On the Kronecker Product $s_{(n-p,p)} * s_\lambda$

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Abstract

The Kronecker product of two Schur functions s_λ and s_μ, denoted $s_\lambda * s_\mu$, is defined as the Frobenius characteristic of the tensor product of the irreducible representations of the symmetric group indexed by partitions of n, λ and μ, respectively. The coefficient, $g_{\lambda,\mu,\nu}$, of s_ν in $s_\lambda * s_\mu$ is equal to the multiplicity of the irreducible representation indexed by ν in the tensor product. In this paper we give an algorithm for expanding the Kronecker product $s_{(n-p,p)} * s_\lambda$ whenever $\lambda_1 - \lambda_2 \geq 2p$. As a consequence of this algorithm we obtain a formula for the coefficients $g_{\lambda,\mu,\nu}$ in terms of Littlewood-Richardson coefficients which does not involve cancellations. We also show that the coefficients in the expansion of $s_{(n-p,p)} * s_\lambda$ are stable. Moreover, we obtain a simple combinatorial interpretation for $g_{\lambda,(n-p,p),\nu}$ if λ is not a partition inside the $2(p-1) \times 2(p-1)$ square.

Introduction

Let χ^λ and χ^μ be the irreducible characters of S_n (the symmetric group on n letters) indexed by the partitions λ and μ of n. The *Kronecker product* $\chi^\lambda\chi^\mu$ is defined by $(\chi^\lambda\chi^\mu)(w) = \chi^\lambda(w)\chi^\mu(w)$ for all $w \in S_n$. Hence, $\chi^\lambda\chi^\mu$ is the character that corresponds to the diagonal action of S_n on the tensor product of the irreducible representations indexed by λ and μ. Then we have

$$\chi^\lambda\chi^\mu = \sum_{\nu \vdash n} g_{\lambda,\mu,\nu}\chi^\nu,$$

where $g_{\lambda,\mu,\nu}$ is the multiplicity of χ^ν in $\chi^\lambda\chi^\mu$. Hence the $g_{\lambda,\mu,\nu}$ are non-negative integers.

By means of the Frobenius map one can define the Kronecker (internal) product on the Schur symmetric functions by

$$s_\lambda * s_\mu = \sum_{\nu \vdash n} g_{\lambda,\mu,\nu} s_\nu.$$

A reasonable formula for decomposing the Kronecker product is unavailable, although the problem has been studied since the early twentieth century. In recent years Lascoux [La], Remmel [R], Remmel and Whitehead [RWd] and Rosas [Ro] derived closed formulas for

Kronecker products of Schur functions indexed by two row shapes or hook shapes. Gessel [Ge] obtained a combinatorial interpretation for zigzag partitions.

More general results include a formula of Garsia and Remmel [GR-1] which decomposes the product of homogeneous symmetric functions with a Schur function. Dvir [D] and Clausen and Meier [CM] have found bounds for the largest part and the maximal number of parts in a constituent of a product. Bessenrodt and Kleshchev [BK] have looked at the problem of determining when the decomposition of the Kronecker product has one or two constituents.

In 1937 Murnaghan [M] noticed that for large n the Kronecker product did not depend on the first part of the partitions λ and μ. That is, if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)})$ is a partition of n (written $\lambda \vdash n$) and $\bar{\lambda} = (\lambda_2, \ldots, \lambda_{\ell(\lambda)})$ denotes the partition obtained by removing the first part of λ, then there exists an n such that $g_{(n-|\bar{\lambda}|,\bar{\lambda}),(n-|\bar{\mu}|,\bar{\mu}),(n-|\bar{\nu}|,\bar{\nu})} = g_{(m-|\bar{\lambda}|,\bar{\lambda}),(m-|\bar{\mu}|,\bar{\mu}),(m-|\bar{\nu}|,\bar{\nu})}$ for all $m \geq n$. In this case we say that $g_{\lambda,\mu,\nu}$ is *stable*. Vallejo [V] has recently found a bound for n for the stability of $g_{\lambda,\mu,\nu}$. In this paper we show that $g_{(n-p,p),\lambda,\nu}$ is stable for all ν if $\lambda_1 - \lambda_2 \geq 2p$.

There is a simple algorithm for the decomposition of $s_{(n-1,1)} * s_\lambda$ whenever $\lambda_1 - \lambda_2 \geq 2$.

First Step: Everywhere possible delete zero or one box from $\bar{\lambda}$ such that the resulting diagram corresponds to a partition.

Second step: To each diagram $\beta \neq \bar{\lambda}$ obtained in the first step, everywhere possible add zero or one box so that the resulting diagram corresponds to a partition. And to $\beta = \bar{\lambda}$ add everywhere possible one box.

Finally, we complete the resulting diagrams $\bar{\nu}$ obtained in the second step such that $\nu = (n - |\bar{\nu}|, \bar{\nu})$ is a partition of n. Then $s_{(n-1,1)} * s_\lambda$ is equal to the sum of the Schur functions corresponding to all diagrams ν obtained via the remove/add steps above.

We generalize this algorithm for the Kronecker product $s_{(n-p,p)} * s_\lambda$ whenever $\lambda_1 - \lambda_2 \geq 2p$. We use the algorithm to obtain a close formula for $g_{\lambda,\mu,\nu}$ as well as bounds for the size of ν_1 and ν_2. Our main tools are the Garsia-Remmel identity [GR-1, Lemma 6.3] and the Remmel-Whitney algorithm for multiplying Schur functions [RWy].

