Solution Spaces of H-Systems and the Ore-Sato Theorem*

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Abstract

An *H-system* is a system of first-order linear homogeneous difference equations for a single unknown function T, with coefficients which are polynomials with complex coefficients. We consider solutions of H-systems which are of the form $T: \operatorname{dom}(T) \to \mathbb{C}$ where either $\operatorname{dom}(T) = \mathbb{Z}^d$, or $\operatorname{dom}(T) = \mathbb{Z}^d \setminus S$ and S is the set of integer singularities of the system. It is shown that any natural number is the dimension of the solution space of some H-system, and that in the case $d \geq 2$ there are H-systems whose solution space is infinite-dimensional. The relationships between dimensions of solution spaces in the two cases $\operatorname{dom}(T) = \mathbb{Z}^d$ and $\operatorname{dom}(T) = \mathbb{Z}^d \setminus S$ are investigated. Finally we give an appropriate formulation of the Ore-Sato theorem on possible forms of solutions of H-systems in this setting.

Résumé

Par un *H-système* nous désignons un système des équations aux differences linéaires homogènes pour une seule fonction inconnue T, à coefficients polynomiaux sur le corps des nombres complexes. Nous considérons les solutions des H-systèmes de la forme $T: \operatorname{dom}(T) \to \mathbb{C}$ où soit $\operatorname{dom}(T) = \mathbb{Z}^d$, soit $\operatorname{dom}(T) = \mathbb{Z}^d \setminus S$, et S est l'ensemble

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des singularités entières du système. Nous montrons que chaque nombre naturel est égal à la dimension de l'éspace des solutions d'un H-système, et que dans le cas $d \geq 2$ il y a des H-systèmes dont la dimension de l'éspace des solutions est infinie. Les rélations entre les dimensions des éspaces des solutions dans les cas $\mathrm{dom}(T) = \mathbb{Z}^d$ et $\mathrm{dom}(T) = \mathbb{Z}^d \setminus S$ sont recherchées. Enfin nous présentons une formulation propre du théorême d'Ore-Sato sur les formes possibles des solutions des H-systèmes.

1 Introduction

Linear homogeneous recurrence equations with polynomial coefficients and systems of such equations play a significant role in combinatorics and in the theory of hypergeometric functions; the question of the dimension of the space of solutions of such systems is of great importance for many problems.

Let $n_1, \ldots, n_d$ be variables ranging over the integers and E_{n_i} the corresponding shift operators, acting on functions (sequences) of $n_1, \ldots, n_d$ by $E_{n_i} f(n_1, \ldots, n_i) = f(n_1, \ldots, n_i + 1, \ldots, n_d)$, $i = 1, \ldots, d$. We consider H-systems, i.e., systems of equations of the form $f_i\, E_{n_i} T = g_i T$, where $f_i, g_i \in \mathbb{C}[n_1, \ldots, n_d] \setminus \{0\}$ for $i = 1, \ldots, d$. The notion of singular points (singularities) of such systems can be defined in the usual way. Such singularities make obstacles (sometimes insuperable) for continuation of partial solutions of the system on all of $\mathbb{Z}^d$.

In this paper we consider two spaces of solutions of H-systems: the space V_1 of solutions defined everywhere on $\mathbb{Z}^d$, and the space V_2 of solutions that are defined at all nonsingular points of $\mathbb{Z}^d$ (more precisely, if W is the set of all solutions of a given system that are defined at least at all non-singular elements of $\mathbb{Z}^d$, then V_2 contains the restrictions of all elements of W to the set of all non-singular elements of $\mathbb{Z}^d$). In Sections 3 and 4 we investigate the dimensions of the spaces V_1, V_2. It is well known [6] that if (in the case $d = 1$) one considers the germs of sequences at infinity (i.e., classes of sequences which agree from some point on), then the dimension of the solution space is 1. However, the situation is different with $\dim V_1$ and $\dim V_2$. In Section 3 we prove for the case $d = 1$ that if the equation has singularities then $1 \leq \dim V_1 < \dim V_2 < \infty$, and for any integers s, t such that $1 \leq s < t$ there exists an equation with $\dim V_1 = s$ and $\dim V_2 = t$ (the case where there is no singularity is trivial: $\dim V_1 = \dim V_2 = 1$). In turn, in Section 4 we show that in the case $d > 1$ the possibilities are even richer: for any $s, t \in \mathbb{Z}_+ \cup \{\infty\}$ there exists an H-system with $\dim V_1 = s$ and $\dim V_2 = t$.

In Section 5 we revisit the Sato-Ore theorem [5, 7] and show that, contrary to some interpretations in the literature (e.g., [3, 4]), this theorem does not imply that any solution of an H-system is of the form

$$R(n_1, \ldots, n_d) \frac{\prod_{i=1}^{p} \Gamma(a_{i,1} n_1 + \cdots + a_{i,d} n_d + \alpha_i)}{\prod_{j=1}^{q} \Gamma(b_{j,1} n_1 + \cdots + b_{j,d} n_d + \beta_j)}\, u_1^{n_1} \cdots u_d^{n_d}, \tag{1}$$

where $R \in \mathbb{C}(x_1, \ldots, x_d)$, $a_{ik}, b_{jk} \in \mathbb{Z}$, and $\alpha_i, \beta_j \in \mathbb{C}$ (for the case when the solution of the system is holonomic, and R is required to be a polynomial, we have already noted this in [2]). Finally we give an appropriate corollary of the Ore-Sato theorem on possible forms of solutions of systems under consideration.

