Proposition 4.3 (1) *This map $\mathcal{C}_0$ is a surjection from $W(\widetilde{A}_{N-1})$ to $\mathcal{C}[N]$.*

(2) *For $\pi \in W(\widetilde{A}_{N-1})$, π fixes $\emptyset$ if and only if $\pi \in W(\widetilde{A}_{N-1})_{J_0}$. Moreover, for $\pi \in W(\widetilde{A}_{N-1})$, we have*

$$\mathcal{C}_0(\pi) = \pi^{J_0} \cdot \emptyset = \mathcal{C}_0(\pi^{J_0}).$$

Proof. The statement (1) is clear. So it is sufficient to show (2). As for the proof of (2), we need some arguments, see G. James–A. Kerber [6] or M. Nanba [11]. □

Corollary 4.4 *The restriction of $\mathcal{C}_0$ to $W(\widetilde{A}_{N-1})^{J_0}$ is a bijection from $W(\widetilde{A}_{N-1})^{J_0}$ to $\mathcal{C}[N]$.* □

For this bijection, A. Lascoux deduced the following result:

Theorem 4.5 (A. Lascoux [7])
For $\pi, \tau \in W(\widetilde{A}_{N-1})^{J_0}$,

$$\pi \leq \tau \iff \mathcal{C}_0(\pi) \subseteq \mathcal{C}_0(\tau).$$

□

A. Lascoux proved this theorem by using Theorem 2.3 in Section 2.1 for $J = J_0$.

5 The Bruhat ordering on $W(\widetilde{C}_n)^{I_0}$

By Theorem 3.3 and Lemma 3.4, $W(\widetilde{C}_n)^{I_0}$ is the intersection of a Coxeter group $W(\widetilde{C}_n)$ $(= W(\widetilde{A}_{2n-1})_\sigma)$ and $W(\widetilde{A}_{2n-1})^{J_0}$. Then Theorem 1.2 implies that the Bruhat ordering on $W(\widetilde{C}_n)^{I_0}$ is the restriction of the Bruhat ordering on $W(\widetilde{A}_{2n-1})^{J_0}$ to $W(\widetilde{C}_n)^{I_0}$. Therefore the Bruhat ordering on $W(\widetilde{C}_n)^{I_0}$ can be described by using the map $\mathcal{C}_0$ on $W(\widetilde{A}_{2n-1})^{J_0}$. In this section, we give a combinatorial description of the Bruhat ordering on $W(\widetilde{C}_n)^{I_0}$.

We study a relation between the automorphism σ and the map $\mathcal{C}_0$. For a Young diagram λ, let λ^c be the conjugate diagram of λ.

Proposition 5.1 *For $s_i \in S(\widetilde{A}_{2n-1})$ and an $2n$-core λ, then we have $(s_i \cdot \lambda)^c = s_{2n-i} \cdot \lambda^c$. That is, $(s_i \cdot \lambda)^c = \sigma(s_i) \cdot \lambda^c$.*

Proof. Since the action of s_i to λ has three ways as the case may be, we will show this proposition for each case. It is obvious that the color of the box (q, p) in λ^c is $2n - i$ if the box (p, q) is colored i in λ.

Case 1. If $s_i \cdot \lambda = \lambda$, then λ can't have either removable or addable i-corner. This implies that λ^c can't have either removable or addable $(2n-i)$-corner. Therefore we have $s_{2n-i} \cdot \lambda^c = \lambda^c$ and

$$(s_i \cdot \lambda)^c = \lambda^c = s_{2n-i} \cdot \lambda^c.$$

Case 2. If $\lambda \supsetneq s_i \cdot \lambda$, then λ has some removable i-corners and the action s_i to λ is removing them from λ. Therefore λ^c has some removable $(2n-i)$-corners and the action of s_{2n-i} to λ^c is removing them from λ^c. In particular, the component of each removable $(2n-i)$-corner in λ^c is conjugate to the one of the removable i-corners in λ. This forces $(s_i \cdot \lambda)^c$ to be $s_{2n-i} \cdot \lambda^c$.

Case 3. If $\lambda \subsetneq s_i \cdot \lambda$, then the action of s_i to λ is adding boxes to all addable i-corners in λ. By the same argument as Case 2, we can deduce that $(s_i \cdot \lambda)^c = s_{2n-i} \cdot \lambda^c$.

$\square$

Corollary 5.2 *For $\pi \in W(\widetilde{A}_{N-1})$,*

$$\mathcal{C}_0(\sigma(\pi)) = \mathcal{C}_0(\pi)^c.$$

$\square$

Then we have a combinatorial description of the Bruhat ordering on $W(\widetilde{C}_n)^{I_0}$ from Lemma 3.4, Theorem 4.5 and Corollary 5.2:

Theorem 5.3 *Let $\mathrm{Sym}\mathcal{C}[2n] = \{\lambda \in \mathcal{C}[2n] \ : \ \lambda^c = \lambda\}$ and $\leq_\sigma$ be the Bruhat ordering on $W(\widetilde{C}_n)^{I_0}$ with respect to $S(\widetilde{C}_n)$. Then we have*

(1) *The map $\mathcal{C}_0 : W(\widetilde{C}_n)^{I_0} \longrightarrow \mathrm{Sym}\mathcal{C}[2n]$ is a bijection.*