We also give a combinatorial interpretation for the coefficient of s_ν in $s_{(n-p,p)} * s_\lambda$, if $\lambda_1 \geq 2p-1$ or $\ell(\lambda) \geq 2p-1$, in terms of what we call *Kronecker Tableaux*. In particular, our combinatorial interpretation holds for all λ if $n > (2p-2)^2$. Our analysis involves studying the Schur positivity of the symmetric function $s_{\lambda/\alpha} s_\alpha - s_{\lambda/\beta} s_\beta$, where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{\ell(\alpha)})$ with $\alpha_1 > \alpha_2$ and $\beta = (\alpha_1 - 1, \alpha_2, \ldots, \alpha_{\ell(\alpha)})$. We prove that this symmetric function is Schur positive if and only if $\lambda_1 \geq 2\alpha_1 - 1$. This result is then used to give a combinatorial interpretation for $g_{(n-p,p),\lambda,\nu}$ whenever λ is not a partition that fits in the $(2p-2) \times (2p-2)$ square.

Summary of results

1) The (modified) Remmel-Whitney algorithms.

The *reverse lexicographic filling* of μ, $rl(\mu)$, is a filling of the Young diagram μ with the numbers $1, 2, \ldots, |\mu|$ so that the numbers are entered in order from right to left and top to bottom.

Definition: A tableau T is (λ, μ)-*compatible* if it contains $|\lambda|$ unlabelled boxes and $|\mu|$ labelled boxes (with labels $1, 2 \ldots, |\mu|$) and all of the following conditions are satisfied:

(a) T contains $|\lambda|$ unlabelled boxes in the shape λ. They are positioned in the upper-left corner of T.

(b) The labelled boxes in T are in increasing order in each row from left to right and in each column from top to bottom. If one box of T is labelled, so are all the boxes in the same row that are to the right of it.

(c) If a box labelled $i+1$ occurs immediately to the left of the box labelled i in $rl(\mu)$, then in T the label $i+1$ occurs weakly above and strictly to the right of i.

(d) If the box labelled y occurs immediately below the box labelled x in $rl(\mu)$, then in T the label y occurs strictly below and weakly to the left of x.

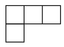
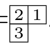
Remmel and Whitney showed that $c^{\nu}_{\lambda\mu}$ is the number of (λ, μ)-compatible tableaux of shape ν [RWy].

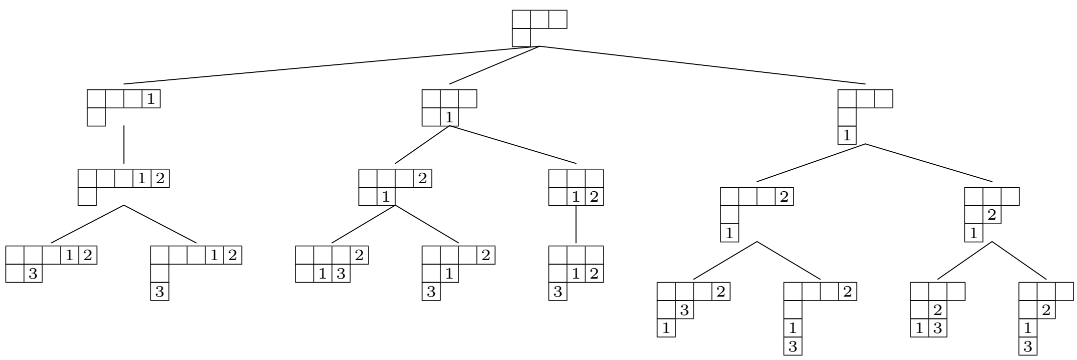
Multiplication: $s_\lambda s_\mu$ - Add$[\mu]$ to λ. Computing $s_\lambda s_\mu = \sum_{|\nu|=|\lambda|+|\mu|} c^{\nu}_{\lambda\mu} s_\nu$:

(1) To the Young diagram λ add a box labelled 1 everywhere possible so that the rows are weakly increasing in size.

(2) We add each subsequent number so that, at each step, the conditions of the definition of (λ, μ)-compatible tableau are satisfied.

In this way we obtain a tree. The leaves of this tree are the elements of the multi-set Add$[\mu]$ to λ. They are the summands in the decomposition of $s_\lambda s_\mu$.

Example: The decomposition of $s_\lambda s_\mu$, where $\lambda = (3, 1)$, $\mu = (2, 1)$: $\lambda =$  and $rl(\mu) =$ .



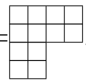
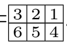
Hence $s_\lambda s_\mu = s_{(5,2)} + s_{(5,1,1)} + s_{(4,3)} + 2s_{(4,2,1)} + s_{(3,3,1)} + s_{(4,1,1,1)} + s_{(3,2,2)} + s_{(3,2,1,1)}$.

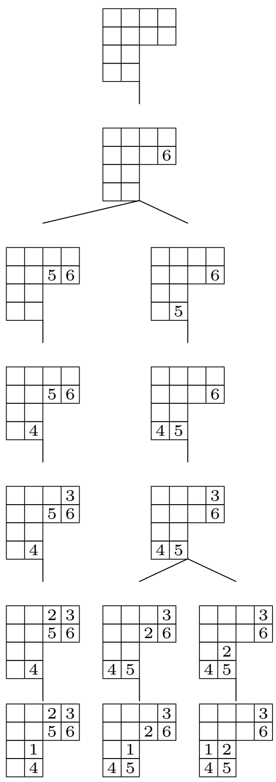
Skew: $s_{\lambda/\mu}$ **- Delete**$[\mu]$ **from** λ. Computing $s_{\lambda/\mu} = \sum_{|\nu|=|\lambda|-|\mu|} c^{\lambda}_{\mu\,\nu} s_\nu$:

(1) Form the reverse lexicographic filling of μ.