We write $p \perp q$ if polynomials $p, q \in \mathbb{C}[x_1, \ldots, x_d]$ are relatively prime. We call a set $A \subseteq \mathbb{Z}^d$ *algebraic* if there is a polynomial $p \in \mathbb{C}[x_1, \ldots, x_d] \setminus \{0\}$ which vanishes on A.

Definition 1 *Let E denote the shift operator corresponding to x, so that $Ef(x) = f(x+1)$ for every $f \in \mathbb{C}(x)$. A rational function $u \in \mathbb{C}(x)$ is* shift-reduced *if there are $a, b \in \mathbb{C}[x]$ such that $u = a/b$ and $a \perp E^k b$ for all $k \in \mathbb{Z}$.*

Theorem 1 *For every rational function $F \in \mathbb{C}(x)$ there are rational functions $u, v \in \mathbb{C}(x)$ such that*

(i) $F = u \cdot \frac{Ev}{v}$,

(ii) *u is shift-reduced.*

Definition 2 *If u, v, F are as in Theorem 1, (u, v) is a* rational normal form *of F.*

Theorem 2 *Let (u, v) and (u_1, v_1) be two rational normal forms of $F \in \mathbb{C}(x) \setminus \{0\}$. Write $u = p/q$ and $u_1 = p_1/q_1$ where $p, q, p_1, q_1 \in \mathbb{C}[x]$, $p \perp q$, and $p_1 \perp q_1$. Then $\deg p = \deg p_1$ and $\deg q = \deg q_1$.*

For proofs of Theorems 1 and 2, see [1].

2 H-systems and their solution spaces

Definition 3 *An H-*system[1] *is a system of equations*

$$f_i(n_1, \ldots, n_d) T(n_1, \ldots, n_i + 1, \ldots, n_d) = g_i(n_1, \ldots, n_d) T(n_1, \ldots, n_i, \ldots, n_d), \quad \text{for } i = 1, 2, \ldots, d, \tag{2}$$

where $f_i, g_i \in \mathbb{C}[n_1, \ldots, n_d] \setminus \{0\}$ and $f_i \perp g_i$. We say that a d-variate sequence T (i.e., a function $T : \mathrm{dom}(T) \to \mathbb{C}$) is a solution *of (2) if (2) is satisfied for all $(n_1, \ldots, n_i, \ldots, n_d) \in \mathrm{dom}(T)$ such that $(n_1, \ldots, n_i + 1, \ldots, n_d) \in \mathrm{dom}(T)$ as well.*

Definition 4 *Let A be an H-system of the form (2).*

A d-tuple $(n_1, \ldots, n_d) \in \mathbb{Z}^d$ is a trailing integer singularity *of A if there exists i, $1 \le i \le d$, such that $g_i(n_1, \ldots, n_d) = 0$. A d-tuple $(n_1, \ldots, n_d) \in \mathbb{Z}^d$ is a* leading integer singularity *of A if there exists i, $1 \le i \le d$, such that $f_i(n_1, \ldots, n_{i-1}, n_i - 1, n_{i+1}, \ldots, n_d) = 0$. A d-tuple $(n_1, \ldots, n_d) \in \mathbb{Z}^d$ is an* integer singularity *of A if it is a leading or a trailing integer singularity of A.*

Let $S(A)$ denote the set of all integer singularities of A. Denote by $V_1(A)$ the $\mathbb{C}$-linear space of all solutions of A which are defined at all elements of $\mathbb{Z}^d$, and by $V_2(A)$ the $\mathbb{C}$-linear space of all solutions of A which are defined at all elements of $\mathbb{Z}^d \setminus S(A)$.

[1] The prefix "H" refers to Jakob Horn and to the adjective "hypergeometric" as well.

We consider only integer singularities here, therefore we will drop the adjective "integer" in the sequel. Sometimes we will also drop the name of the H-system, and will write V_1, V_2 instead of $V_1(A)$, $V_2(A)$.

Definition 5 *Call the two d-tuples $(n_1, \ldots, n_d), (n'_1, \ldots, n'_d) \in \mathbb{Z}^d$* adjacent *if $\sum_{i=1}^d |n_i - n'_i| = 1$. Call a finite sequence $t_1, \ldots, t_k \in \mathbb{Z}^d$ a* path *from t_1 to t_k if t_i is adjacent to t_{i+1} for all $i = 1, \ldots, k-1$. Given an H-system A, we define* components *induced by A on $\mathbb{Z}^d$ as the equivalence classes of the following equivalence relation $\sim$ in $\mathbb{Z}^d$: $t' \sim t''$ iff there exists a path from t' to t'' which contains no singularity of A. If T is a solution of an H-system A, then its* constituent *is the sequence that is the restriction of T on a component induced by A.*

Definition 6 *Rational functions $F_1, \ldots, F_d \in \mathbb{C}(n_1, \ldots, n_d)$ are* compatible *if*

$$(E_{n_j} F_i) F_j = F_i (E_{n_i} F_j)$$

for all $1 \le i \le j \le d$.

