Random Strict Partitions and Pfaffian

Sho Matsumoto

Abstract

The shifted Schur measure is a measure on the set of all strict partitions, which is defined by Schur Q-functions. We study distributions of parts of strict partitions. We prove that the correlation function of the measure is given by a Pfaffian in two ways. In the first way, we use commutation relations of operators on an exterior algebra. In the second way, the idea of random point processes is used. As an application, we prove that limit distributions of parts of random strict partitions with respect to specialized shifted Schur measures are given by the Airy ensemble.
2000 Mathematics Subject Classification : 60C05, 05E05,
Key Words : Schur Q-function, strict partition, shifted Schur measure, Pfaffian, Pfaffian point process.

1 Introduction

For a partition (or equivalently, a Young diagram) λ, we denote by f^λ the number of standard Young tableaux of shape λ. The Plancherel measure for the symmetric group $\mathfrak{S}_N$ assigns to each partition λ of N the probability

$$\mathrm{P}_{\mathrm{Plan},N}(\lambda) = \frac{(f^\lambda)^2}{N!}.$$

It is closely related to Ulam's problem for the length $\ell(\pi)$ of the longest increasing subsequence of a random permutation π with respect to the uniform measure $\mathrm{P}_{\mathrm{uniform},N}$ on $\mathfrak{S}_N$, see the survey [AD]. Namely, via the Robinson correspondence (see e.g. [S]), we have

$$\mathrm{P}_{\mathrm{uniform},N}(\{\pi \in \mathfrak{S}_N \ : \ \ell(\pi) = h\}) = \mathrm{P}_{\mathrm{Plan},N}(\{\lambda \in \mathcal{P}_N \ : \ \lambda_1 = h\}),$$

where $\mathcal{P}_N$ is the set of all partitions of N and λ_1 is the largest part of a partition $\lambda = (\lambda_1, \lambda_2, \dots)$. To other parts λ_j, we can also give combinatorial sense related with increasing sequences.

In order to see distributions of λ_j, the correlation function of the poissonized Plancherel measure is calculated in [BOO]. This correlation function is expressed as a determinant. Via the determinantal expression, it is proved in [BOO, Jo2, O1] (see also [BDJ, Jo1, O3, R]) that, as $N \to \infty$, limit distributions of scaled λ_j are described as the Airy ensemble (see §5). In particular, the limit distribution of λ_1 is expressed as the Tracy-Widom distribution F_{GUE} for the Gaussian unitary ensemble.

The Schur measure is the measure on all partitions, which gives each partition λ the probability $s_\lambda(\mathbf{x})s_\lambda(\mathbf{y})$. Here $s_\lambda(\mathbf{x})$ (resp. $s_\lambda(\mathbf{y})$) is the Schur function corresponding to λ in variables $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \dots)$ (resp. $\mathbf{y} = (\mathbf{y}_1, \mathbf{y}_2, \dots)$). The poissonized Plancherel measure is obtained as a specialization of the Schur measure. In [O2], the correlation function of the Schur measure is calculated and expressed as a determinant as similar as that of the poissonized Plancherel measure is.

In this note, we study random strict partitions. A strict partition is a partition with distinct parts. The shifted Schur measure, introduced in [TW2], is a measure on the set of all strict partitions, which is defined by Schur Q-functions instead of Schur functions (see Definition 1). We prove that, with respect to the shifted Schur measure, the correlation function of random variables $\lambda_1, \lambda_2, \dots$ is expressed by a Pfaffian (see Theorem 1). The Pfaffian of a $2m$ by $2m$ skew-symmetric matrix $B = (b_{ij})_{1 \le i,j \le 2m}$ is defined by

$$\mathrm{pf}(B) = \sum_{\sigma \in \mathfrak{F}_{2m}} \mathrm{sgn}(\sigma) \prod_{j=1}^{m} b_{\sigma(2j-1)\sigma(2j)},$$

where $\mathfrak{F}_{2m}$ is the subset of $\mathfrak{S}_{2m}$ given by

$$\mathfrak{F}_{2m} = \{\sigma = (\sigma(1), \sigma(2), \dots, \sigma(2m)) \in \mathfrak{S}_{2m} \ : \\ \sigma(2j-1) < \sigma(2j) \ (1 \le j \le m), \ \sigma(1) < \sigma(3) < \cdots < \sigma(2m-1)\}.$$

The correlation function is calculated in two ways. First, it is obtained via a representation of a Heisenberg algebra on an exterior algebra. We express the correlation function as a matrix element of an operator on the exterior algebra by using annihilation and creation operators. This idea is used by Okounkov [O2] for the Schur measure. Second, it is a more direct way and we use the idea in [BR]. Since the Schur Q-function has a Pfaffian expression, we can regard the shifted Schur measure as a Pfaffian point process on the set of all non-negative integers. Then our problem is translated into the problem to calculate the inverse of a matrix explicitly. We state the outlines of these proofs in §3 and §4, respectively.

Further, we are interested in limit distributions of λ_j. We see that a limit theorem of the shifted Schur measure is also given by using the Airy ensemble as same as the Schur measure and the Plancherel measure (see Theorem 9). The special case is closely related to the length of the longest ascent pair for a permutation.

Throughout the paper, we denote the set of all positive integers and non-negative integers by $\mathbb{Z}_{>0}$ and $\mathbb{Z}_{\ge 0}$, respectively.