(2) Starting with the Young diagram λ we will label its outermost boxes with the numbers $1, 2, \ldots, |\mu|$ in decreasing order, starting with $|\mu|$, in the following way. At every step, the diagram obtained from λ by deleting the labelled boxes must be a Young diagram. Suppose the position (i, j) in $rl(\mu)$ is labelled x. If $j > 1$, let x^- be the label in position $(i, j-1)$ in $rl(\mu)$. If $i < \ell(\mu)$, let x^+ be the label in position $(i+1, j)$ in $rl(\mu)$. In λ, x will be placed to the left and weakly below (to the SW) of x^- and above and weakly to the right (to the NE) of x^+.

From each of the diagrams obtained (with $|\mu|$ labelled boxes) we remove all labelled boxes. The resulting diagrams are the elements in the multi-set Delete$[\mu]$ from λ. They are the summands in the decomposition of $s_{\lambda/\mu}$.

Example: The decomposition of $s_{\lambda/\mu}$, $\lambda = (4, 4, 2, 2)$, $\mu = (3, 3)$: $\lambda =$ , $rl(\mu) =$ .



Hence $s_{\lambda/\mu} = s_{(2,2,1,1)} + s_{(3,2,1)} + s_{(3,3)}$.

2) Algorithm for computing $s_{(n-p,p)} * s_\lambda$

If $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$, we denote by $\bar{\mu}$ the partition $\bar{\mu} = (\mu_2, \ldots, \mu_k)$. We will follow the philosophy of [M], and attempt to work with the partition $\bar{\mu}$ instead of μ whenever possible. Knowing that $\mu \vdash n$, μ_1 is completely determined by $\bar{\mu}$.

Let p be a positive integer and λ a partition of n such that $\lambda_1 - \lambda_2 \geq 2p$. We consider the subset of partitions of p contained in λ: $S_\lambda = \{\alpha \vdash p \,|\, \alpha \subseteq \lambda\}$.

Algorithm: For every $\alpha \in S_\lambda$ form the following set of Young diagrams:

$$Q(\alpha) = \bigcup_{j=0}^{\alpha_1} \{\nu |\, \nu \text{ is obtained by removing a horizontal strip with } j \text{ boxes from } \alpha\}$$

$$= \bigcup_{j=0}^{\alpha_1} \text{ Delete } [(j)] \text{ from } \alpha$$

For each $\alpha \in S_\lambda$ perform the following two steps:

(1) Remove$[\alpha]$**:** For each $\delta \in Q(\alpha)$ perform *Delete[δ] from* $\bar{\lambda}$. Record all diagrams obtained from *Delete[δ] from* $\bar{\lambda}$, with multiplicity, in the multi-set $D(\alpha)$. Denote by $d_{\alpha\lambda\beta}$ the multiplicity of β in $D(\alpha)$. If $\alpha_1 > \alpha_2$, let $D'(\alpha)$ be the submulti-set of $D(\alpha)$ of diagrams obtained by performing *Delete[δ] from* $\bar{\lambda}$ whenever $\delta_1 = \alpha_1$. Denote the multiplicity of $\beta \in D'(\alpha)$ by $d'_{\alpha\lambda\beta}$. If $\alpha_1 = \alpha_2$, set $d'_{\alpha\lambda\beta} = 0$.

(2) Add$[\alpha]$**:** For each (distinct) $\beta \in D(\alpha)$,

(a) If $d'_{\alpha\lambda\beta} = 0$, then for each $\gamma \in Q(\alpha)$ with $\gamma_1 = \alpha_1$ perform *Add[γ] to* β. The multiplicity of each resulting diagram is multiplied by $d_{\alpha\lambda\beta}$.

(b) If $0 < d'_{\alpha\lambda\beta} = d_{\alpha\lambda\beta}$, then for each $\gamma \in Q(\alpha)$ perform *Add[γ] to* β. The multiplicity of each resulting diagram is multiplied by $d_{\alpha\lambda\beta}$.

(c) If $0 < d'_{\alpha\lambda\beta} < d_{\alpha\lambda\beta}$, then for each $\gamma \in Q(\alpha)$ perform *Add[γ] to* β. For each $\gamma \in Q(\alpha)$ with $\gamma_1 = \alpha_1$ the multiplicity of each resulting diagram is multiplied by $d_{\alpha\lambda\beta}$. And for each γ such that $\gamma_1 < \alpha_1$ the multiplicity of each resulting diagram is multiplied by $d'_{\alpha\lambda\beta}$.

Finally, we record all diagrams obtained in step (2), for every β, in a multi-set R_α.

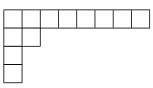
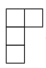
Note: Whenever we perform *Delete[η] from* η, the empty diagram, denoted ϵ, will be recorded. Thus, if $\alpha = (p)$, then $\epsilon \in Q(\alpha)$. Similarly, in the **Remove**$[\alpha]$ step, if $\delta = \bar{\lambda} \in Q(\alpha)$, then $\epsilon \in D(\alpha)$.

If $\eta = (\eta_1, \ldots, \eta_{\ell(\eta)}) \in R_\alpha$, let $\tilde{\eta} = (\eta_0, \eta_1, \ldots, \eta_{\ell(\eta)})$, where $\eta_0 = n - |\eta|$. Thus $\tilde{\eta} \vdash n$.

Theorem 1: Let p be a positive integer and λ a partition of n such that $\lambda_1 - \lambda_2 \geq 2p$. Then

$$s_{(n-p,p)} * s_\lambda = \sum_{\alpha \in S_\lambda} \sum_{\eta \in R_\alpha} s_{\tilde{\eta}}.$$

Example: We will perform the algorithm for $s_{(n-p,p)} * s_\lambda$ in the case when $n = 12$, $p = 3$ and $\lambda = (8, 2, 1, 1)$. Since $\lambda_1 - \lambda_2 = 8 - 2 = 6 \geq 2p$, the condition of the algorithm is satisfied. The Young diagrams for λ and $\bar{\lambda}$ are

$\lambda =$  and $\bar{\lambda} =$ .