Note that a single rational function (corresponding to the case $d = 1$) is always compatible.

Proposition 1 *Let A be an H-system of the form (2) where $g_1/f_1, \ldots, g_d/f_d$ are compatible rational functions. Then $\dim V_2$ is equal to the number of components induced by A.*

Proof: To each component C_i induced by A on $\mathbb{Z}^d$ we assign a solution T_i of (2) which is 1 at a selected point $p_i \in C_i$, and 0 on all the remaining components. The values of T_i on the remaining points of C_i are uniquely determined by (2). It is clear that the set of all T_i is a basis for V_2. $\square$

3 Dimensions of solution spaces: The univariate case

When $d = 1$ the system (2) is of the form

$$f(n)T(n+1) = g(n)T(n) \tag{3}$$

where $f(n), g(n) \in \mathbb{C}[n] \setminus \{0\}$ and $f(n) \perp g(n)$.

Example 1 ($\dim V_1 = 1$, $\dim V_2 = k$) Consider the recurrence

$$T(n+1) = p_k(n)\ T(n) \tag{4}$$

where $k \ge 1$ and $p_k(n) = \prod_{i=0}^{k-2}(n - 2i + 1)$. Here we use the convention that a product is 1 if its lower limit exceeds its upper limit. Clearly the set of singularities of (4) is $\{2i - 1;\ i = 0, 1, \ldots, k-2\}$, so $\dim V_2 = k$. To compute $\dim V_1$, note that any solution $T(n)$ of (4) defined for all $n \in \mathbb{Z}$ is a constant multiple of

$$F_k(n) = \begin{cases} (-1)^{(k-1)n} / \prod_{i=0}^{k-2}(2i - n - 1)!, & n < 0, \\ 0, & n \ge 0. \end{cases}$$

Therefore $\dim V_1 = 1$. $\blacksquare$

Example 2 ($\dim V_1 = m$, $\dim V_2 = m+1$) Now consider the recurrence

$$q_m(n+1)\ T(n+1) = q_m(n)\ T(n) \tag{5}$$

where $m \geq 1$ and $q_m(n) = \prod_{i=1}^m (n+2i+1)$. The set of singularities is $\{-(2i+1);\ i = 1,2,\ldots,m\}$, so $\dim V_2 = m+1$. Let $T(n)$ be a solution of (5) defined for all $n \in \mathbb{Z}$. By substituting $n = -2(i+1)$ for $i = 1,2,\ldots,m$ into (5), we see that $T(n) = 0$ for these values of n. Likewise, by substituting $n = -3$ into (5), we find that $T(-2) = 0$. Using (5) it follows by induction on n that $T(n) = 0$ for all $n \leq -2(m+1)$ and for all $n \geq -2$ as well. On the other hand, it is easy to check that

$$G_m^{(i)}(n) = \delta_{n,-(2i+1)}$$

(where δ is the Kronecker delta) is a solution of (5) for $i = 1,2,\ldots,m$. Therefore $\dim V_1 = m$. ■

Before describing the general situation we need a definition and a lemma.

Definition 7 *Let A be an H-system of the form (3). An interval of integers*

$$I = \{k, k+1, \ldots, k+m\}, \quad m \geq 0, \tag{6}$$

is a segment of singularities *of A if $I \subseteq S(A)$ while $k-1, k+m+1 \notin S(A)$.*

Lemma 1 *Each segment of singularities (6) of equation (3) is of (at least) one of the following types:*

(i) all elements of the segment are trailing singularities;

(ii) all elements of the segment are leading singularities;

(iii) there exists j, $0 \leq j < m$, such that $k, k+1, \ldots, k+j$ are leading singularities, while $k+j+1, \ldots, k+m$ are trailing singularities.

Proof: If $u \in \mathbb{Z}$ is a trailing singularity and $u+1$ a leading singularity of (3) then $f(u) = g(u) = 0$, contrary to the assumption $f \perp g$. So any segment of singularities of (3) consists of a (possiby empty) interval of leading singularities followed by a (possiby empty) interval of trailing singularities. □

Theorem 3 *Let S denote the set of singularities of equation (3).*
a) If $S = \emptyset$ then $\dim V_1 = \dim V_2 = 1$.
b) If $S \neq \emptyset$ then $1 \leq \dim V_1 < \dim V_2 < \infty$.

Proof: a) This is clear.

b) There is only a finite set of components induced on $\mathbb{Z}$ by (3), therefore $\dim V_2 < \infty$.

