[1]

Abstract

We will consider the realization of the Littlewood-Richardson rule for the outer product of symmetric group characters using generating functions. It allows to prove the rationality of multiplicity series of some PI-algebras.

MSC: Primary 05E10; Secondary 05A15, 16R10

Keywords: multiplicity series, generating functions, Young diagrams, representation of symmetric groups, PI-algebras

On the Littlewood-Richardson rule applying

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November 20, 2004.

Introduction

The subject we will consider for the most part concerns the combinatorics of Young diagrams and Young tableaus. As a combinatorial object Young diagrams have a wide application in various fields of mathematics, in particular in the representation theory of symmetric groups ([10]) which is also actively used. One of the field of applying of the symmetric group representation theory is the PI-theory (see [4], [9], [12], [13]).

We will consider associative algebras over a field of zero characteristic. Let $F\langle X\rangle$ be the free associative algebra over a field F of zero characteristic with a countable set of generators $X = \{x_1, x_2, \dots\}$. Let $f(x_1, \dots, x_n) \in F\langle X\rangle$ be any associative noncommutative polynomial on variables $x_1, \dots, x_n$. They say an associative algebra A over a field F satisfies the polynomial identity $f(x_1, \dots, x_n) \equiv 0$ if $f(a_1, \dots, a_n) = 0$ holds in A for any $a_i \in A$. The algebra satisfying some nontrivial polynomial identity is called a PI-algebra. For example, a commutative associative algebra is a PI-algebra because it satisfies the identity $xy - yx \equiv 0$, also a nilpotent algebra is PI, it satisfies the identity $x_1 \cdots x_n \equiv 0$ for some natural n. It is well known ([4], [9], [12], [13]) all polynomial identities of an associative PI-algebra A form a T-ideal of the algebra $F\langle X\rangle$ (i.e., an ideal invariant under all endomorphisms of $F\langle X\rangle$). We will denote by $\Gamma = T[A]$ the T-ideal of polynomial identities of A and by $Var(A)$ the variety of all associative algebras over the field F satisfying all polynomial identities of the algebra A.

*The work is partially supported by grants RFFI 04-01-00739

Let us consider the multilinear part $P_n(A) = P_n/(P_n \bigcap T[A])$ of the relatively free algebra $F\langle X\rangle/T[A]$ which is left FS_n- module ([4], [13]). Here $P_n = \langle x_{\sigma(1)} \cdots x_{\sigma(n)} | \sigma \in S_n\rangle$. The S_n-character of $P_n(A)$ $\chi_n(A) = \sum_{\lambda\vdash n} m_\lambda \chi_\lambda$ is called the n-th cocharacter of A (respectively of T[A] or of Var(A)). We will consider the multiplicity series for the algebra A

$$f_A(t_1, t_2, \dots) = \sum_{\lambda=(\lambda_1,\dots,\lambda_k)} m_\lambda t_1^{\lambda_1} \cdots t_k^{\lambda_k}.$$

It is well known ([1]) in the case of a finitely generated algebra the height of Young diagrams in the cocharacter decomposition formula is restricted. It means in this case the number of variables t_i of the multiplicity series f_A is finite. We will consider only this case to make the presentation of the methods simpler.

Let A_1 and A_2 be any finitely generated PI-algebras over a field F of zero characteristic, $\Gamma_1 = T[A_1]$ and $\Gamma_2 = T[A_2]$ be correspondingly their ideals of polynomial identities. Let us consider the multiplicity series for these algebras

$f^{(1)}(t_1, \dots, t_k) = \sum_\lambda m_\lambda^{(1)} t_1^{\lambda_1} \cdots t_k^{\lambda_k}$ - the multiplicity series for the algebra A_1 and the T-ideal Γ_1,

$f^{(2)}(t_1, \dots, t_r) = \sum_\lambda m_\lambda^{(2)} t_1^{\lambda_1} \cdots t_r^{\lambda_r}$ - the multiplicity series for the algebra A_2 and the T-ideal Γ_2.