We have $S_\lambda = \{\alpha \vdash 3 \mid \alpha \leq \lambda\} = \left\{ \yng(3) \, , \yng(2,1) \, , \yng(1,1,1) \right\}$

$\alpha =$ $\yng(3)$: From α remove j boxes, $0 \leq j \leq 3$, no two in the same column.

$$Q(\alpha) = \{ \yng(3) \, , \yng(2) \, , \yng(1) \, , \epsilon \}$$

(1) Remove$[\alpha]$**:** For each $\delta \in Q(\alpha)$ perform *Delete[δ] from* $\bar{\lambda}$.

$Delete[\young(321)]$, $Delete[\young(21)]$, $Delete[\young(1)]$, and $Delete[\epsilon]$ *from* $\yng(2,1,1)$. Then we have

$$D(\alpha) = \left\{ \yng(1,1) \, , \yng(1,1,1) \, , \yng(2,1) \, , \yng(2,1,1) \right\} \quad \text{and} \quad D'(\alpha) = \emptyset.$$

(2) Add$[\alpha]$**:** Since $D'(\alpha) = \emptyset$, we have $d'_{\alpha\lambda\beta} = 0$ for all $\beta \in D(\alpha)$. We are in case (a). The only $\gamma \in Q(\alpha)$ with $\gamma_1 = \alpha_1$ is $\gamma = \yng(3)$. For every $\beta \in D(\alpha)$ we perform $Add[\yng(3)]$ *to* β.

$Add[\young(321)]$ *to* $\yng(1,1) = \{(4,1), (3,1,1)\}$;

$Add[\young(321)]$ *to* $\yng(1,1,1) = \{(4,1,1), (3,1,1,1)\}$;

$Add[\young(321)]$ *to* $\yng(2,1) = \{(5,1), (4,2), (4,1,1), (3,2,1)\}$;

$Add[\young(321)]$ *to* $\yng(2,1,1) = \{(5,1,1), (4,2,1), (4,1,1,1), (3,2,1,1)\}$.

We take the union of these four multi-sets to get:

$R_{\yng(3)} = \{(4,1), (3,1,1), 2(4,1,1), (3,1,1,1), (5,1), (4,2), (3,2,1), (5,1,1), (4,2,1), (4,1,1,1), (3,2,1,1)\}$

$\alpha =$ $\yng(2,1)$: From α remove j boxes, $0 \leq j \leq 2$, no two in the same column.

$$Q(\alpha) = \left\{ \yng(2,1) \, , \yng(1,1) \, , \yng(2) \, , \yng(1) \right\}$$

(1) Remove$[\alpha]$**:** For each $\delta \in Q(\alpha)$ perform *Delete[δ] from* $\bar{\lambda}$.

$Delete[\young(21,3)]$ *from* $\yng(2,1,1) = \{(1)\}$; $Delete[\young(1,2)]$ *from* $\yng(2,1,1) = \{(1,1), (2)\}$;

$Delete[\young(21)]$ *from* $\yng(2,1,1) = \{(1,1)\}$; $Delete[\young(1)]$ *from* $\yng(2,1,1) = \{(2,1), (1,1,1)\}$.

This yields:

$$D(\alpha) = \left\{ \yng(1) \, , 2 \, \yng(1,1) \, , \yng(2) \, , \yng(2,1) \, , \yng(1,1,1) \right\} \quad \text{and} \quad D'(\alpha) = \left\{ \yng(1) \, , \yng(1,1) \right\}.$$

(2) Add[α]: If $\beta = \square$, then we have $d'_{\alpha\lambda\beta} = 1 = d_{\alpha\lambda\beta}$ and we are in case (b). For each $\gamma \in Q(\alpha)$ we perform *Add[γ] to* $\square$.

$Add[\begin{smallmatrix}2&1\\3\end{smallmatrix}]$ *to* $\square = \{(3,1),(2,2),(2,1,1)\}$; $Add[\begin{smallmatrix}1\\2\end{smallmatrix}]$ *to* $\square = \{(2,1),(1,1,1)\}$;

$Add[\begin{smallmatrix}2&1\end{smallmatrix}]$ *to* $\square = \{(3),(2,1)\}$; $Add[\begin{smallmatrix}1\end{smallmatrix}]$ *to* $\square = \{(2),(1,1)\}$.

If $\beta = \begin{smallmatrix}\square\\\square\end{smallmatrix}$, then $d'_{\alpha\lambda\beta} = 1$ and $d_{\alpha\lambda\beta} = 2$. Thus we are in case (c).

For each $\gamma \in Q(\alpha)$ we perform *Add[γ] to* $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ and if $\gamma_1 = \alpha_1$ count the resulting diagrams with multiplicity $d_{\alpha\lambda\beta} = 2$.

$2 \times Add[\begin{smallmatrix}2&1\\3\end{smallmatrix}]$ *to* $\begin{smallmatrix}\square\\\square\end{smallmatrix} = \{2(3,2), 2(3,1,1), 2(2,2,1), 2(2,1,1,1)\}$;

$Add[\begin{smallmatrix}1\\2\end{smallmatrix}]$ *to* $\begin{smallmatrix}\square\\\square\end{smallmatrix} = \{(2,2),(2,1,1),(1,1,1,1)\}$;

$2 \times Add[\begin{smallmatrix}2&1\end{smallmatrix}]$ *to* $\begin{smallmatrix}\square\\\square\end{smallmatrix} = \{2(3,1), 2(2,1,1)\}$;

$Add[\begin{smallmatrix}1\end{smallmatrix}]$ *to* $\begin{smallmatrix}\square\\\square\end{smallmatrix} = \{(2,1),(1,1,1)\}$.