Next we prove that $\dim V_1 < \dim V_2$. First we show that if (6) is a segment of singularities of (3), then the restriction of V_1 to

$$\hat{I} = \{k-1, k, , \ldots, k+m, k+m+1\}$$

has dimension ≤ 1, while the analogous restriction of V_2 obviously has dimension 2. Indeed, if u is a trailing singularity, then any sequence from V_1 vanishes at $u+1$; and if u is a leading singularity, then any sequence from V_1 vanishes at $u-1$. By Lemma 1 we have three possibilities (i), (ii), (iii) for (6). In case (i) we have $T(k+1) = T(k+2) = \cdots = T(k+m+1) = 0$, in case (ii) $T(k-1) = T(k) = \cdots = T(k+m-1) = 0$, in case (iii) $T(k-1) = T(k) = \ldots T(k+j-1) = 0$ and $T(k+j+2) = T(k+j+3) = \cdots = T(k+m+1) = 0$; in each case $T(n)$ can be nonzero at most in two points of $\hat{I}$, however the value at one of them is uniquely determined by the value at the other one. Therefore the dimension of the restricted V_1 is ≤ 1. The same holds for dimension of the restriction of V_1 to the set

$$\{k-v, k-v+1, \ldots, k, k+1, \ldots, k+m, k+m+1, \ldots, k+w\},$$

where $k, k+1, \ldots, k+m$ are singularities, while $k-v, \ldots, k-1$ and $k+m+1, \ldots, k+w$ are not. Gluing together two such restrictions with coinciding, say, $k+m+1, \ldots, k+w$, and non-intersecting singular parts, we get the dimension ≤ 2, while the dimension of the corresponding restriction of V_2 is 3 and so on. This proves that $\dim V_1 < \dim V_2$.

Finally we prove that $\dim V_1 \geq 1$. If there are leading singularities, let n_0 be the largest leading singularity. Set $T(n_0) = 1$ and $T(n) = 0$ for $n < n_0$. None of the points $n > n_0$ is a leading singularity, hence the value of T at $n > n_0$ is uniquely determined by the recurrence (3) and the initial condition $T(n_0) = 1$. If there are no leading singularities, let n_0 be the least trailing singularity. Set $T(n_0) = 1$ and $T(n) = 0$ for $n > n_0$. None of the points $n < n_0$ is a trailing singularity, hence the value of T at $n < n_0$ is uniquely determined by the recurrence (3) and the initial condition $T(n_0) = 1$. In either case V_1 contains a nonzero solution. □

Theorem 4 *For any integers s, t such that $1 \leq s < t$ there exists an equation of the form (3) such that $\dim V_1 = s$ and $\dim V_2 = t$.*

Proof: Consider the recurrence

$$q_m(n+1)\ T(n+1) = p_k(n) q_m(n)\ T(n) \tag{7}$$

where $k, m \geq 1$, $p_k(n)$ is as in Example 1, and $q_m(n)$ is as in Example 2. Here the set of singularities is $\{2i-1;\ i = 0, 1, \ldots, k-2\} \cup \{-(2i+1);\ i = 1, 2, \ldots, m\}$, so $\dim V_2 = k+m$. Let $T(n)$ be a solution of (7) defined for all $n \in \mathbb{Z}$. In exactly the same way as in Example 2 we can see that $T(n) = 0$ for $n = -2, -4, \ldots, -2(m+1)$, $n \leq -2(m+1)$ or $n \geq -2$, and that $G_m^{(i)}(n) = \delta_{n,-(2i+1)}$ is a solution of (7) for $i = 1, 2, \ldots, m$. Therefore $\dim V_1 = m$.

If $1 \leq s < t$, let $m = s$ and $k = t - s$. Then for equation (7), $\dim V_1 = m = s$ and $\dim V_2 = k + m = t$. □

We conclude this section by some remarks on computation of $\dim V_1$ and $\dim V_2$. Let A denote equation (3). According to Proposition 1, $\dim V_2(A)$ is the number of components induced on $\mathbb{Z}$ by A and is thus easy to compute. We claim that $\dim V_1(A)$ equals the dimension of the kernel of a bidiagonal matrix B defined as follows. Let α be the maximum and β the minimum of the integer roots of $f(x)g(x)$; if A has no integer singularities then we can take $\alpha = \beta = 1$. Let B be the $(\alpha - \beta + 1) \times (\alpha - \beta + 2)$ matrix with entries

$$b_{i,j} = \begin{cases} f(\alpha - i + 1), & j = i, \\ -g(\alpha - i + 1), & j = i + 1, \\ 0, & \text{otherwise}, \end{cases}$$

where $1 \leq i \leq \alpha - \beta + 1$ and $1 \leq j \leq \alpha - \beta + 2$. Indeed, any vector v such that $Bv = 0$ can be extended to a solution of A in a unique way. This mapping is an isomorphism between the kernel of B and $V_1(A)$.

Incidentally, this gives an alternative proof of the inequality $\dim V_1 \geq 1$: B has more columns than rows, hence its kernel is nontrivial.

4 Dimensions of solution spaces: The multivariate case

If $d \geq 2$ in (2) then the dimensions of V_1 and/or V_2 can be infinite as shown by the following examples.