Let us denote

$$\eta(\Gamma_1, \Gamma_2) = \sum_{n\geq 0} \sum_{i=0}^{n} \chi_i(\Gamma_1) \hat{\otimes} \chi_{n-i}(\Gamma_2) = \sum_{i\geq 0, j\geq 0} \chi_i(\Gamma_1) \hat{\otimes} \chi_j(\Gamma_2).$$

Here $\chi_i(\Gamma_1)\hat{\otimes}\chi_j(\Gamma_2) = (\chi_i(\Gamma_1)\otimes\chi_j(\Gamma_2)) \uparrow^{S_{i+j}}$ is the outer product of characters and can be computed by the Littlewood-Richardson rule ([10, 11]). We will present an algorithm counting the multiplicity series $f_{\eta(\Gamma_1,\Gamma_2)}(t_1, \dots, t_{k+r})$ for the character $\eta(\Gamma_1, \Gamma_2)$ if the multiplicity series $f^{(1)}$ and $f^{(2)}$ for the T-ideals Γ_1 and Γ_2 are given. The similar formulas for the case of two-variable multiplicity series were introduced and applied in ([5, 6]).

1 The basic transformations.

Let us consider five basic transformations of multiplicity series used in the algorithm.

Let $f(t_1,\dots,t_k)$ be any function depending on k variables $t_1,\dots,t_k$ (notice, it can also depend on another variables). To be short we will use sometimes for a set of variables $t_1,\dots,t_k$ a notation $(t)_k$.

1. $D^{(1)}_{(t)_k\to(z)_k}(f)=\hat{f}(t_1,\dots,t_k,z_1,\dots,z_k)=f(t_1,\dots,t_k)|_{t_i=t_i\cdot z_i,\ i=\overline{1,k}}$;

2. $D^{(2)}_{(t)_k\to(z)_k}(f)=\tilde{f}(z_1,\dots,z_k)=f(t_1,\dots,t_k)|_{t_i=z_i/z_{i-1},z_0=1,\ i=\overline{1,k}}$;

3. $D^{(3)}_{(t)_k\to(z)_k}(f)=\bar{f}(z_1,\dots,z_k)=f(t_1,\dots,t_k)|_{t_i=z_i\cdots z_k,\ i=\overline{1,k}}$;

4. $D^{(4)}_{(t)_k\to(z)_{k+1}}(f)=\breve{f}(z_1,\dots,z_{k+1})=$
$$\frac{\displaystyle\sum_{\substack{1\le i_1<\dots<i_m\le k\\ 0\le m\le k}}(-1)^m(z_{i_1+1}\cdots z_{i_m+1})f((t)_k)\Bigg|_{t_j=\begin{cases}1, j\notin\{i_1,\dots,i_m\}\\ z_{j+1}, j\in\{i_1,\dots,i_m\}\end{cases}}}{(1-z_2)\cdots(1-z_{k+1})},$$

5. $D^{(5)}_{(s)_k,(z)_k}(f)=\frac{1}{(2\pi)^k}\int\limits_0^{2\pi}\cdots\int\limits_0^{2\pi}f((s)_k;(z)_k)\Bigg|_{\substack{s_j=2e^{i\varphi_j},\\ z_j=\frac{1}{2}e^{-i\varphi_j}}}d\varphi_1\dots d\varphi_k.$

Here and later for transformations the superscript enumerates the transformation and the subscript determines the set of variables which are modified.We will omit the subscripts if the sets of variables are not essential for the understanding of a matter.

Examples.