If $\beta = \square\square$, then $d'_{\alpha\lambda\beta} = 0$. We are in case (a). The only $\gamma \in Q(\alpha)$ with $\gamma_1 = \alpha_1$ are $\gamma = \begin{smallmatrix}\square&\square\\\square\end{smallmatrix}$ and $\gamma = \square\square$.

$Add[\begin{smallmatrix}2&1\\3\end{smallmatrix}]$ *to* $\square\square = \{(4,1),(3,2),(3,1,1),(2,2,1)\}$;

$Add[\begin{smallmatrix}2&1\end{smallmatrix}]$ *to* $\square\square = \{(4),(3,1),(2,2)\}$;

If $\beta = \begin{smallmatrix}\square&\square\\\square\end{smallmatrix}$, then $d'_{\alpha\lambda\beta} = 0$. We are in case (a). As before, the only $\gamma \in Q(\alpha)$ with $\gamma_1 = \alpha_1$ are $\gamma = \begin{smallmatrix}\square&\square\\\square\end{smallmatrix}$ and $\gamma = \square\square$.

$Add[\begin{smallmatrix}2&1\\3\end{smallmatrix}]$ *to* $\begin{smallmatrix}\square&\square\\\square\end{smallmatrix} = \{(4,2),(4,1,1),(3,3),2(3,2,1),(3,1,1,1),(2,2,2),(2,2,1,1)\}$;

$Add[\begin{smallmatrix}2&1\end{smallmatrix}]$ *to* $\begin{smallmatrix}\square&\square\\\square\end{smallmatrix} = \{(4,1),(3,2),(3,1,1),(2,2,1)\}$.

If $\beta = \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}$, then $d'_{\alpha\lambda\beta} = 0$. We are in case (a). As before, the only $\gamma \in Q(\alpha)$ with $\gamma_1 = \alpha_1$ are $\gamma = \begin{smallmatrix}\square&\square\\\square\end{smallmatrix}$ and $\gamma = \square\square$.

$Add[\begin{smallmatrix}2&1\\3\end{smallmatrix}]$ *to* $\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix} = \{(3,2,1),(3,1,1,1),(2,2,1,1),(2,1,1,1,1)\}$;

$Add[\begin{smallmatrix}2&1\end{smallmatrix}]$ *to* $\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix} = \{(3,1,1),(2,1,1,1)\}$.

We take the union of all the multi-sets above (from the Add step):

$$\begin{aligned}R_{\begin{smallmatrix}\square&\square\\\square\end{smallmatrix}} = \{&4(3,1), 3(2,2), 4(2,1,1), 3(2,1), 2(1,1,1), (3), (2), (1,1), 4(3,2),\\ &5(3,1,1), 4(2,2,1), 3(2,1,1,1), (1,1,1,1), 2(4,1), (4), (4,2), (4,1,1),\\ &(3,3), 3(3,2,1), 2(3,1,1,1), (2,2,2), 2(2,2,1,1), (2,1,1,1,1)\}\end{aligned}$$

$\boldsymbol{\alpha} = \ydiagram{1,1,1}$: From α remove j boxes, $0 \le j \le 1$, no two in the same column.

$$Q(\alpha) = \left\{ \ydiagram{1,1,1}\, , \ydiagram{1,1} \right\}$$

(1) Remove$[\alpha]$**:** For each $\delta \in Q(\alpha)$ perform *Delete[δ] from* $\bar{\lambda}$.

$Delete[\young(1,2,3)]\ from\ \ydiagram{2,1,1} = \{(1)\}$; $\qquad$ $Delete[\young(1,2)]\ from\ \ydiagram{2,1,1} = \{(2),(1,1)\}$.

This yields:

$$D(\alpha) = \left\{ \ydiagram{1}\, , \ydiagram{1,1}\, , \ydiagram{2} \right\}.$$

(2) Add$[\alpha]$**:** Since $\alpha_1 = \alpha_2$, $d'_{\alpha\lambda\beta} = 0$ for all $\beta \in D(\alpha)$. We are in case (a). For $\alpha = (1,1,1)$, all $\gamma \in Q(\alpha)$ satisfy $\gamma_1 = \alpha_1$. We perform *Add[γ] to β* for all $\gamma \in Q(\alpha)$ and all $\beta \in D(\alpha)$.

$Add[\young(1,2,3)]\ to\ \ydiagram{1} = \{(2,1,1),(1,1,1,1)\}$; $\qquad$ $Add[\young(1,2)]\ to\ \ydiagram{1} = \{(2,1),(1,1,1)\}$;

$Add[\young(1,2,3)]\ to\ \ydiagram{1,1} = \{(2,2,1),(2,1,1,1),(1,1,1,1,1)\}$; $\qquad$ $Add[\young(1,2)]\ to\ \ydiagram{1,1} = \{(2,2),(2,1,1),(1,1,1,1)\}$;

$Add[\young(1,2,3)]\ to\ \ydiagram{2} = \{(3,1,1),(2,1,1,1)\}$; $\qquad$ $Add[\young(1,2)]\ to\ \ydiagram{2} = \{(3,1),(2,1,1)\}$.

We take the union of all the multi-sets above:

$$R_{\ydiagram{1,1,1}} = \{3(2,1,1), 2(1,1,1,1), (2,1), (1,1,1), (2,2,1),$$
$$2(2,1,1,1), (1,1,1,1,1), (2,2), (3,1,1), (3,1)\}$$

Finally, we use Theorem 1 to obtain the decomposition of $s_{(9,3)} * s_{(8,2,1,1)}$. Consider the union of the multi-sets R_α, for all $\alpha \in S_{(8,2,1,1)}$, and "complete" each shape to size 12.