Example 3 ($\dim V_1 = \infty$, $\dim V_2 = 1$) Let A be the system

$$\begin{aligned} (n_1 - 4n_2 + 1)T(n_1 + 1, n_2) &= (n_1 - 4n_2)T(n_1, n_2), \\ (n_1 - 4n_2 - 4)T(n_1, n_2 + 1) &= (n_1 - 4n_2)T(n_1, n_2). \end{aligned}$$

It is easy to check that

$$T_i(n_1, n_2) = \delta_{n_1,4i}\delta_{n_2,i}, \qquad \text{for } i \in \mathbb{Z},$$

are linearly independent solutions of A on all of $\mathbb{Z}^2$, hence $\dim V_1 = \infty$. On the other hand, $S(A) = \{(n_1, n_2);\ n_1 = 4n_2\}$, so A induces a single component on $\mathbb{Z}^2$, and $\dim V_2 = 1$. ∎

Example 4 ($\dim V_1 = 1$, $\dim V_2 = \infty$) Let B be the system

$$\begin{aligned} (n_1 - 4n_2)T(n_1 + 1, n_2) &= (n_1 - 4n_2 + 1)T(n_1, n_2), \\ (n_1 - 4n_2)T(n_1, n_2 + 1) &= (n_1 - 4n_2 - 4)T(n_1, n_2). \end{aligned}$$

It can be shown that any solution of B defined on all $\mathbb{Z}^2$ is a constant multiple of $n_1 - 4n_2$, so $\dim V_1 = 1$. On the other hand, $S(B) = \{(n_1, n_2);\ n_1 - 4n_2 \in \{-4, -1, 1, 4\}\}$, so each of the points $(4i, i)$ for $i \in \mathbb{Z}$ is a separate component of $\mathbb{Z}^2$ induced by B, hence $\dim V_2 = \infty$. ∎

Example 5 ($\dim V_1 = \dim V_2 = \infty$) Let C be the system

$$\begin{aligned} (n_1 - n_2 - 1)(n_1 - n_2 + 1)T(n_1 + 1, n_2) &= (n_1 - n_2)(n_1 - n_2 + 2)T(n_1, n_2), \\ (n_1 - n_2 - 1)(n_1 - n_2 + 1)T(n_1, n_2 + 1) &= (n_1 - n_2)(n_1 - n_2 - 2)T(n_1, n_2). \end{aligned}$$

It is easy to check that

$$T_i(n_1, n_2) = \delta_{n_1,i}\delta_{n_2,i}, \qquad \text{for } i \in \mathbb{Z}, \tag{8}$$

are linearly independent solutions of C on all of $\mathbb{Z}^2$, hence $\dim V_1 = \infty$. As $S(C) = \{(n_1, n_2);\ n_1 - n_2 \in \{-2, 0, 2\}\}$, each of the points $(i, i - 1)$ and $(i, i + 1)$ for $i \in \mathbb{Z}$ is a separate component of $\mathbb{Z}^2$ induced by C, so $\dim V_2 = \infty$ as well. ∎

The following theorem describes the general situation.

Theorem 5 *Let $1 \leq s, t \leq \infty$. Then there exists an H-system such that* $\dim V_1 = s$ *and* $\dim V_2 = t$.

Proof: Let $t \geq 2$ and $p_t(n_1, n_2) = \prod_{i=0}^{t-2}(n_1 - n_2 + 3i)$. Then the set of singularities of

$$\begin{aligned} p_t(n_1+1, n_2)T(n_1+1, n_2) &= p_t(n_1, n_2)T(n_1, n_2), \\ p_t(n_1, n_2+1)T(n_1, n_2+1) &= p_t(n_1, n_2)T(n_1, n_2) \end{aligned}$$

is $S = \{(n_1, n_2);\ n_1 - n_2 \in \{-3i;\ 0 \leq i \leq t-2\}\}$. As in Example 5, the functions (8) are linearly independent solutions of this system on all of $\mathbb{Z}^2$, hence $\dim V_1 = \infty$. On the other hand, the number of components induced on $\mathbb{Z}^2$ is t, so $\dim V_2 = t$.

Let $s \geq 2$ and

$$q_s(n_1, n_2) = \prod_{i=1}^{s-1}((n_1 - 2i)^2 + n_2^2). \tag{9}$$

Then the set of singularities of

$$\begin{aligned} (n_1 - 4n_2)q_{s+1}(n_1+1, n_2)T(n_1+1, n_2) &= (n_1 - 4n_2 + 1)q_{s+1}(n_1, n_2)T(n_1, n_2), \\ (n_1 - 4n_2)q_{s+1}(n_1, n_2+1)T(n_1, n_2+1) &= (n_1 - 4n_2 - 4)q_{s+1}(n_1, n_2)T(n_1, n_2) \end{aligned}$$

is $S = \{(n_1, n_2);\ n_1 - 4n_2 \in \{-4, -1, 1, 4\}\} \cup \{(2i, 0);\ 1 \leq i \leq s\}$. Each of the points $(4i, i)$ for $i \in \mathbb{Z}$ is a separate component, so $\dim V_2 = \infty$. It can be shown that any solution $T(n_1, n_2)$ defined on all $\mathbb{Z}^2$ vanishes everywhere except at the points $(2i, 0)$ where $1 \leq i \leq s$, and that

$$T_i(n_1, n_2) = \delta_{n_1, 2i}\delta_{n_2, 0}, \tag{10}$$

for $i = 1, 2, \ldots, s$, are linearly independent solutions of this system defined on all $\mathbb{Z}^2$. Hence $\dim V_1 = \infty$.