2 Shifted Schur measure

Let $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \dots)$ and $\mathbf{y} = (\mathbf{y}_1, \mathbf{y}_2, \dots)$ be variables. Let $\mathcal{D}$ be the set of all strict partitions;

$$\mathcal{D} = \{\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l) \ : \ l \ge 0, \ \lambda_j \in \mathbb{Z}_{>0} \ (1 \le j \le l), \ \lambda_1 > \lambda_2 > \cdots > \lambda_l > 0\}$$

and $\ell(\lambda)$ be the length of a partition $\lambda \in \mathcal{D}$ (see [Mac]). Set

$$Q_{\mathbf{x}}(z) = \sum_{k=0}^{\infty} Q_{(k)}(\mathbf{x}) z^k = \prod_{j=1}^{\infty} \frac{1 + \mathbf{x}_j z}{1 - \mathbf{x}_j z} = \exp\left(\sum_{n=1,3,5,\dots} \frac{2p_n(\mathbf{x})}{n} z^n \right), \tag{2.1}$$

where $p_n(\mathbf{x}) = \mathbf{x}_1^n + \mathbf{x}_2^n + \cdots$. The Schur Q-function $Q_\lambda(\mathbf{x})$ associated with $\lambda \in \mathcal{D}$ is defined as the coefficient of $z_1^{\lambda_1} \cdots z_n^{\lambda_n}$ in the formal series expansion of

$$Q_{\mathbf{x}}(z_1) \cdots Q_{\mathbf{x}}(z_n) \prod_{1 \le i < j \le n} \frac{z_i - z_j}{z_i + z_j},$$

where $n \geq \ell(\lambda)$ and $\frac{z-w}{z+w} = 1 + 2\sum_{k=1}^{\infty}(-1)^k z^{-k} w^k$. They satisfy the Cauchy-type identity

$$\sum_{\lambda \in \mathcal{D}} 2^{-\ell(\lambda)} Q_\lambda(\mathbf{x}) Q_\lambda(\mathbf{y}) = \prod_{i,j=1}^{\infty} \frac{1+\mathbf{x}_i \mathbf{y}_j}{1-\mathbf{x}_i \mathbf{y}_j}. \tag{2.2}$$

One can define a probability measure on $\mathcal{D}$ via this identity.

Definition 1 ([TW2]). We define the shifted Schur measure by

$$\mathrm{P}_{\mathrm{SS}}(\lambda) = \left(\prod_{i,j=1}^{\infty} \frac{1-\mathbf{x}_i \mathbf{y}_j}{1+\mathbf{x}_i \mathbf{y}_j}\right) 2^{-\ell(\lambda)} Q_\lambda(\mathbf{x}) Q_\lambda(\mathbf{y}) \qquad \text{for } \lambda \in \mathcal{D}.$$

It follows from (2.2) that P_{SS} is a formal probability measure on $\mathcal{D}$; $\sum_{\lambda \in \mathcal{D}} \mathrm{P}_{\mathrm{SS}}(\lambda) = 1$.

Remark 1. The terminology "shifted Schur measure" is used in [TW2] because it is the measure on "shifted" Young diagrams (see §5). However, since the terminology "shifted Schur functions" are already been used in e.g. [OO], the name may confuse.

We are interested in distributions of parts λ_j of random strict partitions $\lambda \in \mathcal{D}$ with respect to the shifted Schur measure P_{SS}. In order to their distributions we study its correlation function. We identify each strict partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ with a finite set $\{\lambda_1, \lambda_2, \dots, \lambda_l\}$ of positive integers. In this sense, we write as $\lambda \supset X$ if the set $\{\lambda_1, \lambda_2, \dots, \lambda_l\}$ contains X for a finite set X of positive integers. Define the correlation function ρ_{SS} by

$$\rho_{\mathrm{SS}}(X) = \mathrm{P}_{\mathrm{SS}}\left(\{\lambda \in \mathcal{D} \ : \ \lambda \supset X\}\right) \qquad \text{for any finite subset } X \subset \mathbb{Z}_{>0}.$$

The following theorem is our main result.

Theorem 1 ([M1, M2]). *For any finite subset $X = \{x_1, x_2, \dots, x_N\} \subset \mathbb{Z}_{>0}$, we have*

$$\rho_{\mathrm{SS}}(X) = \mathrm{pf}(\mathcal{K}(x_i, x_j))_{1 \leq i,j \leq N},$$

where $\mathcal{K}(r,s)$ is a 2 by 2 matrix given by

$$\mathcal{K}(r,s) = \begin{pmatrix} \mathcal{K}_{00}(r,s) & \mathcal{K}_{01}(r,s) \\ \mathcal{K}_{10}(r,s) & \mathcal{K}_{11}(r,s) \end{pmatrix} \qquad \textit{for } r,s \in \mathbb{Z}_{>0}.$$

The each entry is given as the coefficient of a formal power series as follows:

$$\mathcal{K}_{00}(r,s) = -\mathcal{K}_{00}(s,r) = \frac{1}{2}[z^r w^s] \frac{Q_{\mathbf{x}}(z) Q_{\mathbf{x}}(w)}{Q_{\mathbf{y}}(z^{-1}) Q_{\mathbf{y}}(w^{-1})} \frac{z-w}{z+w}, \qquad \textit{if } r < s,$$

where $\frac{z-w}{z+w} = 1 + 2\sum_{k=1}^{\infty}(-1)^k z^{-k} w^k$ and $[z^r w^s]$ stands for the coefficient of $z^r w^s$, and $\mathcal{K}_{00}(r,r) = 0$.

$$\mathcal{K}_{01}(r,s) = -\mathcal{K}_{10}(s,r) = \frac{1}{2}[z^r w^s] \frac{Q_{\mathbf{x}}(z) Q_{\mathbf{y}}(w)}{Q_{\mathbf{y}}(z^{-1}) Q_{\mathbf{x}}(w^{-1})} \frac{zw+1}{zw-1}, \qquad \textit{for any } r \textit{ and } s,$$

where $\frac{zw+1}{zw-1} = 1 + 2\sum_{k=1}^{\infty} z^{-k} w^{-k}$.

$$\mathcal{K}_{11}(r,s) = -\mathcal{K}_{11}(s,r) = \frac{1}{2}[z^r w^s] \frac{Q_{\mathbf{y}}(z) Q_{\mathbf{y}}(w)}{Q_{\mathbf{x}}(z^{-1}) Q_{\mathbf{x}}(w^{-1})} \frac{w-z}{w+z}, \qquad \textit{if } r < s,$$

where $\frac{w-z}{w+z} = 1 + 2\sum_{k=1}^{\infty}(-1)^k z^k w^{-k}$, and $\mathcal{K}_{11}(r,r) = 0$.

This theorem is a generalization of the Cauchy-type identity (2.2)

$$\sum_{\lambda\in\mathcal{D},\ \lambda\supset X} 2^{-\ell(\lambda)} Q_\lambda(\mathbf{x})Q_\lambda(\mathbf{y}) = \left(\prod_{i,j=1}^{\infty} \frac{1+\mathbf{x}_i\mathbf{y}_j}{1-\mathbf{x}_i\mathbf{y}_j}\right)\cdot \mathrm{pf}(\mathcal{K}(x_i,x_j))_{1\le i,j\le N}, \tag{2.3}$$

where $X=\{x_1,\dots,x_N\}\subset\mathbb{Z}_{>0}$. The formula (2.3) is reduced to (2.2) if $X=\emptyset$.