Thus

$s_{(9,3)} * s_{(8,2,1,1)} = 3s_{(7,4,1)} + 7s_{(7,3,1,1)} + 3s_{(6,4,1,1)} + 3s_{(6,3,1,1,1)} + s_{(6,5,1)} + 2s_{(6,4,2)} + 4s_{(6,3,2,1)} + s_{(5,5,1,1)} + s_{(5,4,2,1)} + s_{(5,4,1,1,1)} + s_{(5,3,2,1,1)} + 5s_{(8,3,1)} + 4s_{(8,2,2)} + 7s_{(8,2,1,1)} + 4s_{(9,2,1)} + 3s_{(9,1,1,1)} + s_{(9,3)} + s_{(10,2)} + s_{(10,1,1)} + 4s_{(7,3,2)} + 5s_{(7,2,2,1)} + 5s_{(7,2,1,1,1)} + 3s_{(8,1,1,1,1)} + s_{(8,4)} + s_{(6,3,3)} + s_{(6,2,2,2)} + 2s_{(6,2,2,1,1)} + s_{(6,2,1,1,1,1)} + s_{(7,1,1,1,1,1)}$.

3) Multiplicities in the Kronecker Product

Denote by $c^{\mu}_{\nu\,\eta}$ the Littlewood-Richardson coefficient. If we denote by $T^{\eta}_{\mu/\nu}$ the set of the semistandard Young tableaux of shape μ/ν and type η whose reverse reading word is a lattice permutation, then the cardinality of $T^{\eta}_{\mu/\nu}$ is equal to $c^{\mu}_{\nu\,\eta}$. Let $T^{\eta}_{\mu/\nu}(i,j)$ be the subset of $T^{\eta}_{\mu/\nu}$ of SSYTs of shape μ/ν and type η with label 1 in position (i,j). Note that this multi-subset could be empty. Define

$$a^{\mu}_{\nu\,\eta} := \begin{cases} |T^{\eta}_{\mu/\nu}(2,\nu_1)|, & \text{if } \mu_2 \geq \nu_1 \text{ and } \nu_1 > \nu_2, \\ 0 & \text{otherwise.} \end{cases}$$

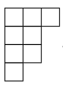
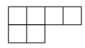
If $\beta = (\beta_1, \beta_2, \ldots, \beta_{\ell(\beta)}) \vdash m < n-p$, let $\hat{\beta} = (n-p-|\beta|, \beta_1, \beta_2, \ldots, \beta_{\ell(\beta)})$ be the partition of $n-p$ obtained from β by adding a first row of the correct size.

Theorem 2: Let n and p be positive integers such that $n \geq 2p$ and let λ be a partition of n with $\lambda_1 - \lambda_2 \geq 2p$. The multiplicity of s_ν in $s_{(n-p,p)} * s_\lambda$ is equal to

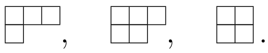
$$\sum_{\substack{\beta \subseteq \bar{\lambda},\, \beta \subseteq \bar{\nu} \\ |\beta| \geq n-\lambda_1-p}} \sum_{\substack{\alpha \vdash p \\ \alpha \subseteq \lambda}} \left(\sum_{\substack{\gamma \in Q(\alpha) \\ \gamma_1 = \alpha_1,\, \gamma \subseteq \bar{\nu} \\ |\gamma| = |\bar{\nu}| - |\beta|}} c^{\lambda}_{\alpha\hat{\beta}} c^{\bar{\nu}}_{\beta\gamma} + \sum_{\substack{\gamma \in Q(\alpha) \\ \gamma_1 < \alpha_1,\, \gamma \subseteq \bar{\nu} \\ |\gamma| = |\bar{\nu}| - |\beta|}} a^{\lambda}_{\alpha\hat{\beta}} c^{\bar{\nu}}_{\beta\gamma} \right).$$

Example: We use the above theorem to determine the multiplicity of $s_{(13,4,2)}$ in the Kronecker product $s_{(15,4)} * s_{(11,3,2,2,1)}$.

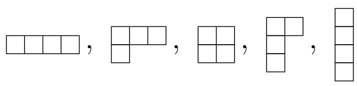
We have $n = 19$, $p = 4$, $\bar{\lambda} = (3,2,2,1)$ and $\bar{\nu} = (4,2)$, i.e

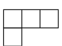
$\bar{\lambda} =$ , $\bar{\nu} =$ .

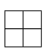
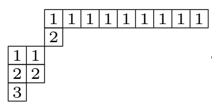
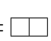
Since $n - \lambda_1 - p = 19 - 11 - 4 = 4$, the first summation in the formula of Theorem 2 runs over all Young diagrams β such that $|\beta| \geq 4$, $\beta \subseteq \bar{\lambda}$ and $\beta \subseteq \bar{\nu}$. Thus β has at most two rows: $\beta = (\beta_1, \beta_2)$ with $\beta_1 \leq 3$ and $\beta_2 \leq 2$. The possible β's in the first summation are



The second summation runs over all Young diagrams α of size $p = 4$ with $\alpha \subseteq \lambda$. They are the elements of

$S_\lambda = \Big\{$  $\Big\}$.

(1) If $\beta =$ , then $\hat{\beta} = (11,3,1) \vdash n - p = 15$. For each α, the inner sums will run over all $\gamma \in Q(\alpha)$ with $|\gamma| = |\bar{\nu}| - |\beta| = 6 - 4 = 2$.