Together with Examples 3 – 5 this proves the assertion in the case when at least one of s, t is infinite.

Now assume that s, t are natural numbers, and let $r_t(n_1, n_2) = \prod_{i=1}^{t-1}(n_1 + 2i + 1)$. Consider the system

$$\begin{aligned} q_s(n_1+1, n_2)T(n_1+1, n_2) &= q_s(n_1, n_2)r_t(n_1, n_2)T(n_1, n_2), \\ q_s(n_1, n_2+1)T(n_1, n_2+1) &= q_s(n_1, n_2)T(n_1, n_2), \end{aligned}$$

where q_s is as in (9). It can be shown that any solution $T(n_1, n_2)$ defined on all $\mathbb{Z}^2$ vanishes for all (n_1, n_2) such that $n_1 > -(2t-1)$ and (n_1, n_2) is not of the form $(2i, 0)$ with $1 \leq i \leq s-1$. Further, a basis of V_1 is given by the s functions $T_i(n_1, n_2)$ for $i = 0, 1, \ldots, s-1$ where

$$T_0(n_1, n_2) = \begin{cases} \frac{(-1)^{(t-1)n_1}}{\prod_{i=1}^{s-1}((n_1-2i)^2+n_2^2)\prod_{i=1}^{t-1}(-n_1-2i-1)!}, & n_1 \leq -(2t-1), \\ 0, & \text{otherwise}, \end{cases}$$

and $T_i(n_1, n_2)$ are as in (10) for $i = 1, 2, \ldots, s-1$. It follows that $\dim V_1 = s$. The set of singularities of this system is $S = \{(2i, 0);\ 1 \leq i \leq s-1\} \cup \{(-(2i+1), j);\ 1 \leq i \leq t-1, j \in \mathbb{Z}\}$, and the number of components induced on $\mathbb{Z}^2$ is t, so $\dim V_2 = t$ as desired. $\square$

We considered the case $d = 2$ here. The corresponding H-systems for the case of an orbitrary $d > 1$ can be obtained by adding equations $E_{n_i}T = T$, $i = 3, \dots, d$, to the systems with $d = 2$.

5 The Ore-Sato theorem and its consequences

The well-known Ore-Sato theorem (see [5], [7]) is commonly believed to imply that any solution of an H-system (2) is of the form (1). We show that this is not so, and give an appropriate corollary of the Ore-Sato theorem on possible forms of solutions of H-systems in our setting.

Definition 8 *Let T be a solution of (2). We write* $\operatorname{supp} T$ *for the* support *of T, i.e., for the set of points in $\mathbb{Z}^d$ where T is defined and does not vanish.*

If (2) has a solution with non-algebraic support, then the rational functions f_i/g_i, $i = 1, \dots, d$, are compatible, and uniquely determined by this solution (see [2]).

Definition 9 *A polynomial $p \in \mathbb{C}[x_1, \dots, x_d]$ is* integer-linear *if $p(x_1, \dots, x_d) = a_0 + a_1x_1 + \dots + a_dx_d$ where $a_1, \dots, a_d \in \mathbb{Z}$.*

The Ore-Sato theorem states (in the case $d = 2$) that for any compatible rational functions $F_1(x, y)$ and $F_2(x, y)$ there are compatible rational functions $G_1(x, y)$ and $G_2(x, y)$ which factor into integer-linear factors, and a rational function $R(x, y)$ such that $F_1(x, y) = G_1(x, y)R(x + 1, y)/R(x, y)$ and $F_2(x, y) = G_2(x, y)R(x, y + 1)/R(x, y)$. The full statement gives a precise description of the integer-linear factors.

In the literature one often encounters the claim that as a corollary of this theorem, any solution of an H-system (2) is of the form (1). For example, in [3, p. 223] one can read: "*From Ore's result it can be deduced that the most general form of A_{mn} is of the form*

$$A_{mn} = R(m, n)\gamma_{mn}a^m b^n$$

where R is a fixed rational function of m and n, a and b are constants, and γ_{mn} is a gamma product (...) that is to say it is of the form

$$\gamma_{mn} = \prod_i \Gamma(a_i + u_im + v_in)/\Gamma(a_i)$$

where the a_i are arbitrary (real or complex) constants, and the u_i and v_i are arbitrary integers which may be positive, negative, or zero." A similar quote can be found in [4, p. 5].

It may be the case that in the literature referred to above the term A_{mn} is implicitly assumed to be nonzero for all $m, n \in \mathbb{Z}$. This possibility is supported by the fact that, e.g., in [3] the corresponding H-system is given in terms of the two quotients $A_{m+1,n}/A_{mn}$ and $A_{m,n+1}/A_{mn}$. But such a severe restriction would preclude many important functions from being hypergeometric, such as the binomial coefficient $T(n_1, n_2) = \binom{n_1}{n_2}$, and all polynomials with integer roots.