Let us consider $f(t_1,t_2,s_1,s_2,s_3)=t_1t_2^2+t_2s_3-\frac{s_1t_2+2t_1}{1-t_1s_1s_2}$,
$g(t_1,t_2,t_3,s_1,s_2,s_3,p_1)=p_1t_2t_3s_2s_3-2p_1t_2t_3s_1$. Then

1. $f_1(t_1,t_2,s_1,s_2,s_3,p_1,p_2)=D^{(1)}_{(t)_2\to(p)_2}(f)=f|_{\substack{t_1=t_1\cdot p_1,\\ t_2=t_2\cdot p_2}}=t_1p_1(t_2p_2)^2+t_2p_2s_3-\frac{s_1t_2p_2+2t_1p_1}{1-t_1p_1s_1s_2}$;

2. $f_2(t_1,t_2,t_3)=D^{(2)}_{(s)_3\to(t)_3}(f)=f(t_1,t_2,s_1,s_2)|_{\substack{s_1=t_1,s_2=t_2/t_1,\\ s_3=t_3/t_2}}=t_1t_2^2+t_2(t_3/t_2)-\frac{t_1t_2+2t_1}{1-t_1^2(t_2/t_1)}=t_1t_2^2+t_3-\frac{t_1t_2+2t_1}{1-t_1t_2}$;

3. $f_3(t_1,t_2,z_1,z_2,z_3) = D^{(3)}_{(s)_3\to(z)_3}(f) = f(t_1,t_2,s_1,s_2,s_3)\big|_{\substack{s_1=z_1z_2z_3,\\ s_2=z_2z_3,s_3=z_3}} = t_1t_2^2 + t_2z_3 - \frac{z_1z_2z_3t_2+2t_1}{1-t_1z_1z_2^2z_3^2}$;

4. $g_1((t)_3,(z)_4,p_1) = D^{(4)}_{(s)_3\to(z)_4}(g) = \frac{1}{(1-z_2)\cdot(1-z_3)\cdot(1-z_4)}\big(g|_{s_1=s_2=s_3=1} - z_2\cdot g|_{\substack{s_1=z_2,\\ s_2=s_3=1}} - z_3\cdot g|_{\substack{s_2=z_3,\\ s_1=s_3=1}} - z_4\cdot g|_{\substack{s_3=z_4,\\ s_1=s_2=1}} + z_2z_3\cdot g|_{\substack{s_1=z_2,s_2=z_3,\\ s_3=1}} + z_2z_4\cdot g|_{\substack{s_1=z_2,s_3=z_4,\\ s_2=1}} + z_3z_4\cdot g|_{\substack{s_2=z_3,s_3=z_4,\\ s_1=1}} - z_2z_3z_4\cdot g|_{\substack{s_1=z_2,s_2=z_3,\\ s_3=z_4}}\big) = \frac{p_1t_2t_3}{(1-z_2)\cdot(1-z_3)\cdot(1-z_4)}\big(-1-z_2+2z_2^2-z_3^2+2z_3-z_4^2+2z_4+z_2z_3^2-2z_2^2z_3+z_2z_4^2-2z_2^2z_4+z_3^2z_4^2-2z_3z_4-z_2z_3^2z_4^2+2z_2^2z_3z_4\big) = p_1t_2t_3(-1-2z_2+z_3+z_4+z_3z_4)$;

5. $g_2(p_1) = D^{(5)}_{(t)_3,(s)_3}(g) = \frac{1}{(2\pi)^3}\int\limits_0^{2\pi}\int\limits_0^{2\pi}\int\limits_0^{2\pi} g\Big|_{\substack{t_j=2e^{i\varphi_j},\\ s_j=\frac{1}{2}e^{-i\varphi_j}}} d\varphi_1 d\varphi_2 d\varphi_3 =$