Example 1. Let $X=\{x\}$. Then Theorem 1 says

$$\sum_{\lambda\in\mathcal{D},\ \lambda\ni x} 2^{-\ell(\lambda)} Q_\lambda(\mathbf{x})Q_\lambda(\mathbf{y}) = \left(\prod_{i,j=1}^{\infty} \frac{1+\mathbf{x}_i\mathbf{y}_j}{1-\mathbf{x}_i\mathbf{y}_j}\right)\cdot \mathcal{K}_{01}(x,x)$$

summed over all strict partitions containing x, where $\mathcal{K}_{01}(r,s)$ is given by

$$\mathcal{K}_{01}(r,s)=\frac{1}{2}\mathcal{J}_r(\mathbf{x},\mathbf{y})\mathcal{J}_s(\mathbf{y},\mathbf{x})+\sum_{n=1}^{\infty}\mathcal{J}_{r+n}(\mathbf{x},\mathbf{y})\mathcal{J}_{s+n}(\mathbf{y},\mathbf{x})$$

with

$$\mathcal{J}_r(\mathbf{x},\mathbf{y})=[z^r]\frac{Q_{\mathbf{x}}(z)}{Q_{\mathbf{y}}(z^{-1})}=\sum_{k=0}^{\infty}(-1)^k Q_{(r+k)}(\mathbf{x})Q_{(k)}(\mathbf{y}).$$

As a corollary of Theorem 1, we can obtain the distribution of the largest part λ_1 of $\lambda\in\mathcal{D}$.

Corollary 2. *For a positive integer h, we have*

$$\mathrm{P}_{\mathrm{SS}}(\lambda_1<h)=\sum_{\lambda_1<h}\mathrm{P}_{\mathrm{SS}}(\lambda)=\mathrm{pf}(J-\mathcal{K})_{\{h,h+1,\dots\}}.$$

Here $\mathrm{pf}(J-\mathcal{K})_{\{h,h+1,\dots,\}}$ *is the Fredholm pfaffian for the kernel $\mathcal{K}$ on $\{h,h+1,\dots\}$ defined by*

$$\mathrm{pf}(J-\mathcal{K})_{\{h,h+1,\dots,\}}=\sum_{n=0}^{\infty}(-1)^n\sum_{h\le x_1<x_2<\cdots<x_n}\mathrm{pf}(\mathcal{K}(x_i,x_j))_{1\le i,j\le n}.$$

3 First Proof of Theorem 1

We state the outline of the first proof of Theorem 1, obtained in [M1].

Let V be a module on $\mathbb{C}[\mathbf{x}_1,\mathbf{x}_2,\dots,\mathbf{y}_1,\mathbf{y}_2,\dots]$ spanned by $\boldsymbol{e}_k$ $(k=1,2,\dots)$. The exterior algebra $\bigwedge V$ is spanned by vectors

$$\boldsymbol{v}_\lambda=\boldsymbol{e}_{\lambda_1}\wedge\boldsymbol{e}_{\lambda_2}\wedge\cdots\wedge\boldsymbol{e}_{\lambda_\ell},$$

where $\lambda=(\lambda_1,\dots,\lambda_\ell)\in\mathcal{D}$ $(\lambda_1>\cdots>\lambda_\ell\ge 1)$. In particular, we have $\boldsymbol{v}_\emptyset=1$ for the empty partition $\emptyset$. We give $\bigwedge V$ the inner product

$$\langle\boldsymbol{v}_\lambda,\boldsymbol{v}_\mu\rangle=\delta_{\lambda,\mu}2^{-\ell(\lambda)}\qquad\text{for }\lambda,\mu\in\mathcal{D}.$$

Putting $\boldsymbol{e}_k^\vee=2\boldsymbol{e}_k$ and $\boldsymbol{v}_\lambda^\vee=\boldsymbol{e}_{\lambda_1}^\vee\wedge\cdots\wedge\boldsymbol{e}_{\lambda_\ell}^\vee=2^\ell\boldsymbol{v}_\lambda$, the bases $(\boldsymbol{v}_\lambda)_{\lambda\in\mathcal{D}}$ and $(\boldsymbol{v}_\lambda^\vee)_{\lambda\in\mathcal{D}}$ are dual to each other. We define the operator ψ_k on $\bigwedge V$ by

$$\psi_k\boldsymbol{v}_\lambda=\boldsymbol{e}_k\wedge\boldsymbol{v}_\lambda$$

for $k \geq 1$ and let ψ_k^* be the adjoint operator of ψ_k with respect to the inner product defined above. The operator ψ_k^* is then explicitly given by

$$\psi_k^* \boldsymbol{v}_\lambda = \sum_{i=1}^{\ell(\lambda)} \frac{(-1)^{i-1}}{2} \delta_{k,\lambda_i} \boldsymbol{e}_{\lambda_1} \wedge \cdots \wedge \widehat{\boldsymbol{e}_{\lambda_i}} \wedge \cdots \wedge \boldsymbol{e}_{\lambda_\ell},$$

where $\widehat{\boldsymbol{e}_k}$ means that $\boldsymbol{e}_k$ is omitted. Also we define the self-adjoint operator S by $S\boldsymbol{v}_\lambda = (-1)^{\ell(\lambda)}\boldsymbol{v}_\lambda$ for any $\lambda \in \mathcal{D}$. Put

$$\tilde{\psi}_k = \begin{cases} \psi_k, & \text{for} \quad k \geq 1, \\ S/2, & \text{for} \quad k = 0, \\ (-1)^k \psi_{-k}^*, & \text{for} \quad k \leq -1. \end{cases}$$

Then they satisfy the following commutation relation

$$\tilde{\psi}_i \tilde{\psi}_j + \tilde{\psi}_j \tilde{\psi}_i = \frac{(-1)^{|i|}}{2} \delta_{i+j,0} \qquad \text{for } i, j \in \mathbb{Z}$$

and give a projection

$$2\psi_k \psi_k^* \boldsymbol{v}_\lambda = \begin{cases} \boldsymbol{v}_\lambda, & \text{if } k \in \{\lambda_1, \dots, \lambda_l\}, \\ 0, & \text{otherwise} \end{cases} \tag{3.1}$$

for $k \geq 1$ and $\lambda = (\lambda_1, \dots, \lambda_l) \in \mathcal{D}$. Therefore $(\prod_{k \in X} 2\psi_k \psi_k^*)\boldsymbol{v}_\lambda$ is equal to $\boldsymbol{v}_\lambda$ if $X \subset \lambda$, or to 0 otherwise.