If $\alpha =$ , then the only SSYT of shape λ/α and type $\hat{\beta} = (11,3,1)$ is . Thus $c^{\lambda}_{\alpha\,\hat{\beta}} = 1$ and, since $\alpha_1 = \alpha_2$, $a^{\lambda}_{\alpha\,\hat{\beta}} = 0$. The only $\gamma \in Q(\alpha)$ with $|\gamma| = 2$ is $\gamma =$ . There is one SSYT of shape $\bar\nu/\beta$ and type $\gamma = (2)$: $\begin{ytableau} \none & 1 \\ 1 \end{ytableau}$. Therefore $c^{\bar\nu}_{\beta\,\gamma} = 1$. Hence, $c^{\lambda}_{\alpha\,\hat\beta}c^{\bar\nu}_{\beta\,(2)} = 1$ *This contributes* 1 *to the multiplicity.*

If $\alpha = \ydiagram{2,1,1}$, then the only SSYT of shape λ/α and type $\hat\beta = (11,3,1)$ is $\begin{ytableau} \none & \none & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \none & 1 & 2 \\ \none & 2 \\ 1 & 3 \\ 2 \end{ytableau}$. Thus $c^{\lambda}_{\alpha\,\hat\beta} = a^{\lambda}_{\alpha\,\hat\beta} = 1$. The only $\gamma \in Q(\alpha)$ with $|\gamma| = 2$ is $\gamma = \ydiagram{1,1}$. There is one SSYT of shape $\bar\nu/\beta$ and type $\gamma = (1,1)$: $\begin{ytableau} \none & 1 \\ 2 \end{ytableau}$. Therefore $c^{\bar\nu}_{\beta\,\gamma} = 1$. Hence, $c^{\lambda}_{\alpha\,\hat\beta}c^{\bar\nu}_{\beta\,(1,1)} = 1$. *This contributes* 1 *to the multiplicity.*
For all other $\alpha \in S_\lambda$ we have $c^{\lambda}_{\alpha\,\hat\beta} = a^{\lambda}_{\alpha\,\hat\beta} = 0$. Hence, they do not contribute to the multiplicity.

(2) If $\beta = \ydiagram{3,2}$, then $\hat\beta = (10,3,2) \vdash n - p = 15$. For each α, the inner sums will run over all $\gamma \in Q(\alpha)$ with $|\gamma| = |\bar\nu| - |\beta| = 6 - 5 = 1$.
If $\alpha = \ydiagram{4}$ then $c^{\lambda}_{\alpha\,\hat\beta} = a^{\lambda}_{\alpha\,\hat\beta} = 0$.

If $\alpha = \ydiagram{3,1}$, $\begin{ytableau} \none & \none & \none & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \none & 1 & 2 \\ 1 & 2 \\ 2 & 3 \\ 3 \end{ytableau}$ is the only SSYT of shape λ/α and type $\hat\beta = (10,3,2)$. Thus $c^{\lambda}_{\alpha\,\hat\beta} = 1$ and $a^{\lambda}_{\alpha\,\hat\beta} = 0$. Since $\alpha_1 = 3$, there is no $\gamma \in Q(\alpha)$ with $\gamma_1 = \alpha_1$ and $|\gamma| = 1$.
If $\alpha = \ydiagram{2,2}$, $\alpha = \ydiagram{2,1,1}$ or $\alpha = \ydiagram{1,1,1,1}$, there is no $\gamma \in Q(\alpha)$ with $|\gamma| = 1$.

(3) Finally, if $\beta = \ydiagram{2,2}$, then $\hat\beta = (11,2,2) \vdash n - p = 15$. For each α, the inner sums will run over all $\gamma \in Q(\alpha)$ with $|\gamma| = |\bar\nu| - |\beta| = 6 - 4 = 2$.

If $\alpha = \ydiagram{3,1}$, $\begin{ytableau} \none & \none & \none & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \none & 1 & 1 \\ 1 & 2 \\ 2 & 3 \\ 3 \end{ytableau}$ is the only SSYT of shape λ/α and type $\hat\beta = (11,2,2)$. Thus $c^{\lambda}_{\alpha\,\hat\beta} = a^{\lambda}_{\alpha\,\hat\beta} = 1$. The shapes $\gamma \in Q(\alpha)$ with $|\gamma| = 2$ are $\gamma = \ydiagram{2}$ and $\gamma = \ydiagram{1,1}$. There is exactly one SSYT of shape $\bar\nu/\beta$ and type $\gamma = (2)$. Thus, for $\gamma = (2)$, $c^{\bar\nu}_{\beta\,\gamma} = 1$. Hence, $c^{\lambda}_{\alpha\,\hat\beta}c^{\bar\nu}_{\beta\,(2)} = 1$. *This contributes* 1 *to the multiplicity.* We also have $c^{\bar\nu}_{\beta\,(1,1)} = 0$

If $\alpha = \ydiagram{2,2}$, then $\begin{ytableau} \none & \none & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \none & \none & 2 \\ 1 & 1 \\ 2 & 3 \\ 3 \end{ytableau}$ is the only SSYT of shape λ/α and type $\hat\beta = (11,2,2)$. Thus $c^{\lambda}_{\alpha\,\hat\beta} = 1$ and, since $\alpha_1 = \alpha_2$, $a^{\lambda}_{\alpha\,\hat\beta} = 0$. The only $\gamma \in Q(\alpha)$ with $|\gamma| = 2$ (and $\gamma_1 = \alpha_1$) is $\gamma = \ydiagram{2}$. As before, there is one SSYT of shape $\bar\nu/\beta$ and type $\gamma = (2)$. Therefore $c^{\bar\nu}_{\beta\,(2)} = 1$. Hence, $c^{\lambda}_{\alpha\,\hat\beta}c^{\bar\nu}_{\beta\,(2)} = 1$. *This contributes* 1 *to the multiplicity.*

If $\alpha = \ydiagram{2,1,1}$, then $\begin{ytableau} \none & \none & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \none & 1 & 2 \\ \none & 2 \\ 1 & 3 \\ 3 \end{ytableau}$ is the only SSYT of shape λ/α and type $\hat\beta = (11,2,2)$. Thus $c^{\lambda}_{\alpha\,\hat\beta} = a^{\lambda}_{\alpha\,\hat\beta} = 1$. The only $\gamma \in Q(\alpha)$ with $|\gamma| = 2$ is $\gamma = \ydiagram{1,1}$. However, $c^{\bar\nu}_{\beta\,(1,1)} = 0$.
For all other $\alpha \in S_\lambda$ we have $c^{\lambda}_{\alpha\,\hat\beta} = a^{\lambda}_{\alpha\,\hat\beta} = 0$.

