ON SOME CLASSES OF PRUDENT WALKS

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ABSTRACT. In this paper we consider a class of walks introduced by Pascal Préa, that we call *prudent* walks: they are self-avoiding because they never try to walk towards points that they already visited.

We write functional equations for counting three classes of such prudent walks with respect to their length. The first one is algebraic and we give an object grammar decomposition that explains this fact directly. For the second one we obtain the algebraic generating function but in an indirect way. Whether the third class has an algebraic generating function remains an open problem.

RÉSUMÉ. Dans cet article nous considérons une classe de chemins introduits par Pascal Préa, que nous appelons les *chemins prudents*, qui sont auto-évitants parce qu'ils ne vont jamais dans la direction d'un point qu'ils ont déjà visité.

Nous écrivons des équations fonctionnelles pour compter trois classes de chemins de ce type en fonction de leur longueur. La première classe est algebrique et nous donnons une explication de ce résultat à l'aide d'une grammaire. Pour la seconde nous obtenons aussi l'algébricité mais de manière indirecte. Le problème de savoir si la troisième classe a une série génératrice algébrique reste ouvert.

1. PRUDENT WALKS

The term *walk* is used to denote a sequence of points $s_0, s_1, \ldots s_n$ in the plane $\mathbb{Z} \times \mathbb{Z}$. A couple (s_i, s_{i+1}) is said to be a *step* of the walk and the number n of steps is called the *length* of the walk. Given $s_i = (x, y)$ then (s_i, s_{i+1}) is:

- an *east* ($\rightarrow$) step if $s_{i+1} = (x+1, y)$
- a *west* ($\leftarrow$) step if $s_{i+1} = (x-1, y)$
- a *north* ($\uparrow$) step if $s_{i+1} = (x, y+1)$
- a *south* ($\downarrow$) step if $s_{i+1} = (x, y-1)$

From now on all the walks will be on the lattice, that is, they are made of east, west, north, and south steps only. We shall concentrate on some families of self-avoiding walks. A self-avoiding walk is a walk that cannot cross itself, i.e. it never visits two times the same point. Counting self-avoiding walks is a well-known open problem. See [5, 6] for some references. For this reason various subclasses of these walks have been introduced and counted. Here we consider a subclass of self-avoiding walks that we call the class of *prudent walks*. As we learned recently from Mireille Bousquet-Mélou, these walks were introduced by Pascal Préa in [7], where he obtained some recurrences for their enumeration.

Definition 1. *A* prudent walk *is a sequence of east, west, north, and south steps running from* $(0,0)$ *to* (n,m)*, with* $n, m \in \mathbb{Z}$*, defined in the following manner:*

- *The empty walk starting from* $(0,0)$ *and ending in* $(0,0)$ *is a prudent walk.*
- *A prudent walk is obtained from another prudent walk by attaching a new step at its end in such a manner that the extension of this step in the sense of its direction never encounters the walk itself.*

In particular this definition implies that a prudent walk is a self-avoiding walk.

Definition 2. *The* prudent box *of a prudent walk is the smallest rectangle including the walk. We remark that for particular walks this rectangle reduces to a line or also to a point in the case of the empty walk.*

In Figure 1 are given examples of prudent walks with their prudent box and of a non prudent walk.

Proposition 1. *The last point of a prudent walk is always on the border of the prudent box.*

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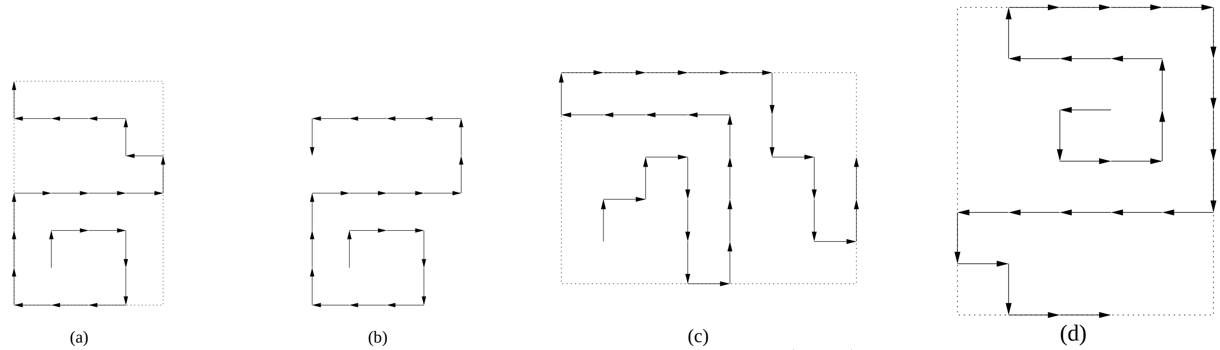


FIGURE 1. Some prudent walks with their prudent boxes (a,c,d) and a walk that is not prudent (b).

Remark that the converse is not true, as illustrated by walk (b) in Figure 1.

In this paper we write functional equations for the generating functions of several classes of prudent walks using the recursive definition of these walks. These equations are equivalent to the recurrences independently obtained by Pascal Préa. For some of these classes we are able to give explicit formulas for the generating functions. First, we deal with prudent walks that can only end on the right side or on the top of their prudent box. For this class, using the kernel method, [1, 2], we compute their generating function which is algebraic. In order to give a direct explanation of this fact we also find an algebraic decomposition [8] (or object grammar [3, 4]) for this class. Then we deal with another subclass of prudent walks, made of walks that can end on the top, right, or bottom side of their prudent box. We show that the generating function of these walks is also algebraic but this result is more complicated to derive. In particular we need several applications of the kernel method and at the moment we do not know a direct algebraic decomposition for this class. The paper is organized as follows: in Section 1 we introduce the first subclass of prudent walks and we give their generating function. In Section 2 we present an object grammar for this class. In Section 3 we introduce the second class of prudent walks and we count them. To conclude, in Section 4 we write a functional equation for the complete class of prudent walks and leave open the problem of knowing whether they have an algebraic generating function.

Given a walk w we define the following parameters (see Figure 2):

- $i(w)$ is the distance between the last point of w and the top of its prudent box;
- $j(w)$ is the distance between the last point of w and the bottom of its prudent box;
- $k(w)$ is the distance between the last point of w and the right side of its prudent box;
- $l(w)$ is the distance between the last point of w and the left side of its prudent box.