Define $\alpha_n = \sum_{k \in \mathbb{Z}} (-1)^k \tilde{\psi}_{k-n} \tilde{\psi}_{-k}$ for any odd integer $n \in 2\mathbb{Z} + 1$. Then they satisfy the Heisenberg relation

$$[\alpha_n, \alpha_m] = \alpha_n \alpha_m - \alpha_m \alpha_n = \frac{n}{2} \delta_{n+m,0}. \tag{3.2}$$

Put

$$\Gamma_\pm(\mathbf{x}) = \exp\left(\sum_{n=1,3,5,\dots} \frac{2p_n(\mathbf{x})}{n} \alpha_{\pm n} \right).$$

It is not hard to obtain the following lemma from commutation relations above.

Lemma 3. *We have*

$$\Gamma_+(\mathbf{x})\boldsymbol{v}_\emptyset = \boldsymbol{v}_\emptyset, \quad (\Gamma_\pm(\mathbf{x}))^* = \Gamma_\mp(\mathbf{x}), \quad \Gamma_+(\mathbf{x})\Gamma_-(\mathbf{y}) = \left(\prod_{i,j=1}^{\infty} \frac{1 + \mathbf{x}_i \mathbf{y}_j}{1 - \mathbf{x}_j \mathbf{y}_j} \right) \Gamma_-(\mathbf{y})\Gamma_+(\mathbf{x}).$$

Further, when we put $\psi(z) = \sum_{k \in \mathbb{Z}} z^k \tilde{\psi}_k$, *we have*

$$[\alpha_n, \psi(z)] = z^n \psi(z), \quad \langle 4\psi(z)\psi(w)\boldsymbol{v}_\emptyset, \boldsymbol{v}_\emptyset \rangle = \frac{z - w}{z + w}, \quad \Gamma_\pm(\mathbf{x})\psi(z) = Q_{\mathbf{x}}(z^{\pm 1})\psi(z)\Gamma_\pm(\mathbf{x}).$$

Using the lemma, we can express the Schur Q-function $Q_\lambda(\mathbf{x})$ as the matrix element of $\Gamma_-(\mathbf{x})$ as follows.

Proposition 4. *For each $\lambda \in \mathcal{D}$, we have $\langle \Gamma_-(\mathbf{x})\boldsymbol{v}_\emptyset, \boldsymbol{v}_\lambda^\vee \rangle = Q_\lambda(\mathbf{x})$.*

From (3.1), Lemma 3, and Proposition 4, the correlation function ρ_{SS} of the shifted Schur measure is expressed as

$$\begin{aligned}\rho_{\mathrm{SS}}(X) &= \prod_{i,j=1}^{\infty}\left(\frac{1-\mathbf{x}_i\mathbf{y}_j}{1+\mathbf{x}_i\mathbf{y}_j}\right)\sum_{\lambda\supset X} 2^{-\ell(\lambda)} Q_\lambda(\mathbf{x})Q_\lambda(\mathbf{y}) \\ &= \prod_{i,j=1}^{\infty}\left(\frac{1-\mathbf{x}_i\mathbf{y}_j}{1+\mathbf{x}_i\mathbf{y}_j}\right)\left\langle \Gamma_+(\mathbf{x})\left(\prod_{k\in X} 2\psi_k\psi_k^*\right)\Gamma_-(\mathbf{y})\boldsymbol{v}_\emptyset, \boldsymbol{v}_\emptyset\right\rangle \\ &= \left\langle\left(\prod_{k\in X} 2\Psi_k\Psi_k^*\right)\boldsymbol{v}_\emptyset, \boldsymbol{v}_\emptyset\right\rangle\end{aligned}$$

with $\Psi_k = G\psi_k G^{-1}$, $\Psi_k^* = G\psi_k^* G^{-1}$, and $G = \Gamma_+(\mathbf{x})\Gamma_-(\mathbf{y})^{-1}$. Now we obtain

Proposition 5. *We have $\rho_{\mathrm{SS}}(X) = \mathrm{pf}(\widetilde{\mathcal{K}}(x,y))_{x,y\in X}$ with*

$$\widetilde{\mathcal{K}}(x,y) = \begin{pmatrix}\widetilde{\mathcal{K}}_{00}(x,y) & \widetilde{\mathcal{K}}_{01}(x,y) \\ \widetilde{\mathcal{K}}_{10}(x,y) & \widetilde{\mathcal{K}}_{11}(x,y)\end{pmatrix}.$$

Here we put $\widetilde{\mathcal{K}}_{00}(x,y) = \langle 2\Psi_x\Psi_y\boldsymbol{v}_\emptyset, \boldsymbol{v}_\emptyset\rangle$, $\widetilde{\mathcal{K}}_{01}(x,y) = -\widetilde{\mathcal{K}}_{10}(y,x) = \langle 2\Psi_x\Psi_y^*\boldsymbol{v}_\emptyset, \boldsymbol{v}_\emptyset\rangle$, and $\widetilde{\mathcal{K}}_{11}(x,y) = \langle 2\Psi_x^*\Psi_y^*\boldsymbol{v}_\emptyset, \boldsymbol{v}_\emptyset\rangle$ for $x,y \in \mathbb{Z}_{>0}$.

Finally, by Lemma 3, we can prove $\mathcal{K}(x,y) = \widetilde{\mathcal{K}}(x,y)$ for all $x,y \in \mathbb{Z}_{>0}$. For example, putting $\Psi(z) = G\psi(z)G^{-1}$,

$$\widetilde{\mathcal{K}}_{00}(x,y) = [z^x w^y]\langle 2\Psi(z)\Psi(w)\boldsymbol{v}_\emptyset, \boldsymbol{v}_\emptyset\rangle = [z^x w^y]\frac{1}{2}\frac{Q_{\mathbf{x}}(z)Q_{\mathbf{x}}(w)}{Q_{\mathbf{y}}(z^{-1})Q_{\mathbf{y}}(w^{-1})}\frac{z-w}{z+w} = \mathcal{K}_{00}(x,y).$$

It completes the proof of Theorem 1.

