A COMBINATORIAL MODEL FOR CRYSTALS OF KAC-MOODY ALGEBRAS (EXTENDED ABSTRACT)

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Abstract. We present a simple combinatorial model for the characters of the irreducible representations of Kac-Moody algebras. This model can be viewed as a discrete counterpart to the Littelmann path model. We describe crystal graphs and give a Littlewood-Richardson rule for decomposing tensor products of irreducible representations.

1. Introduction

We have recently given a combinatorial model for the characters of the irreducible representations of a complex semisimple Lie group G, and for the Demazure characters [LP1]. This model was defined in the context of the equivariant K-theory of the generalized flag variety G/B. Our character formulas were derived from a Chevalley-type formula in $K_T(G/B)$. Our model was based on enumerating certain saturated chains in the *Bruhat order* on the corresponding *Weyl group* W. This enumeration is determined by an *alcove path*, which is a sequence of adjacent alcoves for the *affine Weyl group* W_{aff} of the Langland's dual group $G^\vee$. Alcove paths correspond to decompositions of elements in the affine Weyl group. Our Chevalley-type formula was formulated in terms of a certain *R-matrix*, that is, in terms of a collection of operators satisfying the *Yang-Baxter equation*. This setup allowed us to easily explain the independence of our formulas from the choice of an alcove path.

There are other models for Chevalley-type formulas in $K_T(G/B)$ and for the irreducible characters of G. Most notably, there is the *Littelmann path model*. Littelmann [Li1, Li2, Li3] showed that the characters can be described by counting certain continuous paths in $\mathfrak{h}^*_{\mathbb{R}}$. These paths are constructed recursively, by starting with an initial one, and by applying certain *root operators*. By making specific choices for the initial path, one can obtain special cases which have more explicit descriptions. For instance, a straight line initial path leads to the *Lakshmibai-Seshadri paths* (LS paths). These were introduced before Littelmann's work, in the context of *standard monomial theory* [LS]. They have a nonrecursive description as weighted chains in the Bruhat order on the quotient W/W_λ of the corresponding Weyl group W modulo the stabilizer W_λ of the weight λ; therefore, we will use the term *LS chains* when referring to this description. LS paths were used by Pittie and Ram [PR] to derive a K_T-Chevalley formula. Recently, Gaussent and Littelmann [GL], motivated by the study of Mirković-Vilonen cycles, defined another combinatorial model for the irreducible characters of a complex semisimple Lie group. This model is based on *LS galleries*, which are certain sequences of faces of alcoves for the corresponding affine Weyl group. For each LS gallery, there is an associated Littelmann path, and a saturated chain in the Bruhat order on W/W_λ. In [LP1], we explained the way in which our construction, which was developed independently of LS galleries, is related (although not quite equivalent) to the latter in the case of regular weights.

In this paper, we develop the combinatorial model in [LP1] purely in the context of representation theory, and extend it to complex symmetrizable Kac-Moody algebras. Instead of alcove paths (that make sense only in finite types) we now use λ-*chains*, which are chains of roots satisfying a certain *interlacing property*. Note that Littelmann paths and, in particular, LS paths were also defined in this more general context, but LS galleries were not. In fact, we show that LS paths are a certain limiting case of a special case of our model. The latter can be viewed as a discrete counterpart to the Littelmann path model. We define root operators in our model, and study their properties. This allows us to show that our model

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satisfies the axioms of an *admissible system* of Stembridge [Ste]. Thus, we easily derive character formulas, a Littlewood-Richardson rule for decomposing tensor products of irreducible representations, as well as a branching rule. The approach via admissible systems was already applied to LS chains in [Ste, Section 8]. Compared to the proofs in [GL, Li2, Li3], Stembridge's approach has the advantage of making a part of the proof independent of a particular model for Weyl characters, by using a system of axioms for such models.

Our model has several advantages over the Littelmann path model and its specializations mentioned above. First of all, our formulas are equally simple for all weights, regular and nonregular. Note that the (nonrecursive) construction of LS chains and LS galleries usually involves certain choices that add to their computational complexity. Also, it is harder to work with sequences of lower dimensional faces of alcoves (in the case of LS galleries) than with sequences of roots (in our model). We refer to [LP1] for a discussion showing that the computational complexity of our model is significantly smaller than the one of Littelmann paths (constructed recursively via root operators). Our definition of root operators resembles the one for LS paths, which is simpler than the general definition of root operators for Littelmann paths. We think that our model is easier to work with in explicit computations because, being based on certain chains of roots, it has a stronger combinatorial nature than Littelmann paths and, in particular, LS chains. Indeed, even for LS chains, we do need their description as piecewise-linear paths in order to define root operators.