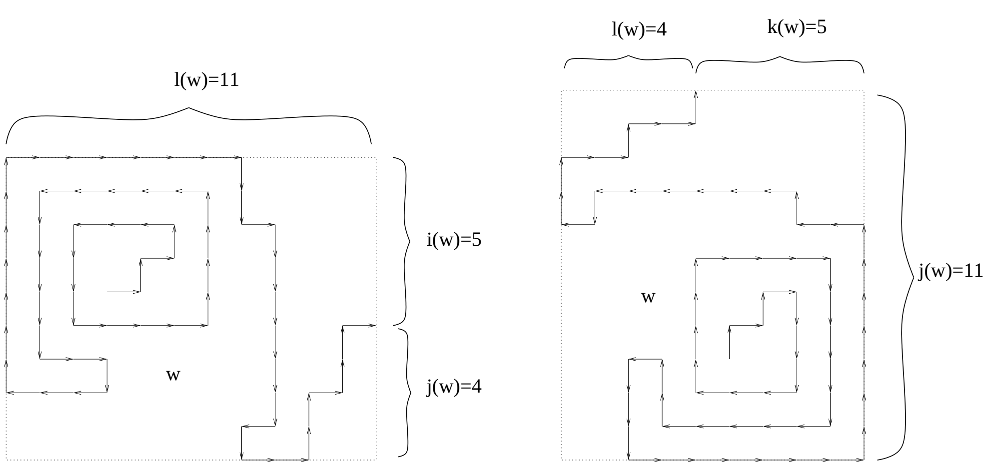


FIGURE 2. The parameters $i(w)$, $j(w)$, $k(w)$, $l(w)$.

Definition 3. *A prudent walk of the first type is a prudent walk avoiding the following subsequences of steps: a west step followed by a south step* $\overset{\leftarrow}{\downarrow}$ *and a south step followed by a west step* $\overset{\downarrow}{\leftarrow}$.

Let us denote by P^1 the class of prudent walks of the first type. See Figure 1(c) for an example.

Remark 1. *The set of prudent walks of the first type is symmetric with respect to the main diagonal. That is, the set is invariant by the following transformation:* $(\leftarrow) \mapsto (\downarrow)$, $(\downarrow) \mapsto (\leftarrow)$, $(\rightarrow) \mapsto (\uparrow)$, $(\uparrow) \mapsto (\rightarrow)$ *for the steps and Bottom* $\mapsto$ *Left, Right* $\mapsto$ *Top, Left* $\mapsto$ *Bottom, Top* $\mapsto$ *Right for the border of the prudent box.*

In order to count the number of walks of P^1 according to their length, the following proposition is useful:

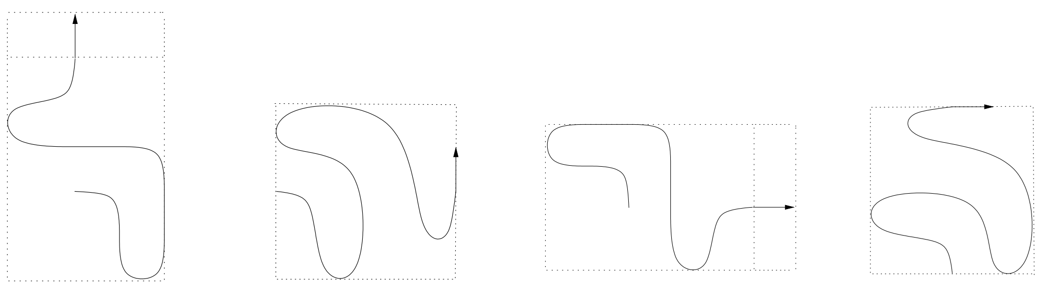
Proposition 2. *A prudent walk w of the first type always ends on the right side or on the top of its prudent box B.*

Counting prudent walks of the first type. We are interested in determining the generating function of the walks of the first type according to their length. However we will need to count them also with respect to $i(w)$ and $k(w)$. We denote by

$$p^1(u, v; t) = \sum_{w \in P^1} u^{k(w)} v^{i(w)} t^{|w|}$$

such a generating function. In order to compute it we distinguish the following subclasses:

- H_n is the subclass of P^1 formed by the prudent walks ending with a $\uparrow$ step, and such that their last step moved the top of the previous prudent box, i.e. the prudent box at the previous step. In particular this means that $i(w) = 0$ for each $w \in H_n$. See Figure 3.
- G_n is the subclass of P^1 formed by the prudent walks ending with a $\uparrow$ step, and such that their prudent box is the same as the one at the previous step. In particular this means that $k(w) = 0$ for each $w \in G_n$. See Figure 3.



H$_n$ G$_n$ H$_e$ G$_e$

Figure 3. The classes H_n, G_n, H_e and G_e.

- H_e is the subclass of P^1 formed by the prudent walks ending with a $\rightarrow$ step, and such that their last step $\rightarrow$ moved the right border of the previous prudent box. In particular this means that $k(w) = 0$ for each $w \in H_e$. See Figure 3.
- G_e is the subclass of P^1 formed by the prudent walks ending with a $\rightarrow$ step, and such that their prudent box is the same as the one at the previous step. In particular this means that $i(w) = 0$ for each $w \in G_e$. See Figure 3.
- F_o is the subclass of P^1 formed by the prudent walks having as last step a $\leftarrow$ step, therefore ending in the top of B. Then $i(w) = 0$ for each $w \in F_o$. See Figure 4.
- F_s is the subclass of P^1 formed by the prudent walks whose last step is a $\downarrow$ step, therefore ending in the right side of B. Then $k(w) = 0$ for each $w \in F_s$. See Figure 4.

We respectively denote by $h_n(u; t)$, $g_n(v; t)$, $h_e(v; t)$, $g_e(u; t)$, $f_o(u; t)$, $f_s(v; t)$ the generating functions of the previous defined classes, where the variable u marks the parameter $k(w)$, v marks $i(w)$, and t marks the length. Also note that the indices n, s, e, o consistently indicate the direction of the last step of the walks (o stands for *ovest*).

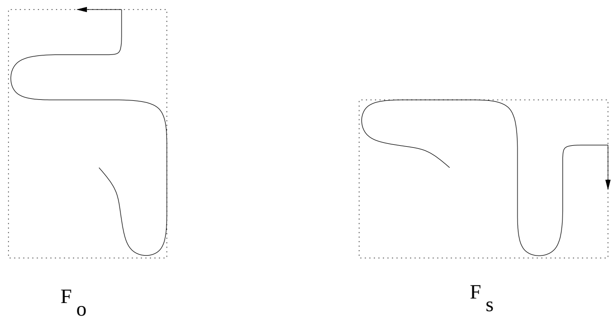


FIGURE 4. The classes F_o and F_s.