Remark 2. The discussion of this section is very related to the fermion Fock space and vertex operators. Proposition 5 is essentially obtained from the fermion Wick formula, see e.g. [Ji] and [Y].

4 Second Proof of Theorem 1

In this section, we give another proof of Theorem 1, which is obtained in [M2], via a random point process. We recall some fundamental facts of the Pfaffian point process formulated in [BR]. Let $\mathfrak{X}$ be a countable set. Let L be a map

$$L : \mathfrak{X}\times\mathfrak{X} \to \mathfrak{gl}_2(\mathbb{C})\ ;\ (x,y) \mapsto L(x,y) = \begin{pmatrix} L_{00}(x,y) & L_{01}(x,y) \\ L_{10}(x,y) & L_{11}(x,y)\end{pmatrix}$$

such that $L_{ij}(x,y) = -L_{ji}(y,x)$ for any $i,j \in \{0,1\}$ and $x,y \in \mathfrak{X}$. Such L is called a skew-symmetric matrix kernel on $\mathfrak{X}$, see [R, So]. We regard the map L as an operator on the Hilbert space $\ell^2(\mathfrak{X}) \oplus \ell^2(\mathfrak{X})$. Then L is a matrix whose blocks are indexed by elements in $\mathfrak{X}\times\mathfrak{X}$. For any finite subset $X = \{x_1, \cdots, x_n\} \subset \mathfrak{X}$, we denote by $L[X]$ the $2n$ by $2n$ skew-symmetric matrix $(L(x_i, x_j))_{1 \le i,j \le n}$. Let J be the skew-symmetric matrix kernel determined by

$$J(x, y) = \delta_{x,y} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Let $\mathfrak{P}(\mathfrak{X})$ be the set of all finite subsets in $\mathfrak{X}$. We define the Pfaffian point process π_L on $\mathfrak{X}$ determined by L as the probability measure on $\mathfrak{P}(\mathfrak{X})$ given by

$$\pi(X) = \pi_L(X) = \frac{\mathrm{pf}(L[X])}{\mathrm{pf}(J + L)} \qquad \text{for } X \in \mathfrak{P}(\mathfrak{X}).$$

Then its correlation function is expressed as $\rho_L(X) = \sum_{Y \in \mathfrak{P}(\mathfrak{X}),\ Y \supset X} \pi(Y) = \mathrm{pf}(K[X])$, where $K = J + (J + L)^{-1}$.

More generally, let $\mathfrak{Y}$ be a subset in $\mathfrak{X}$ such that $\mathfrak{Y}^c = \mathfrak{X} \setminus \mathfrak{Y}$ is finite. Then we can define the conditional Pfaffian point process on $\mathfrak{Y}$ by

$$\pi_{L,\mathfrak{Y}}(X) = \frac{\mathrm{pf}(L[X \cup \mathfrak{Y}^c])}{\mathrm{pf}(J[\mathfrak{Y}] + L)} \qquad \text{for } X \in \mathfrak{P}(\mathfrak{Y}). \tag{4.1}$$

Here we identify $J[\mathfrak{Y}]$ with the block matrix $\begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix}$, where the blocks correspond to the partition $\mathfrak{X} = \mathfrak{Y} \sqcup \mathfrak{Y}^c$. The correlation function is given by $\rho_{L,\mathfrak{Y}}(X) = \sum_{Y \in \mathfrak{P}(\mathfrak{Y}),\ Y \supset X} \pi_{L,\mathfrak{Y}}(Y) = \mathrm{pf}(K[X])$ for $X \in \mathfrak{P}(\mathfrak{Y})$, where

$$K = J[\mathfrak{Y}] + (J[\mathfrak{Y}] + L)^{-1}\Big|_{\mathfrak{Y} \times \mathfrak{Y}}. \tag{4.2}$$

The shifted Schur measure is regarded as a conditional Pfaffian point process as follows. We may assume that the number of variables $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{y}_1, \mathbf{y}_2, \ldots$ is finite. Let $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$ and $\mathbf{y} = (\mathbf{y}_1, \ldots, \mathbf{y}_n)$, where n is even. We define the bijective map ϕ from $\mathcal{D}$ to $\mathfrak{P}^{\mathrm{even}}(\mathbb{Z}_{\ge 0}) = \{X \in \mathfrak{P}(\mathbb{Z}_{\ge 0}) \ : \ \#X \text{ is even}\}$ by

$$\phi(\lambda) = \begin{cases} \{\lambda_1, \ldots, \lambda_{\ell(\lambda)}\}, & \text{if } \ell(\lambda) \text{ is even}, \\ \{\lambda_1, \ldots, \lambda_{\ell(\lambda)}, 0\}, & \text{if } \ell(\lambda) \text{ is odd}. \end{cases}$$

The following proposition is proved from the fact that the Schur Q-function is expressed as the quotient of Pfaffians (see [N], also [Mac, III-8]).

Proposition 6. *Define a skew-symmetric matrix kernel L on $\mathfrak{X} = \{1, 2, \ldots, n\} \sqcup \mathfrak{Y}$ by*

$$L = \begin{pmatrix} \mathcal{V} & \mathcal{W}\eta^{-\frac{1}{2}} \\ -\eta^{-\frac{1}{2}}\,{}^t\mathcal{W} & O \end{pmatrix},$$

where $\mathfrak{Y} = \mathbb{Z}_{\ge 0}$, $\mathcal{V} = (\mathcal{V}(i,j))_{1 \le i,j \le n}$ and $\mathcal{W} = (\mathcal{W}(i,r))_{1 \le i \le n,\ r \in \mathbb{Z}_{\ge 0}}$. Their blocks are given by

$$\mathcal{V}(i,j) = \begin{pmatrix} -\frac{\mathbf{x}_i - \mathbf{x}_j}{\mathbf{x}_i + \mathbf{x}_j} & 0 \\ 0 & \frac{\mathbf{y}_i - \mathbf{y}_j}{\mathbf{y}_i + \mathbf{y}_j} \end{pmatrix}, \qquad \mathcal{W}(i,r) = \begin{pmatrix} -\mathbf{x}_i^r & 0 \\ 0 & \mathbf{y}_i^r \end{pmatrix}.$$