(2) *For $w, v \in W(\widetilde{C}_n)^{I_0}$,*

$$w \leq_\sigma v \iff \mathcal{C}_0(w) \subseteq \mathcal{C}_0(v).$$

$\square$

6 The Bruhat ordering on $W(\widetilde{C}_n)$

In this section, we use the combinatorial description of the Bruhat ordering on $W(\widetilde{C}_n)^{I_0}$ in the previous section, to find a combinatorial description of the Bruhat ordering on the whole group $W(\widetilde{C}_n)$ ($= W(\widetilde{A}_{2n-1})_\sigma$). In order to attain this purpose, the following theorem is very useful:

Theorem 6.1 (V. Deodhar [4])
Let $\mathcal{H}$ be a family of subsets of S such that $\emptyset \notin \mathcal{H}$ and $\bigcap_{J \in \mathcal{H}} J = \emptyset$. Then $w, v \in W$ are $w \leq v$ in the Bruhat order on W if and only if $w^J \leq v^J$ in the Bruhat order on W^J for all $J \in \mathcal{H}$. $\square$

Now, we consider a Coxeter system $(W(\widetilde{A}_{N-1}), S(\widetilde{A}_{N-1}))$ of type $\widetilde{A}_{N-1}$.

Let $J_k = S(\widetilde{A}_{N-1}) \setminus \{s_k\}$, for each integer k with $1 \leq k \leq N-1$, then by the same argument in case of J_0, $W(\widetilde{A}_{N-1})^{J_k}$ consists of all affine permutations which satisfy

$$\pi(k+1) < \pi(k+2) < \cdots < \pi(N) < \pi(1+N) < \pi(2+N) < \cdots < \pi(k+N).$$

Therefore, for $\tau \in W(\widetilde{A}_{N-1})$, the window of τ^{J_k} satisfies the following condition:

(i) $\tau^{J_k}(1) < \tau^{J_k}(2) < \cdots < \tau^{J_k}(k)$ and $\tau^{J_k}(k+1) < \tau^{J_k}(k+2) < \cdots < \tau^{J_k}(N)$,

(ii) $\{\tau^{J_k}(1), \tau^{J_k}(2), \cdots, \tau^{J_k}(k)\} = \{\tau(1), \tau(2), \cdots, \tau(k)\}$ and $\{\tau^{J_k}(k+1), \tau^{J_k}(k+2), \cdots, \tau^{J_k}(N)\} = \{\tau(k+1), \tau(k+2), \cdots, \tau(N)\}$.

Now, we consider a Coxeter graph automorphism ω of $S(\widetilde{A}_{N-1})$ as follows:

$$\omega(s_i) = \begin{cases} s_{N-1} & \text{if } i = 0, \\ s_{i-1} & \text{otherwise.} \end{cases}$$

Then we can prove the following Proposition by using the induction with respect to the length:

Proposition 6.2 *This automorphism ω extends to a group automorphism on $W(\widetilde{A}_{N-1})$ as follows: For $\pi \in W(\widetilde{A}_{N-1})$,*

$$\omega(\pi) = [\pi(2)-1, \pi(3)-1, \cdots, \pi(N)-1, \pi(1)-1+N].$$

□

Moreover, it is obvious that $\omega(J_k) = J_{k-1}$ for $1 \leq k \leq N-1$. Then Proposition 2.1 and 2.2 imply the following proposition:

Proposition 6.3 *Let k be an integer with $0 \leq k \leq N-1$, then the map $\omega^k : W(\widetilde{A}_{N-1})^{J_k} \to W(\widetilde{A}_{N-1})^{J_0}$ is an order preserving bijection.* □

For each integer k with $0 \leq k \leq N-1$, we define the bijection map $\mathcal{C}_k : W(\widetilde{A}_{N-1})^{J_k} \to \mathcal{C}[N]$ by

$$\mathcal{C}_k = \mathcal{C}_0 \circ \omega^k.$$

Then Theorem 6.1 and Proposition 6.3 lead to the following theorem:

Theorem 6.4 *Let $\leq$ be the Bruhat ordering on $W(\widetilde{A}_{N-1})$ with respect to $S(\widetilde{A}_{N-1})$. For $\pi, \tau \in W(\widetilde{A}_{N-1})$,*

$$\pi \leq \tau \iff \mathcal{C}_k(\pi^{J_k}) \subseteq \mathcal{C}_k(\tau^{J_k})$$

for $0 \leq k \leq N-1$. □

Let $N = 2n$, we consider the Coxeter graph automorphism σ defined by (3.3). Since $\sigma(J_k) = J_{2n-k}$, we have

$$\omega^k \circ \sigma = \sigma \circ \omega^{2n-k}$$

for $0 \leq k \leq 2n-1$. Therefore if $\pi \in W(\widetilde{C}_n)$, then we have

$$\mathcal{C}_{2n-k}(\pi^{J_{2n-k}}) = \mathcal{C}_k(\pi^{J_k})^c \tag{6.1}$$

for $0 \leq k \leq 2n-1$. In particular, we should note that the equation (6.1) is sufficient for $\pi \in W(\widetilde{C}_n)$.

Finally, we can give a combinatorial description of the Bruhat ordering on $W(\widetilde{C}_n)$, i.e., a Coxeter group of type $\widetilde{C}_n$:

Theorem 6.5 *Let $\leq_\sigma$ be the Bruhat ordering on $W(\widetilde{C}_n)$ with respect to $S(\widetilde{C}_n)$. For $w, v \in W(\widetilde{C}_n) \subset W(\widetilde{A}_{2n-1})$,*

$$w \leq_\sigma v \iff \mathcal{C}_k(w^{J_k}) \subseteq \mathcal{C}_k(v^{J_k})$$

for all integers k with $0 \leq k \leq n$. □

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