Lemma 1. *The set* $S^1 = \{H_n, G_n, H_e, G_e, F_o, F_s\}$ *is a partition of the set* P^1*. Consequently* $p^1(u,v;t) = h_n(u;t) + g_n(v;t) + h_e(v;t) + g_e(u;t) + f_o(u;t) + f_s(v;t)$.

Proof. Let us take a walk w in P^1. If it ends with a $\downarrow$(*resp.* $\leftarrow$) step then it belongs to F_s (*resp.* F_o). If it ends with a $\uparrow$ (*resp.* $\rightarrow$) step then it belongs to H_n or G_n (*resp.* H_e or G_e) if the walk obtained by removing this step ends in the top (*resp.* right side) of its box or not. Hence the set S^1 is a partition of P^1. The equation for $p^1(u,v;t)$ is a direct consequence of this fact. □

Now, by using the recursive definition of prudent walk, we write equations for the previous generating functions. We start from those depending only on the variables u and t. The others are obtained from these ones by symmetry.

- *The equation for* $h_n(u;t)$*.* The walks of the class H_n end with a $\uparrow$ step which move the top of their previous prudent box. Hence this class is formed by the step $\uparrow$ itself and by the walks obtained by adding a $\uparrow$ step to walks ending on the top of their prudent box. The latter walks belong to the following classes: H_n; the walks belonging to G_n and ending in the top right angle; the walks belonging to H_e and ending in the top right angle; G_e; F_o. The addition of a $\uparrow$ step to these walks increases their length by one. Therefore we have the following equation:

$$h_n(u;t) = t\ (h_n(u;t) + g_n(0;t) + h_e(0;t) + g_e(u;t) + f_o(u;t) + 1) \tag{1}$$

- *The equation for* $g_e(u;t)$*.* G_e is formed of walks obtained by adding a $\rightarrow$ step to walks ending in the top side of their prudent box, with exclusion of the top right angle. The latter walks belong to the following classes: G_e without the walks ending in the top right angle; H_n without the walks ending in the top right angle. By adding a $\rightarrow$ step, the length of these walks increases by one and their distance to the right side diminishes by one. Therefore we have the following equation:

$$g_e(u;t) = \frac{t}{u}\ (\ g_e(u;t) - g_e(0;t)\ +\ h_n(u;t) - h_n(0;t)\ ) \tag{2}$$

- *The equation for* $f_o(u;t)$*.* This class is obtained by adding a $\leftarrow$ step to walks ending in the leftmost occupied position of the top of their prudent box (otherwise we would not respect the condition of prudent walk). Then the equation for $f_o(u)$ involves: H_n; F_o; the empty walk. The operation of adding a $\leftarrow$ step increases both the length of these walks and their distance to the right side of their prudent box by one. Therefore we have the following equation:

$$f_o(u;t) = t\,u\ (h_n(u;t) + f_o(u;t) + 1)$$

In order to simplify the system formed of these three equations we replace the expression $g_n(0;t) + h_e(0;t)$ by $I(t)$ in (1) and (2). Then the system becomes:

$$\begin{aligned} h_n(u;t) &= t\ (h_n(u;t) + I(t) + g_e(u;t) + f_o(u;t) + 1) \\ g_e(u;t) &= \tfrac{t}{u}\ (g_e(u;t) + h_n(u;t) - I(t)) \\ f_o(u;t) &= t\,u\ (h_n(u;t) + f_o(u;t) + 1) \end{aligned}$$

By solving this system with respect to $h_n(u;t)$, $g_e(u;t)$, and $f_o(u;t)$ we obtain the following:

$$\begin{aligned} h_n(u;t) &= t\, \frac{(I(t)\ (-2ut^2+u^2t-u+2t)-u+t)}{K(u;t)} \\ g_e(u;t) &= t\, \frac{(I(t)\ (1-2t-ut+ut^2)-t)}{K(u;t)} \\ f_o(u;t) &= -t\,u\, \frac{\left(I(t)(-2t^2+ut)+u-t^2-t\right)}{K(u;t)}, \end{aligned}$$

with $K(u;t) = -u + t + ut + u^2t - ut^2 - ut^3$. By symmetry we also have expressions for $h_e(v;t)$, $g_n(v;t)$, and $f_s(v;t)$, obtained respectively from $h_n(u;t)$, $g_e(u;t)$, and $f_o(u;t)$ by substituting u with v. By using Lemma 1 we obtain the generating function of the class P^1 in terms of $I(t)$:

$$p^1(1,1;t) = \frac{2\,t\,(2+t+t\,I(t))}{1-2t-t^2} \tag{3}$$

We apply the kernel method [1, 2], for instance on the equation for $g_e(u;t)$:

$$g_e(u;t)K(u;t) = t\ (I(t)\ (1-2t-ut+ut^2)-t). \tag{4}$$

The kernel $K(u;t)$ has a unique root $U_0(t)$ which is a formal power series:

$$U_0(t) = \frac{1-t+t^2+t^3-\sqrt{(1-t^4)(1-2t-t^2)}}{2t}$$

Substituting $u = U_0(t)$ in (4) the left hand side is canceled and we obtain $I(t)$:

$$I(t) = \frac{t}{(1-2t-U_0(t)t+U_0(t)t^2)} \tag{5}$$

By using equation (5) and equation (3) we have

$$p^1(1,1;t) = t\frac{(1-2t-t^2)(3+2t-3t^2)+(1-t)\sqrt{(1-t^4)(1-2t-t^2)}}{(1-2t-t^2)(1-2t-2t^2+2t^3)} \;=\; 4t+10t^2+26t^3+66t^4+168t^5+O(t^6)$$

3. A grammar for prudent walks of the first type

We counted the number of prudent walks according to their length and we obtained an algebraic generating function. Now we give a direct explanation of the algebraicity of this class by determining an algebraic decomposition (or object grammar) for it. As before we denote classes of paths by capital letters and we use the corresponding small letters for the generating functions with respect to the length. We start by distinguishing four subclasses of prudent walks of the first type:

- The class $W_\uparrow$ of walks of the first type beginning with a $\uparrow$ step.
- The class $W_\downarrow$ of walks of the first type beginning with a $\downarrow$ step.
- The class $W_\rightarrow$ of walks of the first type beginning with a $\rightarrow$ step.
- The class $W_\leftarrow$ of walks of the first type beginning with a $\leftarrow$ step.