Further η is the matrix whose block is given by

$$\eta(r,s)=\delta_{rs}\begin{pmatrix}\eta(r) & 0\\ 0 & \eta(r)\end{pmatrix}\qquad \text{for } r,s\in\mathbb{Z}_{\geq}0,$$

where $\eta(r)$ is equal to 1 if $r=0$, or to $\frac{1}{2}$ if $r\geq 1$. Then the conditional Pfaffian point process on $\mathfrak{Y}$ is agree with the shifted Schur measure on $\mathcal{D}$ via the bijection ϕ. Namely,

$$\pi_{L,\mathfrak{Y}}(\phi(\lambda))=\frac{\mathrm{pf}(L[\{1,\dots,n\}\sqcup\phi(\lambda)]}{\mathrm{pf}(J[\mathfrak{Y}]+L)}=\mathrm{P}_{\mathrm{SS}}(\lambda)$$

for any $\lambda\in\mathcal{D}$.

By (4.2), we have to obtain the explicit expression of the correlation kernel $K=J[\mathfrak{Y}]+(J[\mathfrak{Y}]+L)^{-1}\Big|_{\mathfrak{Y}\times\mathfrak{Y}}$. For that purpose, we employ the following lemma.

Lemma 7. *We have*

$$\begin{pmatrix}A & B\\ C & D\end{pmatrix}^{-1}=\begin{pmatrix}-\mathcal{M}^{-1} & \mathcal{M}^{-1}BD^{-1}\\ D^{-1}C\mathcal{M}^{-1} & D^{-1}-D^{-1}C\mathcal{M}^{-1}BD^{-1}\end{pmatrix},$$

where $\mathcal{M}=BD^{-1}C-A$, if D and $\mathcal{M}$ are invertible.

By this lemma, the kernel $K=J[\mathfrak{Y}]+(J[\mathfrak{Y}]+L)^{-1}\Big|_{\mathfrak{Y}\times\mathfrak{Y}}$ is equal to $J[\mathfrak{Y}]\eta^{-\frac{1}{2}}{}^t\mathcal{W}\mathcal{M}^{-1}\mathcal{W}\eta^{-\frac{1}{2}}J[\mathfrak{Y}]$, with $\mathcal{M}=\mathcal{W}\eta^{-\frac{1}{2}}J[\mathfrak{Y}]\eta^{-\frac{1}{2}}{}^t\mathcal{W}-\mathcal{V}$. We may replace K with $-\eta^{-\frac{1}{2}}{}^t\mathcal{W}\mathcal{M}^{-1}\mathcal{W}\eta^{-\frac{1}{2}}$ because $\mathrm{pf}(-JBJ)=\mathrm{pf}(B)$ for a skew-symmetric matrix B. The explicit expression of entries of the inverse $\mathcal{M}^{-1}$ is obtained by a linear algebraic discussion.

Proposition 8. *Write the skew-symmetric matrix kernel $\mathcal{M}^{-1}$ on $\{1,2,\dots,n\}$ as*

$$\mathcal{M}^{-1}(k,l)=\begin{pmatrix}\mathcal{M}^{-1}_{00}(k,l) & \mathcal{M}^{-1}_{01}(k,l)\\ \mathcal{M}^{-1}_{10}(k,l) & \mathcal{M}^{-1}_{11}(k,l)\end{pmatrix}\qquad \text{for } 1\leq k,l\leq n.$$

Then we have

$$\begin{aligned}
\mathcal{M}^{-1}_{00}(k,l)=&\prod_{j=1}^{n}\left(\frac{1-\mathbf{x}_k\mathbf{y}_j}{1+\mathbf{x}_k\mathbf{y}_j}\frac{1-\mathbf{x}_l\mathbf{y}_j}{1+\mathbf{x}_l\mathbf{y}_j}\right)\prod_{\substack{1\leq i\leq n,\\ i\neq k}}\left(\frac{\mathbf{x}_k+\mathbf{x}_i}{\mathbf{x}_k-\mathbf{x}_i}\right)\prod_{\substack{1\leq j\leq n,\\ j\neq l}}\left(\frac{\mathbf{x}_l+\mathbf{x}_j}{\mathbf{x}_l-\mathbf{x}_j}\right)\frac{\mathbf{x}_k-\mathbf{x}_l}{\mathbf{x}_k+\mathbf{x}_l};\\
\mathcal{M}^{-1}_{01}(k,l)=&-\mathcal{M}^{-1}_{10}(l,k)\\
=&\prod_{j=1}^{n}\left(\frac{1-\mathbf{x}_k\mathbf{y}_j}{1+\mathbf{x}_k\mathbf{y}_j}\frac{1-\mathbf{x}_j\mathbf{y}_l}{1+\mathbf{x}_j\mathbf{y}_l}\right)\prod_{\substack{1\leq i\leq n,\\ i\neq k}}\left(\frac{\mathbf{x}_k+\mathbf{x}_i}{\mathbf{x}_k-\mathbf{x}_i}\right)\prod_{\substack{1\leq j\leq n,\\ j\neq l}}\left(\frac{\mathbf{y}_l+\mathbf{y}_j}{\mathbf{y}_l-\mathbf{y}_j}\right)\frac{1+\mathbf{x}_k\mathbf{y}_l}{1-\mathbf{x}_k\mathbf{y}_l};\\
\mathcal{M}^{-1}_{11}(k,l)=&-\prod_{j=1}^{n}\left(\frac{1-\mathbf{x}_j\mathbf{y}_k}{1+\mathbf{x}_j\mathbf{y}_k}\frac{1-\mathbf{x}_j\mathbf{y}_l}{1+\mathbf{x}_j\mathbf{y}_l}\right)\prod_{\substack{1\leq i\leq n,\\ i\neq k}}\left(\frac{\mathbf{y}_k+\mathbf{y}_i}{\mathbf{y}_k-\mathbf{y}_i}\right)\prod_{\substack{1\leq j\leq n,\\ j\neq l}}\left(\frac{\mathbf{y}_l+\mathbf{y}_j}{\mathbf{y}_l-\mathbf{y}_j}\right)\frac{\mathbf{y}_k-\mathbf{y}_l}{\mathbf{y}_k+\mathbf{y}_l}.
\end{aligned}$$

