A Generalization of the Cayley-Hamilton Theorem

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Abstract

We present a generalization of the Cayley-Hamilton Theorem using a collection of immanants which naturally generalize the determinant.

For $f : S_n \to \mathbb{C}$, and $x = (x_{ij})_{1 \leq i,j \leq n}$, define the *f-immanant*, $Imm_f(x)$, by

$Imm_f(x) = \sum_{\sigma \in S_n} f(\sigma) x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}$.

Given $\xi \in \mathbb{C}$ and a positive integer n, define the to be the n^{th} *Temperly-Lieb algebra* $T_n(\xi)$ to be the multiplicative, associative $\mathbb{C}$-algebra with unity 1 generated by $t_1, t_2, \ldots, t_{(n-1)}$ subject to the relations

$$\begin{cases} t_i t_i = \xi t_i, i \in [n-1] \\ t_i t_j t_i = t_i, |i-j| = 1 \\ t_i t_j = t_j t_i, |i-j| > 1. \end{cases}$$

It is well known that the multiplicative monoid generated by $t_1, t_2, \ldots t_{(n-1)}$ is a basis for $T_n(\xi)$. We refer this basis as *the standard basis* and to its elements as *the basis elements of* $T_n(\xi)$. Given basis elements $\tau_1 = t_{i_1} \ldots t_{i_k}$ of $T_r(\xi)$ and $\tau_2 = t_{j_1} \ldots t_{j_l}$ of $T_s(\xi)$, define $\tau_1 \oplus \tau_2$ to be the basis element of $T_{r+s}(\xi)$ given by $\tau_1 \oplus \tau_2 = t_{i_1} \ldots t_{i_k} t_{j_1+r} \ldots t_{j_l+r}$

Let for $i \in [n-1]$, let s_i denote the element of the symmetric group S_n written $(i, i+1)$ in cycle notation. Define a map $\theta : S_n \to T_n(2)$ by mapping s_i into $(t_i - 1)$ for every $i \in [n-1]$. It is easy to check that this induces a well defined homomorphism from S_n into the multiplicative monoid of $T_n(2)$. (see, for example, [1]) For every basis element τ of $T_n(2)$, define a map $f_\tau : S_n \to \mathbb{C}$ by sending σ to the coefficient of τ in the expansion of $\theta(\sigma)$ in the standard basis. The immanants $Imm_{f_\tau}(x)$ induced by the functions f_τ are called the *Temperly-Lieb immanants*.

In [3] and [4] Rhoades and Skandera show that the Temperly-Lieb immanants are totally, monomial, and Schur nonnegative and may be used to study positivity properties of linear combinations of products of matrix minors. In [2], Lam, Postnikov, and Pylavyskyy use these results to resolve several Schur positivity conjectures. In [3], Rhoades and Skandera also give some generalizations of results from linear algebra using these immanants. In this spirit, we give here a generalization of the Cayley-Hamilton theorem.

Let V be a n-dimensional vector space and let $T \in End(V)$. For any ordered basis γ of V and any basis element τ of $T_n(2)$, define the (τ, γ)-*polynomial* to be the polynomial $g_{(\tau,\gamma)}(X) \in \mathbb{C}[X]$ given by

$$g_{\tau,\gamma}(X) = Imm_{f_\tau}(I_n X - [T]_\gamma),$$

where I_n is the $n \times n$ identity matrix. Let β be a rational canonical basis for V with invariant factor degrees $d_1 \leq d_2 \leq \ldots \leq d_k$. For $j \in [k]$, call an ordered basis γ of V (β, j)-*respecting* if the matrix $[T]_\gamma$ is of the form $diag(A, B, C)$, where A is a $d_1 + \ldots + d_{(j-1)}$ matrix and $B = [T|span(\beta_j)]_{\beta_j}$, where β_j is the subset of β corresponding to the j^{th} invariant factor.

We are now ready to state our result.

Theorem. *Let V be a finite dimensional $\mathbb{C}$-vector space and let $T \in End(V)$. Let β be a rational canonical basis for V and let the invariant factor degrees of T be $d_1 \leq d_2 \leq \ldots \leq d_k$. Let $j \in [k]$ and let γ be a $(\beta,$ j$)$-respecting ordered basis of V. Set $s = d_1 + \cdots + d_{(j-1)}$ and $r = d_{(j+1)} + \cdots + d_k$. If τ_1 and τ_2 are basis elements of $T_s(2)$ and $T_r(2)$, respectively, then*

$$rank(g_{\tau_1 \oplus 1 \oplus \tau_2, \gamma}(T)) \leq r.$$

Proof. Define a $\mathbb{C}[X]$-module structure on V by linearly extending the action

$$X \cdot v = T(v) \text{ for all } v \in V.$$

This makes V into a module over a Principal Ideal Domain. Since V is finite dimensional, we have the following isomorphism of $\mathbb{C}[X]$-modules:

$$V \cong \bigoplus_{i=1}^{k} \mathbb{C}[X]/(p_i(X)),$$

where $p_1 | p_2 | \ldots | p_k$. Recall that the polynomials $p_1(X), \ldots, p_k(X)$ are the invariant factors of T. We now compute the $(\tau_1 \oplus\ 1 \oplus \tau_2, \gamma)$-polynomial of T.

Since γ is $(\beta,$ j$)$-respecting, the matrix $[T]_\gamma$ has the form $diag(A, B, C)$, were A is a square matrix of size s and B is the restriction of $[T]_\beta$ to its j^{th} diagonal block. By Proposition 3.15 of Rhoades and Skandera [3], we have that

$$\begin{aligned}
&g_{\tau_1 \oplus\ 1 \oplus \tau_2, \gamma}(X) = \\
&Imm_{f_{\tau_1 \oplus 1 \oplus \tau_2}}(IX - [T]_\gamma) = \\
&Imm_{f_{\tau_1}}(IX - A) Imm_{f_1}(IX - B) Imm_{f_{\tau_2}}(IX - C).
\end{aligned}$$

It is easy to check that for $\sigma \in S_n$, $f_1(\sigma) = (-1)^{inv(\sigma)}$. That is, $Imm_{f_1}(x) = det(x)$. So, the center factor in the above product is equal to $p_j(X)$, which implies that $g_{\tau_1 \oplus\ 1 \oplus \tau_2, \gamma}(X)$ lies in the ideal $(p_j(X))$. However, since we have the chain of divisibilities, $p_1|p_2|\dots|p_k$, $g_{\tau_1 \oplus\ 1 \oplus \tau_2, \gamma}(X)$ also lies in $(p_i(X))$ for every $i < k$. This implies that $g_{\tau_1 \oplus\ 1 \oplus \tau_2, \gamma}(T)$ kills every vector in the subspaces of V corresponding to $\mathbb{C}[X]/(p_i(X))$ for every $i \leq k$. The desired inequality follows.

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References

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