Then we have:

$$P^1 \;=\; W_\uparrow \;+\; W_\downarrow \;+\; W_\leftarrow \;+\; W_\rightarrow$$

Let us consider the class $W_\rightarrow$. It consists of the walk reduced to $\rightarrow$ and of all walks obtained by adding a first initial $\rightarrow$ step to a prudent walk of the first type respectively beginning with a $\uparrow$, a $\downarrow$, or a $\rightarrow$ step. Hence we can write the following grammar:

$$W_\rightarrow \;=\; \rightarrow\cdot W_\rightarrow + \;\rightarrow\cdot W_\uparrow + \;\rightarrow\cdot W_\downarrow + \;\rightarrow$$

By symmetry we have a similar decomposition for $W_\uparrow$, and we have the equalities $w_\uparrow(t) = w_\rightarrow(t)$ and $w_\downarrow(t) = w_\leftarrow(t)$. This also implies that it is sufficient to find a decomposition for the class $W_\leftarrow$ or for the class $W_\downarrow$ to determine $p^1(t)$.

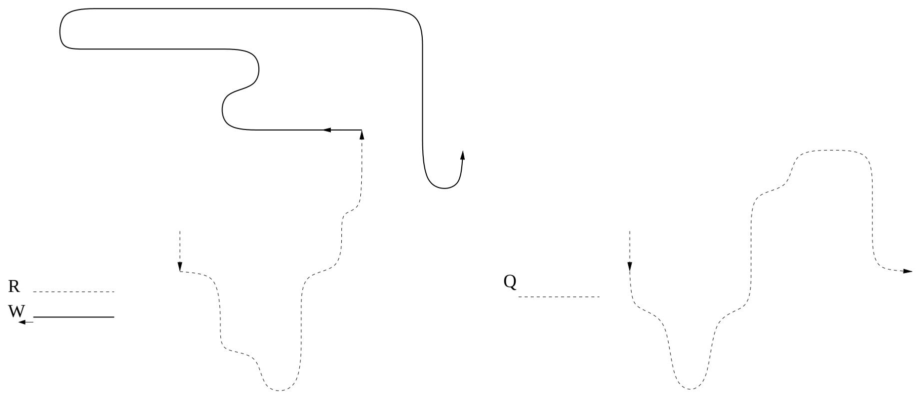


FIGURE 5. The walks belonging to $R \cdot W_{\leftarrow}$ (left) and to Q (right).

3.1. **A decomposition for the class $W_{\downarrow}$.** In order to define this grammar we distinguish the following classes:

i. *The class of walks that contain the subsequence made of a $\uparrow$ step followed by a $\leftarrow$ step.* We divide each walk belonging to this class in two parts after the first $\uparrow$ followed by a $\leftarrow$ (see Figure 5, left hand side):
   - *The first part belongs to the class R:* It is the class of prudent walks of the first type starting with a $\downarrow$ step, not containing any $\leftarrow$ step, ending in the top right angle of their prudent box with a $\uparrow$ step, and such that they moved the top of their prudent box with this last step.
   - *The second part belongs to the class $W_{\leftarrow}$:* It is the class of prudent walks of the first type starting with a $\leftarrow$ step.

ii. *The class of walks of the first type that do not contain any $\leftarrow$ step.* We denote this class of walks by Q (see Figure 5, right hand side).

Then the decomposition for $W_{\downarrow}$ is the following:

$$W_{\downarrow} \;=\; R \cdot W_{\leftarrow} \;+\; Q \tag{6}$$

3.2. **A decomposition for the class R.** We can decompose R in the following manner:

i. The class of walks of R that make a $\downarrow$ step at a later time when they are in the top right angle. In other terms these are the walks of the form $w_0 \cdot \downarrow \cdot w_1$ where w_0 is a walk ending in the top right angle of its box. The decomposition for these walks is the following:

$$\downarrow \cdot C \cdot \uparrow \cdot T \cdot \rightarrow \cdot R$$

where C is essentially the class of Motzkin paths that avoid the subsequences $\uparrow\downarrow$ and $\downarrow\uparrow$,

$$C \;=\; \downarrow \cdot C \cdot \uparrow \quad + \quad \downarrow \cdot C \cdot \uparrow \cdot \rightarrow \cdot C \quad + \quad \downarrow \cdot C \cdot \uparrow \cdot \rightarrow \quad + \quad \rightarrow \cdot C \quad + \quad \rightarrow \tag{7}$$

and T is the class of staircases,

$$T \;=\; \uparrow \cdot T \quad + \quad \rightarrow \cdot T \quad + \quad \epsilon \tag{8}$$

ii. The class of walks of R that after their first step never make anymore a $\downarrow$ step when they are in the top right angle. The decomposition for these walks is the following:

$$\downarrow \cdot C \cdot \uparrow \cdot T \cdot \uparrow$$

In summary the decomposition for R is the following (see Figure 6):

$$R \;=\; \downarrow \cdot C \cdot \uparrow \cdot T \cdot \rightarrow \cdot R \quad + \quad \downarrow \cdot C \cdot \uparrow \cdot T \cdot \uparrow \tag{9}$$

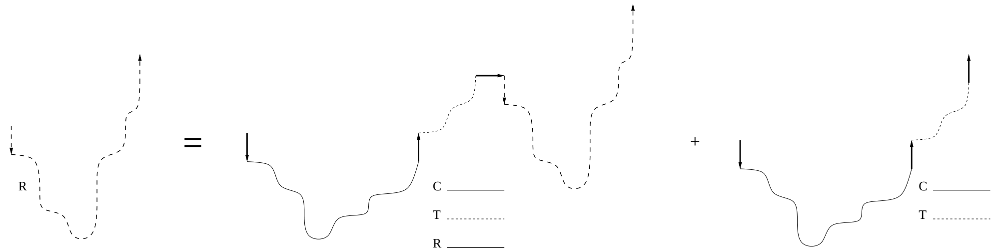


FIGURE 6. The complete decomposition for R.