$\frac{1}{(2\pi)^3}\int\limits_0^{2\pi}\int\limits_0^{2\pi}\int\limits_0^{2\pi}\big(4p_1e^{i\varphi_2}e^{i\varphi_3}\cdot\frac{1}{4}e^{-i\varphi_2}e^{-i\varphi_3} - 2p_1\cdot 4e^{i\varphi_2}e^{i\varphi_3}\cdot\frac{1}{2}e^{-i\varphi_1}\big)d\varphi_1d\varphi_2d\varphi_3 = \frac{1}{(2\pi)^3}\int\limits_0^{2\pi}\int\limits_0^{2\pi}\int\limits_0^{2\pi} p_1d\varphi_1d\varphi_2d\varphi_3 - 4p_1\big(\frac{1}{2\pi}\int\limits_0^{2\pi}e^{-i\varphi_1}d\varphi_1\big)\cdot\big(\frac{1}{2\pi}\int\limits_0^{2\pi}e^{i\varphi_2}d\varphi_2\big)\times$
$\times(\frac{1}{2\pi}\int\limits_0^{2\pi}e^{i\varphi_3}d\varphi_3) = p_1.$

The transformations $D^{(1)}$, $D^{(2)}$, $D^{(3)}$ are the simple substitutions of variables, and $D^{(1)}$ adds the new set of variables, while $D^{(2)}$, $D^{(3)}$ exchange the old variables by the new ones. The transformation $D^{(4)}$ also changes variables and their number increase by 1. The transformation $D^{(5)}$ acts on two sets of variables $s_1,\dots,s_k$ and $z_1,\dots,z_k$.

2 The derived transformations.

We also will use some derived transformations.

1. $S^{(1)}_{(t)_k\to(z)_k} = D^{(2)}_{(s)_k\to(z)_k}\circ D^{(1)}_{(t)_k\to(s)_k}$,
2. $h_z(f) = \frac{1}{1-z_1}f$,
3. $S^{(2)}_{(s)_k\to(z,t)_{k+1}} = D^{(1)}_{(z)_{k+1}\to(t)_{k+1}}\circ h_z\circ D^{(4)}_{(s)_k\to(z)_{k+1}}$,
4. $S^{(3)}_{(t)_k\to(z)_k} = D^{(3)}_{(s)_k\to(z)_k}\circ D^{(1)}_{(t)_k\to(s)_k}$.

Here "$\circ$" denotes the usual composition of maps.

Examples.

Let us take $f(s_1,s_2,s_3)=2s_1s_2^2+s_2s_3$, then

1. $f_1(s_1,s_2,s_3,z_1,z_2,z_3) = S^{(1)}_{(s)_3\to(z)_3}(f) = f(s_1,s_2,s_3)|_{\substack{s_i=s_i(z_i/z_{i-1})\\ i=\overline{1,3},\ z_0=1}} =$
$2s_1z_1(s_2(z_2/z_1))^2+s_2(z_2/z_1)s_3(z_3/z_2)=2s_1s_2^2\frac{z_2^2}{z_1}+s_2s_3\frac{z_3}{z_1}$;

2. $f_2((z)_4,(t)_4)=S^{(2)}_{(s)_3\to(z,t)_4}(f(s_1,s_2,s_3))=$
$$\frac{\left.\sum\limits_{\substack{1\le i_1<\cdots<i_m\le 3\\ 0\le m\le 3}}(-1)^m(\prod\limits_{r=1}^m z_{i_r+1}t_{i_r+1})f((s)_3)\right|_{s_j=\begin{cases}1,j\notin\{i_1,\ldots,i_m\}\\ z_{j+1}t_{j+1},j\in\{i_1,\ldots,i_m\}\end{cases}}}{(1-z_1t_1)(1-z_2t_2)\cdots(1-z_4t_4)} =$$
$\frac{1}{1-z_1t_1}\big(z_4t_4(1+z_3t_3)+2z_2t_2(1+z_3t_3+z_3^2t_3^2)+3+3z_3t_3+2z_3^2t_3^2\big)$;

3. $f_3(s_1,s_2,s_3,z_1,z_2,z_3)=S^{(3)}_{(s)_3\to(z)_3}(f((s)_3))=f(s_1,s_2,s_3)|_{\substack{s_i=s_iz_i\cdots z_3\\ i=\overline{1,3}}} =$
$2(s_1z_1z_2z_3)(s_2z_2z_3)^2+(s_2z_2z_3)(s_3z_3)=2s_1s_2^2z_1z_2^3z_3^3+s_2s_3z_2z_3^2$.