However if we do not adopt this restriction, then there are hypergeometric terms which cannot be written in the form (1), as illustrated by the following examples.

Example 6 Take the H-system

$$\begin{aligned} p(n_1,n_2)T(n_1+1,n_2) &= p(n_1+1,n_2)T(n_1,n_2), \\ p(n_1,n_2)T(n_1,n_2+1) &= p(n_1,n_2+1)T(n_1,n_2), \end{aligned} \tag{11}$$

where $p(n_1,n_2) = (n_1-n_2-1)(n_1-n_2+1)$. It can be checked that any sequence T which satisfies $T(n_1,n_2) = 0$ unless $n_1 = n_2$ is a solution of (11). In particular, the sequence

$$T(n_1,n_2) = \begin{cases} 2^{n_1^2}, & n_1 = n_2, \\ 0, & \text{otherwise}, \end{cases}$$

is a solution of (11), even though it does not have the form (1) because it grows too fast along the diagonal. ■

There are examples which look less artificial and where the solution has a non-algebraic support.

Example 7 In this example we show that $|n_1-n_2|$, although a hypergeometric term, cannot be written in the form (1).

Denote $D(n_1,n_2) = |n_1-n_2|$. Then

$$\begin{aligned} (n_1-n_2)D(n_1+1,n_2) &= (n_1-n_2+1)D(n_1,n_2), \\ (n_1-n_2)D(n_1,n_2+1) &= (n_1-n_2-1)D(n_1,n_2) \end{aligned} \tag{12}$$

for all $n_1,n_2 \in \mathbb{Z}$, so $D(n_1,n_2)$ is a hypergeometric term. Note that when restricted to $n_1,n_2 \geq 0$, it is also holonomic, because its generating function is rational:

$$\sum_{n_1,n_2\geq 0} |n_1-n_2| z_1^{n_1} z_2^{n_2} = \left(\frac{z_1}{(1-z_1)^2} + \frac{z_2}{(1-z_2)^2}\right)\frac{1}{1-z_1z_2}. \tag{13}$$

To compare, the generating function of the polynomial n_1-n_2 is

$$\sum_{n_1,n_2\geq 0} (n_1-n_2) z_1^{n_1} z_2^{n_2} = \left(\frac{z_1}{1-z_1} - \frac{z_2}{1-z_2}\right)\frac{1}{(1-z_1)(1-z_2)}.$$

Let $T(n_1,n_2)$ be a hypergeometric term of the form (1) with $d = 2$, defined for all $n_1,n_2 \geq 0$. Pick $k_0 \in \mathbb{Z}$, $k_0 > 0$, and assume that $T(n,k_0) = |n-k_0|$ for all $n > k_0$. *Then we claim that* $T(n,k_0) = n-k_0$ *for all* $n \geq 0$. Hence $T(n,k_0)$ disagrees with $|n-k_0|$ for all n such that $0 \leq n < k_0$.

To prove the claim, define

$$t(n) \;:=\; T(n,k_0) \;=\; R(n,k_0)\; u_1^n u_2^{k_0} \frac{\prod_{i=1}^p \Gamma(a_{i,1}n + a_{i,2}k_0 + \alpha_i)}{\prod_{j=1}^q \Gamma(b_{j,1}n + b_{j,2}k_0 + \beta_j)}, \quad \text{for all } n \geq 0. \tag{14}$$

We wish to rewrite the right-hand side of (14) in such a way that the coefficient of n in the arguments of the Gamma function will be 1. It is straightforward to verify that for $n \in \mathbb{Z}$, $a \in \mathbb{Z}\setminus\{0\}$ and $z \in \mathbb{C}$ such that $an+z$ is not a nonpositive integer,

$$\Gamma(an+z) = \begin{cases} C(a,z)\, a^{an} \prod_{m=0}^{a-1} \Gamma(n+(z+m)/a), & a > 0, \\ C(a,z)\, a^{an} / \prod_{m=1}^{|a|} \Gamma(n+(z-m)/a), & a < 0,\ z \notin \mathbb{Z}, \end{cases}$$

where $C(a, z) \in \mathbb{C}$ is independent of n. To be able to apply this to (14), we need to show that

(i) for $i = 1, \ldots, p$ and for all $n \geq 0$, the number $a_{i,1}n + a_{i,2}k_0 + \alpha_i$ is not a nonpositive integer,

(ii) for $i = 1, \ldots, p$ and for all $n \geq 0$, if $a_{i,1} < 0$ then $\alpha_i \notin \mathbb{Z}$,

(iii) for $j = 1, \ldots, q$ and for all $n \geq 0$, the number $b_{j,1}n + b_{j,2}k_0 + \beta_j$ is not a nonpositive integer,

(iv) for $j = 1, \ldots, q$ and for all $n \geq 0$, if $b_{j,1} < 0$ then $\beta_j \notin \mathbb{Z}$.