3.3. **A decomposition for the class Q.** The class Q contains all walks with steps $\uparrow$, $\downarrow$, and $\rightarrow$ that start with a $\downarrow$ step and avoid the subsequences $\uparrow\downarrow$ and $\downarrow\uparrow$. This class satisfies:

$$
\begin{array}{lcl}
Q & = & \downarrow \;+\; \downarrow\cdot Q_{\rightarrow} \;+\; \downarrow\cdot Q \\
Q_{\rightarrow} & = & \rightarrow \;+\; \rightarrow\cdot Q_{\rightarrow} \;+\; \rightarrow\cdot Q \;+\; \rightarrow\cdot Q_{\uparrow} \\
Q_{\uparrow} & = & \uparrow \;+\; \uparrow\cdot Q_{\rightarrow} \;+\; \uparrow\cdot Q_{\uparrow}
\end{array}
$$

3.4. **Generating functions.** By using the previous decompositions we obtain the following algebraic equations:

$$
\begin{array}{lcl@{\qquad}lcl}
p^1(t) & = & w_{\rightarrow}(t)+w_{\uparrow}(t)+w_{\downarrow}(t)+w_{\leftarrow}(t) & c(t) & = & t+t^2c(t)+t^3c(t)^2+t^2c(t)+tc(t) \\
w_{\rightarrow}(t) & = & t\,w_{\rightarrow}(t)+t\,w_{\uparrow}(t)+t\,w_{\downarrow}(t)+t & T(t) & = & 1+2tT(t) \\
w_{\downarrow}(t) & = & r(t)w_{\leftarrow}(t)\;+\;Q(t) & r(t) & = & t^3c(t)\,T(t)\,r(t)+t^3c(t)\,T(t) \\
w_{\leftarrow}(t) & = & w_{\downarrow}(t) & q(t) & = & t+tq_{\rightarrow}(t)+tq(t) \\
w_{\uparrow}(t) & = & w_{\rightarrow}(t) & q_{\rightarrow}(t) & = & t+tq_{\rightarrow}(t)+2tq(t)
\end{array}
$$

From this system we can recover the expressions of the previous section.

4. PRUDENT WALKS OF THE SECOND TYPE

We are now interested in counting a wider class of prudent walks. As we saw in Section 2, prudent walks of the first type can only end on the top or on the right side of their prudent box. We introduce a class of prudent walks which can also end on the bottom of their prudent box.

Definition 4. *A prudent walk of the second type is a prudent walk avoiding the following subsequences of steps when the prudent box is not a line: a west step followed by a south step* $\substack{\leftarrow\\ \downarrow}$ *when the walk visits the top of its current prudent box and a west step followed by a north step* $\substack{\uparrow\\ \leftarrow}$ *when the walk visits the bottom of its current prudent box.*

In Figure 1(d) there is an example of prudent walk of the second type with its prudent box.

We want to count the number of these walks according to their length. Similarly to the previous section we have the following:

Proposition 3. *A prudent walk w of the second type always ends on the top, right, or bottom side of its prudent box B.*

Counting prudent walks of the second type. Let us call P^2 the class of prudent walks of the second type. We are interested in determining their generating function according to their length and their distances $i(w)$, $j(w)$, and $k(w)$. We denote by $p^2(u,v,w;t)=\sum_{w\in P^2}u^{k(w)}v^{i(w)}w^{j(w)}t^{|w|}$ this generating function. In order to compute it we distinguish the following subclasses:

- H_n is the subclass of P^2 formed by the prudent walks ending with a $\uparrow$ step, and such that their last step $\uparrow$ moved the top of the previous prudent box, i.e. the prudent box at the previous step. In particular this means that $i(w) = 0$ for each $w \in H_n$ (See Figure 7).
- G_n is the subclass of P^2 formed by the prudent walks ending with a $\uparrow$ step, and such that their prudent box is the same as the one at the previous step. In particular this means that $k(w) = 0$ for each $w \in G_n$ (See Figure 7).
- H_s is the subclass of P^2 formed by the prudent walks ending with a $\downarrow$ step, and such that their last step $\downarrow$ moved the bottom of the previous prudent box. In particular this means that $j(w) = 0$ for each $w \in H_s$. (See Figure 7).
- G_s is the subclass of P^2 formed by the prudent walks ending with a $\downarrow$ step, and such that their prudent box is the same as the one at the previous step. In particular this means that $k(w) = 0$ for each $w \in G_s$ (See Figure 7).

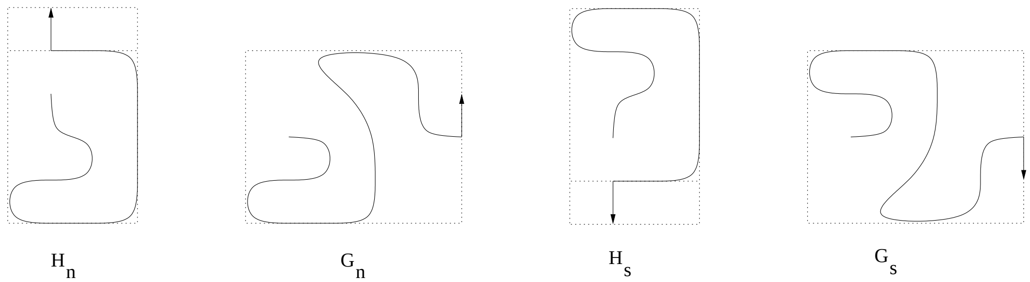


FIGURE 7. The classes H_n, G_n, H_s, and G_s.

- H_e is the subclass of P^2 formed by the prudent walks ending with a $\rightarrow$ step, and such that their last step $\rightarrow$ moved the right side of the previous prudent box. In particular this means that $k(w) = 0$ for each $w \in H_e$ (See Figure 8).
- GT_e is the subclass of P^2 formed by the prudent walks ending with a $\rightarrow$ step on the top of their prudent box, and such that this box is the same as the one at the previous step. In particular this means that $i(w) = 0$ for each $w \in GT_e$ (See Figure 8).
- GB_e is the subclass of P^2 formed by the prudent walks ending with a $\rightarrow$ step on the bottom of their prudent box and such that this box is the same as the one at the previous step. In particular this means that $j(w) = 0$ for each $w \in GB_e$ (See Figure 8).

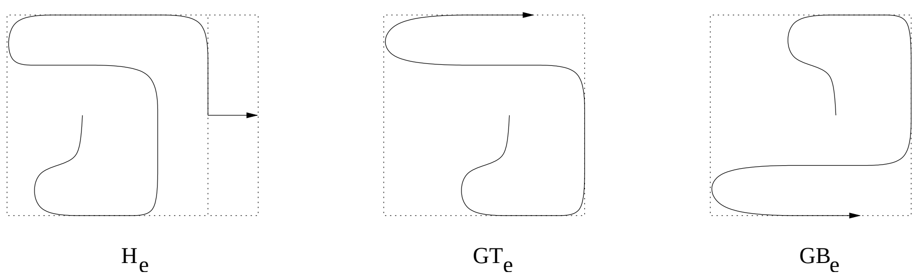


FIGURE 8. The classes H_e, GT_e, and GB_e.