3 The algorithm.

Before counting the multiplicity series $f_{\eta(\Gamma_1,\Gamma_2)}$ we need to modify the first generating function $f^{(1)}(t_1,\ldots,t_k)$.

<u>The 1-st stage.</u>

On the entrance we have a function $F_{11}=f^{(1)}(t_1,\ldots,t_k)$.

1. $F_{12}=S^{(1)}_{(t)_k\to(s)_k}(F_{11})$,

2. $F_{21}=S^{(2)}_{(s)_k\to(\alpha,t)_{k+1}}(F_{12})$.

<u>The j-th stage ($2\le j\le r$).</u>

On the entrance we have a function $F_{j1}((t)_{k+j-1};(\varepsilon)_{j-2};(\alpha)_{k+j-1})$ (F_{21} does not depend on ε).

1. $F_{j2}=S^{(1)}_{(t)_{k+j-1}\to(s)_{k+j-1}}(F_{j1})$,

2. $F_{j3}=S^{(3)}_{(\alpha)_{k+j-1}\to(y)_{k+j-1}}(F_{j2})$,

3. $F_{j4}=D^{(4)}_{(s)_{k+j-1}\to(p)_{k+j}}(F_{j3})$,

4. $F_{j5} = S^{(3)}_{(p)_{k+j}\to(s)_{k+j}}(F_{j4})$,

5. $F_{j6} = D^{(4)}_{(y)_{k+j-1}\to(z)_{k+j}}(F_{j5})$,

6. $F_{j7} = D^{(5)}_{(s)_{k+j},(z)_{k+j}}(F_{j6})$,

7. $F_{j8} = D^{(1)}_{(p)_{k+j}\to(t)_{k+j}}(F_{j7})$,

8. $F_{j9} = F_{j8}((t)_{k+j};(\varepsilon)_{j-2};\alpha_1,\dots,\alpha_{k+j-1};(p)_{k+j})|_{\alpha_1=\dots=\alpha_{k+j-1}=\varepsilon_{j-1}}$,

9. $F_{j+11}((t)_{k+j};(\varepsilon)_{j-1};(\alpha)_{k+j}) = F_{j9}((t)_{k+j};(\varepsilon)_{j-1};p_1,\dots,p_{k+j})|_{p_i=\alpha_i}$.

Now we can go to the next $(j+1)$-th stage.

When we have finished the last r-th stage and obtain the function $F_{r+11}((t)_{k+r};(\varepsilon)_{r-1},(\alpha)_{k+r})$ we can find the multiplicity series $f_{\eta(\Gamma_1,\Gamma_2)}$.

$$F^*((t)_{k+r};(\varepsilon)_r) = F_{r+11}((t)_{k+r};(\varepsilon)_{r-1};\alpha_1,\dots,\alpha_{k+r})|_{\alpha_1=\dots=\alpha_{k+r}=\varepsilon_r},$$

$$f_{\eta(\Gamma_1,\Gamma_2)}(t_1,\dots,t_{k+r}) = D^{(5)}_{(\varepsilon)_r,(s)_r}\big(F^*((t)_{k+r},(\varepsilon)_r)\cdot f^{(2)}((s)_r)\big). \qquad (1)$$

4 On the rationality of some multiplicity series.

The next statement is obvious.

Lemma 1 *The algebra of rational functions is closed under the basic transformations $D^{(1)}$, $D^{(2)}$, $D^{(3)}$, $D^{(4)}$.*

We will call such transformations by rational transformations.