Assertion (i) is obvious, for otherwise $T(n, k_0)$ would be undefined at such n. Assertion (ii) holds for the same reason, since if $a_{i,1} < 0$ and $\alpha_i \in \mathbb{Z}$ for some i, then $a_{i,1}n + a_{i,2}k_0 + \alpha_i$ would be a nonpositive integer for all large enough n. If $b_{j,1}n + b_{j,2}k_0 + \beta_j$ is a nonpositive integer for some j and some $n \neq k_0$, then t vanishes at this n, contrary to the fact that $t(n) = |n - k_0| \neq 0$ for all $n \neq k_0$. If $b_{j,1}k_0 + b_{j,2}k_0 + \beta_j$ is a nonpositive integer then (depending on the sign of $b_{j,1}$) at least one of $b_{j,1}(k_0 - 1) + b_{j,2}k_0 + \beta_j$ and $b_{j,1}(k_0 + 1) + b_{j,2}k_0 + \beta_j$ is also a nonpositive integer, something we have just ruled out. This proves (iii). Assertion (iv) holds for the same reason, since if $b_{j,1} < 0$ and $\beta_j \in \mathbb{Z}$ for some j, then $b_{j,1}n + b_{j,2}k_0 + \beta_j$ would be a nonpositive integer for all large enough n.

Therefore the univariate term t can be written in the form

$$t(n) = r(n)\, v^n \frac{\prod_{i=1}^{p'} \Gamma(n + \gamma_i)}{\prod_{j=1}^{q'} \Gamma(n + \delta_j)}, \quad \text{for all } n \geq 0, \tag{15}$$

where $r \in \mathbb{C}(x)$ and $v, \gamma_i, \delta_j \in \mathbb{C}$. If $\gamma_i - \delta_j \in \mathbb{Z}$ then $\Gamma(n + \gamma_i)/\Gamma(n + \delta_j)$ is a rational function of n, hence (15) can be rewritten as

$$t(n) = s(n)\, v^n \frac{\prod_{i=1}^{p''} \Gamma(n + \varepsilon_i)}{\prod_{j=1}^{q''} \Gamma(n + \zeta_j)}, \quad \text{for all } n \geq 0, \tag{16}$$

where $s \in \mathbb{C}(x)$, $\varepsilon_i, \zeta_j \in \mathbb{C}$, and none of the differences $\varepsilon_i - \zeta_j$ is an integer. It follows that

$$g(x) \;:=\; v \frac{\prod_{i=1}^{p''} (x + \varepsilon_i)}{\prod_{j=1}^{q''} (x + \zeta_j)} \;\in\; \mathbb{C}(x)$$

is a shift-reduced rational function (see Definition 1). Let $f(x) := (x+1-k_0)/(x-k_0) \in \mathbb{C}(x)$. For $n > k_0$ we have

$$\frac{t(n+1)}{t(n)} \;=\; \frac{|n + 1 - k_0|}{|n - k_0|} \;=\; \frac{n + 1 - k_0}{n - k_0} \;=\; f(n)$$

and

$$\frac{t(n+1)}{t(n)} \;=\; g(n)\, \frac{s(n+1)}{s(n)}.$$

The two rational functions $f(x)$ and $g(x)\, s(x+1)/s(x)$ agree infinitely often, so they are equal. Since g is shift-reduced, both $(1, x-k_0)$ and $(g(x), s(x))$ are rational normal forms of f. Now Theorem 2 implies that $g(x) = 1$. Comparing this with the definition of $g(x)$, we see that $v = 1$ and $p'' = q'' = 0$. From (16) it follows that $s(n) = t(n)$ for all $n \geq 0$, therefore $s(n) = n - k_0$ for all $n > k_0$. As the two rational functions $s(x)$ and $x - k_0$ agree infinitely often, they are equal. But then $t(n) = n - k_0$ for all $n \geq 0$, which proves our claim.

We could have chosen a specific value for k_0 (such as $k_0 = 1$, say), but by doing so our result would be slightly weaker. In geometric language, we have shown that as soon as a hypergeometric term $T(n_1, n_2)$ agrees with $|n_1 - n_2|$ on *any* horizontal (or, by symmetry, vertical) line which contains integer points on both sides of the line $n_1 = n_2$, then it cannot have the form (1). It seems that, under some additional conditions, this could be generalized from $|n_1 - n_2|$ to $|R(n_1, n_2)|$ where $R \in \mathbb{C}(x_1, x_2)$, and to horizontal (or vertical) lines containing integer points (p_1, p_2) and (n_1, n_2) with $R(p_1, p_2) > 0$ and $R(n_1, n_2) < 0$.

In the theory of multivariate hypergeometric series, H-systems are used to specify coefficients for such series. The simple rational function on the right-hand side of (13) has series expansion whose coefficients satisfy the H-system (12), however are not of the form (1). ∎

The following statement is a corollary of the Ore-Sato theorem.

Corollary 1 *Any constituent (see Definition 5) of a solution with non-algebraic support of an H-system (2) is of the form (1).*

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