Therefore the multiplicity of $s_{(13,4,2)}$ in $s_{(15,4)} * s_{(11,3,2,2,1)}$ equals 4.

Proposition 3: Let n and p be positive integers with $n \geq 2p$ and let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)})$ be a partition of n with $\lambda_1 - \lambda_2 \geq 2p$. Consider the partition $\nu = (\nu_1, \nu_2, \ldots, \nu_{\ell(\nu)})$ of n. If the multiplicity $g_{(n-p,p),\lambda,\nu}$ of s_ν in $s_{(n-p,p)} * s_\lambda$ is non-zero, then $\lambda_1 - p \leq \nu_1 \leq \lambda_1 + p$. Moreover, if $\lambda_2 < p$ and $g_{(n-p,p),\lambda,\nu} \neq 0$, then $\lambda_1 - p \leq \nu_1 \leq \lambda_1 + \lambda_2$.

Proposition 4: Let n and p and $\lambda \vdash n$ be as in the previous proposition, i.e. $\lambda_1 - \lambda_2 \geq 2p$. Consider the partition $\nu = (\nu_1, \nu_2, \ldots, \nu_{\ell(\nu)})$ of n. If $\nu_2 > \lambda_2 + p$, then the multiplicity $g_{(n-p,p),\lambda,\nu}$ of s_ν in $s_{(n-p,p)} * s_\lambda$ is equal to zero. Moreover, if $\nu = (\lambda_1 - p, \lambda_2 + p, \lambda_3, \ldots, \lambda_{\ell(\lambda)})$, then $g_{(n-p,p),\lambda,\nu} = 1$.

If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)}) \vdash n$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_{\ell(\mu)}) \vdash m$, we say that λ is less than μ *in lexicographic order*, and write $\lambda <_l \mu$, if there is a non-negative integer k such that $\lambda_i = \mu_i$ for all $i = 1, 2, \ldots, k$ and $\lambda_{k+1} < \mu_{k+1}$. Note that the lexicographic order is a total order on the set of all partitions.

Corollary 5: Let n and p be positive integers such that $n \geq 2p$ and let $\lambda \vdash n$ such that $\lambda_1 - \lambda_2 \geq 2p$. The smallest partition in lexicographic order $\nu \vdash n$ such that s_ν appears in the decomposition of $s_{(n-p,p)} * s_\lambda$ is the partition whose parts are $\lambda_1 - p, \lambda_2, \ldots, \lambda_{\ell(\lambda)}, p$, reordered to form a partition. Moreover, this s_ν appears with multiplicity 1.

4) Stability of Kronecker coefficients

Theorem 6: Given an arbitrary partition $\bar{\lambda} = (\lambda_2, \lambda_3, \ldots, \lambda_{\ell(\lambda)})$, let n be a positive integer such that $n \geq 2p + |\bar{\lambda}| + \lambda_2$. Then $g_{(n-p,p),(n-|\bar{\lambda}|,\bar{\lambda}),(n-|\bar{\nu}|,\bar{\nu})} = g_{(m-p,p),(m-|\bar{\lambda}|,\bar{\lambda}),(m-|\bar{\nu}|,\bar{\nu})}$ for all $m \geq n$ and all partitions $\nu \vdash n$.

5) Combinatorial interpretation of the Kronecker coefficients

A SSYT T of shape λ/α and type $\nu - \alpha$ whose reverse reading word is an α-lattice permutation (i.e. in any initial factor $a_1 a_2 \cdots a_j$, $1 \leq j \leq n$, the number of $i's + \alpha_i \geq$ the number of $(i+1)'s + \alpha_{i+1}$) is called a *Kronecker Tableau* of shape λ/α and type $(\nu - \alpha)$ if

(I) $\alpha_1 = \alpha_2$ or

(II) $\alpha_1 > \alpha_2$ and any one of the following two conditions is satisfied:

  (i) The number of 1's in the second row of λ/α is exactly $\alpha_1 - \alpha_2$.

  (ii) The number of 2's in the first row of λ/α is exactly $\alpha_1 - \alpha_2$.

Denote by $k_{\alpha\nu}^{\lambda}$ the number of Kronecker tableaux of shape λ/α and type $\nu - \alpha$.

Theorem 7: Let n and p be positive integers such that $n \geq 2p-1$. Let $\lambda = (\lambda_1, \ldots, \lambda_{\ell(\lambda)}) \vdash n$

such that $\lambda_1 \geq 2p-1$. If ν is a partition of n, the multiplicity of s_ν in $s_{(n-p,p)} * s_\lambda$ equals

$$\sum_{\substack{\alpha \vdash p \\ \alpha \subseteq \lambda}} k^\lambda_{\alpha\nu},$$

where $\alpha \subseteq \lambda$ means $\ell(\alpha) \leq \ell(\lambda)$ and $\alpha_i \leq \lambda_i$ for all $1 \leq i \leq \ell(\alpha)$.

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