- FT_o is the subclass of P^2 formed by the prudent walks having as last step a $\leftarrow$ step, ending in the top of B and such that B is not reduced to a line. Then $i(w) = 0$ for each $w \in FT_o$ (See Figure 9).
- FB_o is the subclass of P^2 formed by the prudent walks having as last step a $\leftarrow$ step, ending in the bottom of B and such that B is not reduced to a line. Then $j(w) = 0$ for each $w \in FB_o$ (See Figure 9).
- F_o is the subclass of P^2 whose prudent walks are made up just by $\leftarrow$ steps (See Figure 9).

We respectively denote by $h_n(u, w; t)$, $g_n(v, w; t)$, $h_s(u, v; t)$, $g_s(v, w; t)$, $h_e(v, w; t)$, $gt_e(u, w; t)$, $gb_e(u, v; t)$, $ft_o(u, w; t)$, $fb_o(u, v; t)$, $f_o(u; t)$ the generating functions of the previous defined classes with the variables marking as for $p^2(u, v, w; t)$.

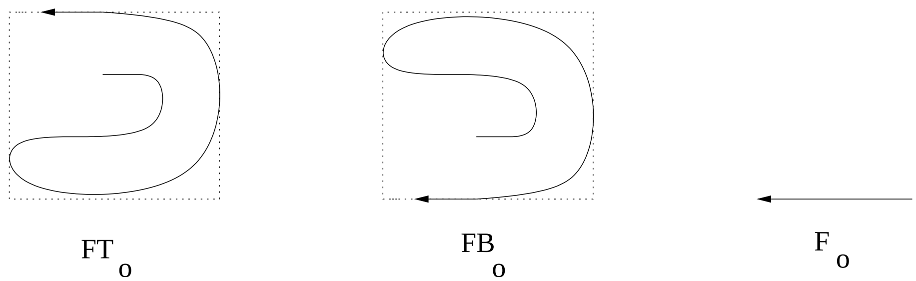


FIGURE 9. The classes FT_o, FB_o, and F_o.

Lemma 2. *The set* $S^2 = \{H_n, G_n, H_s, G_s, H_e, GT_e, GB_e, FT_o, FB_o, F\}$ *is a partition of the set* P^2*. Consequently* $p^2(u,v,w;t) = h_n(u,w;t) + g_n(v,w;t) + h_s(u,v;t) + g_s(v,w;t) + h_e(v,w;t) + gt_e(u,w;t) + gb_e(u,v;t) + ft_o(u,w;t) + fb_o(u,v;t) + f_o(u;t)$.

Proof. A prudent walk w in P^2 can finish with a $\downarrow$, $\leftarrow$, $\uparrow$, $\rightarrow$ step. If it ends with a $\leftarrow$ step then we have the following cases: its prudent box is a line, then it belongs to F_o; it ends on the top of its prudent box, then it belongs to belongs to FT_o; it ends on the bottom of its prudent box, then it belongs to FB_o. If it ends with an $\uparrow$ step then it belongs to H_n or G_n (*resp.* H_s or G_s) if the ending point at the previous step was in the top (*resp.* bottom) of the box or not. If it ends with a $\rightarrow$ step then it belongs to H_e if the ending point at the previous step was in the right side of its prudent box. Otherwise it belongs to GT_e(*resp.* GB_e) if its last step is on the top (*resp.* bottom) of its prudent box. Then the set S^2 is a partition of P^2 and the result on $p^2(u,v,w;t)$ is a direct consequence of this fact. □

Now we can write equations for the generating functions of elements of S^2 by using the recursive definition of a prudent walk. Let us write the equations for $h_n(u,w;t)$, $g_n(v,w;t)$, $h_e(v,w;t)$, $gt_e(u,w;t)$, $ft_o(u,w;t)$, $f_o(u;t)$. The others are obtained from these ones by symmetry.

- *The equation for* $h_n(u,w;t)$. These walks, ending with a $\uparrow$ step, moved the top of the previous prudent box. Therefore this class is formed by $\uparrow$ itself and by the walks obtained by adding a $\uparrow$ step to walks ending in the top of their prudent box. Such walks belong to the following classes: H_n; the walks belonging to G_n and ending on the top right angle; the walks belonging to H_e and ending on the top right angle; GT_e; FT_o; F_o. Adding a $\uparrow$ step to these walks increases their length by one and their distance from the bottom of the prudent box by one. Then we have the following equation:

$$h_n(u,w;t) = tw\left(h_n(u,w;t) + g_n(0,w;t) + h_e(0,w;t) + gt_e(u,w;t) + ft_o(u,w;t) + f_o(u;t) + 1\right)$$

- *The equation for* $g_n(v,w;t)$. G_n is formed of walks obtained by adding a $\uparrow$ step to walks ending in the right side of their prudent box, with exclusion of the top right angle. Such walks belong to the following classes: G_n, with the exclusion of walks ending in the top right angle; H_e, with the exclusion of walks ending in the top right angle. With the addition of a $\uparrow$ step the length of the walks increases by one, their distance to the top of the prudent box diminishes by one, and their distance to the bottom of the prudent box increases by one. Then we have the following equation:

$$g_n(v,w;t) = \frac{tw}{v}\left(g_n(v,w;t) - g_n(0,w;t) + h_e(v,w;t) - h_e(0,w;t)\right)$$

- *The equation for* $h_e(u,w;t)$. These walks, ending with a $\rightarrow$ step, moved the right side of the previous prudent box. Therefore this class is formed by $\rightarrow$ itself and by the walks obtained by adding a $\rightarrow$ step to walks ending in the right of their prudent box. Such walks belong to the following classes: H_e; the walks belonging to GT_e and ending on the top right angle; the walks belonging to GB_e and ending on the bottom right angle; the walks belonging to H_n and ending on the top right angle; G_n; the walks belonging to H_s and ending on the bottom right angle; the walks belonging to G_s. Adding a $\rightarrow$ step to these walks increases their length by one. Then we have the following equation:

$$h_e(v,w;t) = t\left(h_e(v,w;t) + gt_e(0,w;t) + gb_e(0,v;t) + h_n(0,w;t) + g_n(v,w;t) + h_s(0,v;t) + g_s(v,w;t) + 1\right)$$