Finally, we must prove $\mathcal{K}(r,s) = K(r,s)$. Now we assume $\mathbf{x}_1,\dots,\mathbf{x}_n,\mathbf{y}_1,\dots,\mathbf{y}_n$ are $2n$ distinct complex numbers in the unit open disc. Then by changing variables and the residue theorem we obtain

$$\begin{aligned}\mathcal{K}_{00}(r,s) =& \frac{1}{2}\frac{1}{(2\pi\sqrt{-1})^2}\iint_{|z|>|w|>1}\frac{Q_{\mathbf{x}}(z)Q_{\mathbf{x}}(w)}{Q_{\mathbf{y}}(z^{-1})Q_{\mathbf{y}}(w^{-1})}\frac{z-w}{z+w}\frac{dzdw}{z^{r+1}w^{s+1}}\\ =& -2\sum_{k,l=1}^{n}\mathbf{x}_k^r\mathbf{x}_l^s\prod_{j=1}^{n}\left(\frac{1-\mathbf{x}_k\mathbf{y}_j}{1+\mathbf{x}_k\mathbf{y}_j}\frac{1-\mathbf{x}_l\mathbf{y}_j}{1+\mathbf{x}_l\mathbf{y}_j}\right)\prod_{\substack{1\le i\le n,\\ i\ne k}}\left(\frac{\mathbf{x}_k+\mathbf{x}_i}{\mathbf{x}_k-\mathbf{x}_i}\right)\prod_{\substack{1\le j\le n,\\ j\ne l}}\left(\frac{\mathbf{x}_l+\mathbf{x}_j}{\mathbf{x}_l-\mathbf{x}_j}\right)\frac{\mathbf{x}_k-\mathbf{x}_l}{\mathbf{x}_k+\mathbf{x}_l}\\ =& -2\sum_{k,l=1}^{n}\mathbf{x}_k^r\mathcal{M}_{00}^{-1}(k,l)\mathbf{x}_l^s = K_{00}(r,s).\end{aligned}$$

Here the contour in the integral above is $\{z\ :\ |z|=r_1\}\times\{w\ :\ |w|=r_2\}$, where $1+\epsilon>r_1>r_2>1$ and $\epsilon>0$ is very small. Similarly, we can prove $\mathcal{K}_{01}(r,s)=K_{01}(r,s)$ and $\mathcal{K}_{11}(r,s)=K_{11}(r,s)$. Though we have assumed that $\mathbf{x}_i$, $\mathbf{y}_j$ belong to the unit open disc, it is in fact unnecessary, see e.g. [BR]. It completes the proof of Theorem 1 again.

5 Limit Distribution

We study limit distributions of λ_j on special conditions.

We consider the random point process on $\mathbb{R}$ (see the Appendix in [BOO]) whose correlation functions $\rho_{\mathrm{Airy}}(X) = \mathrm{P}_{\mathrm{Airy}}(\{Y\subset\mathbb{R}\ :\ \#Y<\infty,\ X\subset Y\})$ are given by $\rho_{\mathrm{Airy}}(X) = \det(\mathcal{K}_{\mathrm{Airy}}(x_i,x_j))_{1\le i,j\le k}$ for any finite subset $X=\{x_1,\dots,x_k\}\subset\mathbb{R}$. Here $\mathcal{K}_{\mathrm{Airy}}$ is the Airy kernel defined by

$$\mathcal{K}_{\mathrm{Airy}}(x,y) = \int_0^\infty \mathrm{Ai}(x+z)\mathrm{Ai}(y+z)\mathrm{d}z,$$

where $\mathrm{Ai}(x)$ is the Airy function

$$\mathrm{Ai}(x) = \frac{1}{2\pi\sqrt{-1}}\int_{\infty e^{-\pi\sqrt{-1}/3}}^{\infty e^{\pi\sqrt{-1}/3}}\exp\left(\frac{z^3}{3}-xz\right)\mathrm{d}z.$$

Let $\zeta^{\mathrm{Ai}} = (\zeta_1^{\mathrm{Ai}} > \zeta_2^{\mathrm{Ai}} > \cdots)\in\mathbb{R}^\infty$ be its random configuration. The random variables ζ_j^{Ai} are called the Airy ensemble. It is known that the Airy ensemble describes the behavior of the scaled eigenvalues of a large hermitian matrix from the Gaussian unitary ensemble, see [TW1].

We consider the specializations of the shifted Schur measure satisfying the following analytic assumptions.

(0) *Let $\theta>0$. We specialize power-sum symmetric functions as $p_k(\mathbf{x}) = p_k(\mathbf{y}) = p_k^\theta$, where $k=1,3,5,\dots$ and $p_k^\theta\in\mathbb{R}$ satisfies $\lim_{\theta\to+\infty}p_k^\theta/\theta = d_k\ge 0$.*

(1) *There exists an $\epsilon=\epsilon(\theta)>0$ such that the power series $g^\theta(z) := 2\sum_{k\ge1:\mathrm{odd}}\frac{p_k^\theta}{k}z^k$ is holomorphic on $|z|<1+\epsilon$.*

(2) *Put $g(z) := 2\sum_{k\ge1:\mathrm{odd}}\frac{d_k}{k}z^k$. Then the series $g(1) = 2\sum_{k\ge1:\mathrm{odd}}\frac{d_k}{k}$ converges. Further $g(z)$ can be extended as a holomorphic function around $z=1$.*

Then we have the following theorem.

Theorem 9 ([M1]). *Let $\mathrm{P}^{\theta}_{\mathrm{SS}}$ be the shifted Schur measure obtained by the specialization such that satisfies assumptions (0), (1) and (2). Put $b_1 = 2g'(1)$ and $b_2 = g'''(1) + 3g''(1) + g'(1)$. Then, as $\theta \to \infty$, random variables*

$$\frac{\lambda_j - b_1\theta}{(b_2\theta)^{1/3}}, \qquad j = 1, 2, \dots,$$

converge to the Airy ensemble, in joint distribution.