Corollary 2 *The compositions $S^{(1)}$, $S^{(2)}$, $S^{(3)}$ of rational transformations are also rational.*

Definition 3 *We will call a rational function $f = \frac{P}{Q}$, (P, Q are polynomials) specific on variables $(s)_m$ and $(z)_m$ if the denominator Q has a form $Q = \prod_{j=1}^{d}(1-\omega_j)$, where all ω_j are words on variables and for any $i = 1,\dots,m$ and for all $j = 1,\dots,d$ $\deg_{s_i}\omega_j + \deg_{z_i}\omega_j \le 1$.*

Lemma 4 *The image of the transformation $D^{(5)}_{(s)_m,(z)_m}$ of a rational function specific on variables $(s)_m,(z)_m$ is also a rational function.*

Theorem 5 *If f_{A_1}, f_{A_2} are rational functions specific on all variables then $f_{\eta(\Gamma_1,\Gamma_2)}$ is also a rational function, specific on all variables.*

Evidently it is enough to be sure that all functions on the entrance of the transformation D^5 (on the 6-th step of any stage and in (1)) remain specific on all corresponding variables.

Theorem 6 *If the multiplicity series f_{Γ_1}, f_{Γ_2} for T-ideals Γ_1 and Γ_2 are rational specific functions then the multiplicity series f_Γ for the product $\Gamma = \Gamma_1 \cdot \Gamma_2$ of the T-ideals is a rational specific function.*

Proof The theorem follows from the previous theorem and the Berele and Regev formula for the cocharacter of the product of T-ideals [3]. We can realize this formula for the multiplicity series as follows

$$f_\Gamma((t)_{k+r}) = f_{\Gamma_1} + f_{\Gamma_1} + B(f_{\eta(\Gamma_1,\Gamma_2)}) - f_{\eta(\Gamma_1,\Gamma_2)}, \qquad \text{where}$$

$$\check{f}((t)_m,(s)_m) = S^{(1)}_{(t)_k \to (s)_k}(f((t)_m)), \qquad \text{and}$$

$$B(f((t)_m)) = (\sum_{i=1}^{m+1} t_i) f((t)_m) - \sum_{i=1}^{m} t_{i+1} \cdot \check{f}((t)_m,(s)_m) \Bigg|_{s_j = \begin{cases} 1, j \neq i, \\ 0, j = i. \end{cases}}$$

The transformation B here is evidently rational. □

Theorem 7 *Any minimal variety of associative algebras over a field of zero characteristic of the matrix type not greater than 2 and generated by a finitely generated algebra has a rational multiplicity series.*

Proof By [8] the ideal of polynomial identities Γ of such variety is the product of T-ideals $\Gamma = \prod_{j=1}^{m} \Gamma_j$, where any $\Gamma_j = T[M_2(F)]$ is the ideal of identities of full matrix algebra of the 2-nd order over the base field, or $\Gamma_j = \{[x,y]\}^T$ is the commutator ideal. Then in the first case the multiplicity series for Γ_j can be obtained using the description of multiplicities for 2×2 matrices given by V.Drensky [7]

$$f_{M_2(F)} = \frac{1}{(1-t_1)^2(1-t_1t_2)^2(1-t_1t_2t_3)^2(1-t_1t_2t_3t_4)} - \frac{1}{(1-t_1)^2(1-t_1t_2)} - \frac{t_1t_2t_3 + t_1t_2t_3t_4 - 1}{(1-t_1)}.$$

In the second case the multiplicity series is trivial $f_{\{[x,y]\}^T} = \frac{1}{1-t_1}$. It is obvious in the both cases the multiplicity series are rational specific on all variables functions. Then by the previous theorem the multiplicity series for a product Γ of these T-ideals is also rational. $\square$

Taking into account this result, the results of V.Drensky and G.K.Genov [5, 6] and the rationality of a Hilbert series of any relatively free algebra [2] the question whether any associative PI-algebra over a field of zero characteristic has a rational multiplicity series becomes quite natural.

Notice at the end the presenting algorithm also can be applied for counting of exact formulas for cocharacters of some PI-algebras using some mathematical software.

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