- *The equation for* $gt_e(u,w;t)$. GT_e is formed of walks obtained by adding a $\rightarrow$ step to walks ending in the top side of their prudent box, with exclusion of the top right angle. Such walks belong to the following classes: GT_e, with the exclusion of walks ending in the top right angle; H_n, with the exclusion of walks ending in the top right angle. With the addition of a $\rightarrow$ step the length of the walks increases by one and their distance to the right of the prudent box diminishes by one. Then we have the following equation:

$$gt_e(u,w;t) = \frac{t}{u}\left(gt_e(v,w;t) - gt_e(0,w;t) + h_n(u,w;t) - h_n(0,w;t)\right)$$

- *The equation for $ft_o(u,w)$* This class is formed by the walks obtained by adding a $\leftarrow$ step to walks ending on the top of their prudent box. Observe that by adding a $\leftarrow$ step to walks G_n ending in the top right angle we do not respect the condition of prudent walk. Then the classes involved in the equation for $ft_o(u)$ are the following: H_n; FT_o. By adding a $\leftarrow$ step both the length of the walks and their distance to the right of the prudent box increase by one. Therefore we have the following equation:
$$ft_o(u,w;t) = t\,u\left(h_n(u,w;t) + ft_o(u,w;t)\right)$$
- *The equation for $f_o(u)$* This is the class $\{\leftarrow\}^+$, therefore:
$$f_o(u;t) = \frac{tu}{1-tu}$$

The other equations are obtained by symmetry with respect to the horizontal axis. In order to simplify the system of these equations, we introduce the following series:

$$L(w;t) = h_n(0,w;t) + gt_e(0,w;t), \text{ or equivalently, } L(v;t) = h_s(0,v;t) + gb_e(0,v;t)$$

$$M(w;t) = h_e(0,w;t) + g_n(0,w;t), \text{ or equivalently, } M(v;t) = h_e(v,0;t) + g_s(v,0;t)$$

Then the system becomes:

$$\begin{aligned}
h_n(u,w;t) &= tw\left(h_n(u,w;t) + gt_e(u,w;t) + ft_o(u,w;t) + f_o(u;t) + M(w;t) + 1\right)\\
g_n(v,w;t) &= \tfrac{tw}{v}\left(h_e(v,w;t) + g_n(v,w;t) - M(w;t)\right)\\
h_e(v,w;t) &= t\left(h_e(v,w;t) + g_s(v,w;t) + g_n(v,w;t) + L(w;t) + L(v;t) + 1\right)\\
gt_e(u,w;t) &= \tfrac{t}{u}\left(h_n(u,w;t) + gt_e(u,w;t) - L(w;t)\right)\\
gb_e(u,v;t) &= \tfrac{t}{u}\left(h_s(u,v;t) + gb_e(u,v;t) - L(v;t)\right)\\
h_s(u,v;t) &= tv\left(h_s(u,v;t) + gb_e(u,v;t) + fb_o(u,v;t) + f_o(u;t) + M(v;t) + 1\right)\\
g_s(v,w;t) &= \tfrac{tv}{w}\left(h_e(v,w;t) + g_s(v,w;t) - M(v;t)\right)\\
ft_o(u,w;t) &= tu(h_n(u,w;t) + ft_o(u,w;t))\\
fb_o(u,v;t) &= tu(h_s(u,v;t) + fb_o(u,v;t))\\
f_o(u;t) &= \tfrac{tu}{1-tu}
\end{aligned}$$

Observe that in this system of linear equations all the series can be expressed in terms of $L(w;t)$, $L(v;t)$, $M(w;t)$, $M(v;t)$. In particular we have:

$$ft_o(u,w;t) = \frac{t^2\,u\,w((1-tu)(1-t)M(w;t) - t(1-ut)L(w;t) - t + u)}{(1-tu)(-tu^2 + t^2u + u + t^3\,w\,u - u\,t\,w - t)}$$

and

$$g_n(v,w;t) = \frac{t\,w\,(t^2v - t\,w + t\,(tv - w)\,(L(w;t) + L(v;t)) + (w - t\,w - t\,v)\,M(w;t) + t^2v\,M(v;t))}{-w\,v + t\,w^2 + w\,t\,v + t\,v^2 - t^2\,v\,w - t^3\,v\,w}$$

In order to compute $p^2(1,1,1;t)$ it is sufficient to determine $L(1;t)$ and $M(1;t)$. We first apply the kernel method to the equation for $ft_o(u,w;t)$. The kernel is $(-tu^2+t^2u+u+t^3\,w\,u-u\,t\,w-t)$. It has two roots and we denote by $U_0(w,t)$ the one which is a formal power series in t. Then

$$U_0(w,t)=\frac{1-t\,w+t^2+t^3\,w-\sqrt{(1-t^2)(1-t-wt-wt^2)(1+t-wt+wt^2)}}{2t}$$

Canceling the kernel by substituting $u=U_0(w;t)$ implies that:

$$(1-tU_0(w,t))(1-t)M(w;t)-t(1\,-tU_0(w,t))L(w;t)-t+U_0(w,t)=0 \tag{10}$$

Now, by setting respectively $v=1$ and $w=1$ in $g_n(v,w;t)$ we obtain the following system:

$$\begin{aligned}
g_n(1,w;t) &= \frac{t\,w\,(t^2-t\,w+t\,(t\,-\,w)\,(L(w;t)+L(1;t))+(w-t\,w\,-t)\,M(w;t)+t^2\,M(1;t))}{-w+t\,w^2+w\,t+t-t^2\,w-t^3\,w}\\
g_n(v,1;t) &= \frac{t\,(t^2v-t+t\,(tv\,-1)\,(L(1;t)+L(v;t))+(1-t\,-t\,v)\,M(1;t)+t^2v\,M(v;t))}{-v+t+\,t\,v+t\,v^2-t^2\,v\,-t^3\,v}
\end{aligned}$$

Observe that the two equations have the same denominator up to substituting $w=v$. Then they have a common kernel $K(v;t)=-v+t+\,t\,v+t\,v^2-t^2\,v\,-t^3\,v$.