This limit theorem is obtained in [TW2] for only λ_1 and a specialization $p_k^{\theta} = \theta\alpha^k$ with $0 < \alpha < 1$. We now give the simplest example of this theorem. Specialize as $p_k(\mathbf{x}) = p_k(\mathbf{y}) = \delta_{1,k}\sqrt{\frac{\xi}{2}}$ with $\xi > 0$. Then the shifted Schur measure provides (see [M1])

$$\mathrm{P}^{\xi}_{\mathrm{PSP}}(\lambda) = e^{-\xi}\xi^{|\lambda|}2^{|\lambda|-\ell(\lambda)}\left(\frac{g^{\lambda}}{|\lambda|!}\right)^2 \qquad \text{for } \lambda \in \mathcal{D}, \tag{5.1}$$

where $|\lambda|$ stands for the weight $|\lambda| = \sum_{j\geq 1}\lambda_j$, and g^{λ} is the number of the standard shifted tableaux of shifted shape $\mathrm{Sh}(\lambda)$. Here $\mathrm{Sh}(\lambda)$ is the shifted Young diagram associated with a strict partition λ, which is obtained by replacing the i-th row to the right by $i-1$ boxes for $i \geq 1$ from the Young diagram corresponding to λ. A standard shifted tableau T of shifted shape $\mathrm{Sh}(\lambda)$ is an assignment of $1, 2, \dots, |\lambda|$ to each box in $\mathrm{Sh}(\lambda)$ such that entries in T are increasing across rows and down columns. For example,

1	2	4	6
	3	5	8
		7	

is a standard shifted tableau of shape $\lambda = (4, 3, 1)$. Then we have the

Corollary 10. *With respect to the probability measure $\mathrm{P}^{\xi}_{\mathrm{PSP}}$ defined in (5.1), random variables*

$$\frac{\lambda_j - 2\sqrt{2\xi}}{(2\xi)^{\frac{1}{6}}}, \qquad j = 1, 2, \dots,$$

converge to the Airy ensemble as $\xi \to \infty$. In particular, the limit distribution of λ_1 is given by

$$\lim_{\xi\to\infty} \mathrm{P}^{\xi}_{\mathrm{PSP}}\left(\frac{\lambda_1 - 2\sqrt{2\xi}}{(2\xi)^{\frac{1}{6}}} < s\right) = \mathrm{P}_{\mathrm{Airy}}(\zeta_1 < s) =: F_{\mathrm{GUE}}(s) \qquad \textit{for } s \in \mathbb{R}.$$

The distribution function $F_{\mathrm{GUE}}(s)$ is called the Tracy-Widom distribution, see e.g. [BOO, Jo1, Jo2]. By Corollary 10, we have $\lambda_1 \sim 2\sqrt{2\xi}$ as $\xi \to \infty$. Now the largest part λ_1 describes the length of the longest ascent pair (see [HH]) for random permutations with respect to the uniform measure of symmetric groups. Therefore we give a solution of an analogue of Ulam's problem.

References

[AD] D. Aldous and P. Diaconis, Longest increasing subsequences: from patience sorting to the Baik-Deift-Johansson theorem, Bull. Amer. Math. Soc. **36** (1999), 413–432.

[BDJ] J. Baik, P. A. Deift, and K. Johansson, On the distribution of the length of the longest increasing subsequence in a random permutation, J. Amer. Math. Soc. **12** (1999), 1119–1178.

[BOO] A. Borodin, A. Okounkov, and G. Olshanski, Asymptotics of Plancherel measures for symmetric groups, J. Amer. Math. Soc. **13** (2000), 481–515.

[BR] A. Borodin and E. Rains, Eynard-Mehta theorem, Schur process, and their pfaffian analogs, math-ph/0409059.

[HH] P. N. Hoffman and J. F. Humphreys, Projective Representations of the Symmetric Groups: Q-Functions and Shifted Tableaux, Clarendon Press, Oxford, 1992.

[Ji] N. Jing, Vertex operators, symmetric functions, and the spin group Γ_n, J. Algebra **138** (1991), 340–398.

[Jo1] K. Johansson, Shape fluctuations and random matrices, Commun. Math. Phys. **209** (2000), 437–476.

[Jo2] ——, Discrete orthogonal polynomial ensembles and the Plancherel measure, Ann. of Math. (2) **153** (2001), 259–296.

[Mac] I. G. Macdonald, Symmetric Functions and Hall Polynomials, 2nd Edition, Oxford, 1995.

[M1] S. Matsumoto, Correlation functions of the shifted Schur measure, J. Math. Soc. Japan, To appear, math.CO/0312373.

[M2] ——, α-Pfaffian, pfaffian point process and shifted Schur measure, Linear Alg. Appl., To appear, math.CO/0411277.

[N] J. J. C. Nimmo, Hall-Littlewood symmetric functions and the BKP equation, J. Phys. (A) **23** (1990), 751–760.

[O1] A. Okounkov, Random matrices and random permutations, Internat. Math. Res. Notices (2000), 1043–1095.

[O2] ——, Infinite wedge and random partitions, Selecta Math. (N.S.) **7** (2001), 57–81.

[O3] ——, The uses of random partitions, math-ph/0309015.

[OO] A. Okounkov and G. Olshanski, Shifted Schur functions, St. Petersburg Math. J. **9** (1998), 239–300.

[R] E. M. Rains, Correlation functions for symmetrized increasing subsequences, math.CO/0006097.

[S] B. E. Sagan, The symmetric group: Representations, combinatorial algorithms, and symmetric functions, Wadsworth & Brooks / Cole, Pacific Grove, Calif., 1991.

[So] A. Soshnikov, Janossy densities. II. pfaffian ensemble, J. Stat. Phys. **113** (2003), 611–622.

[TW1] C. A. Tracy and H. Widom, Level-spacing distributions and the Airy kernel, Comm. Math. Phys. **159** (1994), 151–174.

[TW2] ——, A limit theorem for shifted Schur measures, Duke Math. J. **123** (2004), 171–208.

[Y] Y. You, DKP and MDKP hierarchy of soliton equations, Phys. D **50** (1991), 429–462.

Sho Matsumoto
Graduate School of Mathematics, Kyushu University.
Hakozaki Fukuoka 812-8581, Japan.
e-mail : shom@math.kyushu-u.ac.jp