We apply the kernel method to $g_n(1,w;t)$ and $g_n(v,1;t)$, taking the unique formal power series $V_0(t)$ that is solution of $K(v;t)$:

$$V_0(t)=\frac{1-t+t^2+t^3-\sqrt{(1-t^4)(1-2t-t^2)}}{2t} \tag{11}$$

Observe that $U_0(1;t)=V_0(t)$ (remark that it is also equal to the $U_0(t)$ of Section 2). Then, by canceling the kernel in the expressions of $g_n(1,w;t)$ and $g_n(v,1;t)$ we get

$$t^2-tV_0(t)+t\,(t-V_0(t))\,(L(V_0(t);t)+L(1;t))+(V_0(t)-tV_0(t)-t)\,M(V_0(t);t)+t^2\,M(1;t)=0 \tag{12}$$

$$t^2V_0(t)-t+t\,(tV_0(t)-1)\,(L(1;t)+L(V_0(t);t))+(1-t-tV_0(t))\,M(1;t)+t^2V_0(t)M(V_0(t);t)=0 \tag{13}$$

Equations (12), (13), equation (10) with $w=1$, and equation (10) with $w=V_0(t)$ form a system of linear equations that determines $L(1;t)$, $M(1;t)$, $L(V_0(t);t)$, $M(V_0(t);t)$. Finally, by substituting $L(1;t)$ and $M(1;t)$ in the equation for $p^2(1,1,1;t)$ (see Lemma 2) we obtain the following expression for $p^2(1,1,1;t)$:

$$p^2(1,1,1;t)=$$

$$\frac{2\,(-4\,t^4+2\,t^3-2\,t^2-t+2\,t^6-3\,t^5+t^7+1)\,t\,U_0(1)\,U_0(V_0)-2\,(t^4+2\,t^3-4\,t-2\,t^2+1)\,t\,U_0(1)-2\,t^3\,(t^2+t^3-1-3\,t)\,U_0(V_0)+2\,(t+2)\,(-1+t)\,t}{(t^2+2\,t-1)\,(t\,U_0(t)-1)\,(t\,U_0(V_0)-1)\,(t-U_0(V_0)+1-U_0(1))}$$

where $U_0(1)=U_0(1;t)$ and $U_0(V_0)=U_0(V_0(t);t)$. In particular $p^2(1,1,1;t)$ is algebraic of degree 4. The first terms of this series are $4t+12t^2+34t^3+90t^4+236t^5+612t^6+1580t^7+O(t^8)$

5. The class of prudent walks

Now we consider the complete class of prudent walks, i.e. prudent walks without restrictions on the subsequences of steps (see Figure 2). Let

$$p(u,u',v,w;t)=\sum_{w\in P}u^{k(w)}u'^{l(w)}v^{i(w)}w^{j(w)}t^{|w|}$$

be the generating function of P with respect to the distances and the length.

Observe that these prudent walks are symmetric with respect to all directions. Then, in order to count the walks of the class P, we just need to know the generating functions for the following classes:

- H_n is the subclass of P formed by the prudent walks ending with a $\uparrow$ step, and such that their last step moved the top of the previous prudent box. In particular, this means that $i(w) = 0$ for each $w \in H_n$.
- GR_n is the subclass of P formed by the prudent walks ending with a $\uparrow$ step on the right side of their prudent box, and such that their prudent box is the same as the one at the previous step. In particular, this means that $k(w) = 0$ for each $w \in G_n$.

In order to write equations for H_n and GR_n we need to introduce some other classes, as we did for walks of first and second types. However these classes are all symmetric to H_n or GR_n, so we omit their definition. The equations for the associated generating functions $h_n(u, u', w)$ and $gr_n(u', v, w)$ are obtained by using the same arguments as for walks of the first and second types:

$$\begin{aligned} h_n(u,u',w;t) &= t\,w\,(1 + h_n(u,u',w;t) + gr_n(u',0,w;t) + , gl_n(u,0,w;t) \\ &\quad + h_o(u,0,w;t) + gt_o(u,u',w;t) + h_e(u',0,w;t) + gt_e(u,u',w;t)) \\ gr_n(u',v,w;t) &= \tfrac{t\,w}{v}(gr_n(u',v,w;t) - gr_n(u',0,w;t) + h_e(u',v,w;t) - h_e(u',0,w;t)) \end{aligned}$$

Equations for all other classes could be obtained by symmetry from these two ones. Alternatively, by using directly the symmetries to express all generating functions in terms of h_n and gr_n, the two previous equations can be rewritten:

$$\begin{aligned} h_n(u,u',w;t) &= t\,w\,(1 + h_n(u,u',w;t) + gr_n(u',0,w;t) + gr_n(u,0,w;t) \\ &\quad + h_n(0,w,u;t) + gr_n(w,u',u;t) + h_n(w,0,u';t) + gr_n(w,u,u';t)) \\ gr_n(u',v,w;t) &= \tfrac{t\,w}{v}(gr_n(u',v,w;t) - gr_n(u',0,w;t) + h_n(w,v,u';t) - h_n(w,0,u';t)) \end{aligned} \tag{14}$$

This system entirely define the series $h_n(u, u', w; t)$ and $gr_n(u', v, w; t)$. We also have that:

$$p(1,1,1,1;t) = 4\,h_n(1,1,1;t) + 8\,gr_n(1,1,1;t)$$

Then, by using this equation and (14) we can compute the first terms of $p(1, 1, 1, 1; t)$:

$$4t \; + 12\,t^2 \; + \; 36\,t^3 \; + \; 100\,t^4 \; + \; 276\,t^5 \; + \; 748\,t^6 \; + \; 2012\,t^7 \; + \; 5356\,t^8 + O(t^9)$$

However, we were not able to find a solution for system (14) by similar methods to those in the previous sections.

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References

[1] C. Banderier, M. Bousquet-Mélou, A. Denise, P. Flajolet, D. Gardy, and D. Gouyou-Beauchamps. Generating functions for generating trees. *Discrete Math.*, 2002.
[2] M. Bousquet-Mélou and M. Petkovsek. Linear recurrences with constant coefficients: the multivariate case. *Discrete Math.*, 2000.
[3] I Dutour and J.M. Fédou. Object grammars and random generation. *Discrete Math. and Theoret. Comput. Sci.*, 2:47–61, 1998.
[4] I Dutour and J.M. Fédou. Object grammars and bijections. *Theoret. Comput. J.*, 290:1915–1929, 2003.
[5] B. Hayes. How to avoid yourself. *Amer. Sci.*, 86, 1998.
[6] N. Madras and G. Slade. *The self-avoiding walk.* Probability and its Applications. Birkhäuser Boston Inc., Boston, MA, 1993.
[7] P. Préa. Exterior self-avoiding walks on the square lattice. manuscript.
[8] R. Stanley. *Enumerative Combinatorics*, volume 2. Cambridge University Press, 1999.

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