We believe that the properties of our model discussed in this paper represent just a small fraction of a rich combinatorial structure yet to be explored. We will investigate it in a forthcoming paper [LP2].

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2. Preliminaries

In this section, we briefly recall the general setup for complex symmetrizable Kac-Moody algebras and their representations. We refer to [Kac, Ku] for more details.

Let V be a finite-dimensional real vector space with a nondegenerate symmetric bilinear form $\langle \cdot\,,\,\cdot \rangle$, and let $\Phi \subset V$ be a crystallographic root system of rank r with simple roots $\{\alpha_1, \ldots, \alpha_r\}$. By this, we mean that Φ is the set of real roots of some complex symmetrizable Kac-Moody algebra. The finite root systems of this type are the root systems of semisimple Lie algebras.

Given a root α, the corresponding *coroot* is $\alpha^\vee := 2\alpha/\langle \alpha, \alpha \rangle$. The collection of coroots $\Phi^\vee := \{\alpha^\vee \mid \alpha \in \Phi\}$ forms the *dual root system.* For each root α, there is a reflection $s_\alpha : V \to V$ defined by $s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle\, \alpha$. More generally, for any integer k, one can consider the affine hyperplane $H_{\alpha,k} := \{\lambda \in V \mid \langle \lambda, \alpha^\vee \rangle = k\}$, and let $s_{\alpha,k}$ denote the corresponding reflection, that is, $s_{\alpha,k} : \lambda \mapsto s_\alpha(\lambda) + k\alpha$.

The *Weyl group* W is the subgroup of $GL(V)$ generated by the reflections s_α for $\alpha \in \Phi$. In fact, the Weyl group W is a Coxeter group, which is generated by the *simple reflections* $s_1, \ldots, s_r$ corresponding to the simple roots $s_p := s_{\alpha_p}$, subject to the *Coxeter relations*: $(s_p)^2 = 1$ and $(s_p s_q)^{m_{pq}} = 1$; here the relations of the second type correspond to the distinct p, q in $\{1, \ldots, r\}$ for which the dihedral subgroup generated by s_p and s_q is finite, in which case m_{pq} is half the order of this subgroup. The Weyl group is finite if and only if Φ is finite.

An expression of a Weyl group element w as a product of generators $w = s_{p_1} \cdots s_{p_l}$ which has minimal length is called a *reduced decomposition* for w; its length $\ell(w) = l$ is called the *length* of w. For $u, w \in W$, we say that u *covers* w, and write $u \gtrdot w$, if $w = u s_\beta$, for some $\beta \in \Phi^+$, and $\ell(u) = \ell(w) + 1$. The transitive closure "$>$" of the relation "$\gtrdot$" is called the *Bruhat order* on W.

Let us note that Φ can be characterized by the following axioms:

(R1) $\{\alpha_1, \ldots, \alpha_r\}$ is a linearly independent set.
(R2) $\langle \alpha_p, \alpha_p \rangle > 0$ for all $p = 1, \ldots, r$.

(R3) $\langle \alpha_p, \alpha_q^\vee \rangle \in \mathbb{Z}_{\leq 0}$ for all distinct simple roots α_p and α_q.
(R4) $\Phi = \bigcup_{p=1}^{r} W\alpha_p$.

Let $\Phi^+ \subset \Phi$ be the set of positive roots, that is, the set of roots in the nonnegative linear span of the simple roots. Then Φ is the disjoint union of Φ^+ and $\Phi^- := -\Phi^+$. We write $\alpha > 0$ (respectively, $\alpha < 0$) for $\alpha \in \Phi^+$ (respectively, $\alpha \in \Phi^-$), and we define $\mathrm{sgn}(\alpha)$ to be 1 (respectively, -1). We also use the notation $|\alpha| := \mathrm{sgn}(\alpha)\alpha$.

The lattice of (integral) *weights* Λ is given by $\Lambda := \{\lambda \in V \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \text{ for any } \alpha \in \Phi\}$. The set Λ^+ of *dominant weights* is given by $\Lambda^+ := \{\lambda \in \Lambda \mid \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for any } \alpha \in \Phi^+\}$. If we replace the weak inequalities above with strict ones, we obtain the strongly dominant weights. It is known that every W-orbit in V has at most one dominant member. The *fundamental weights* $\omega_1, \ldots, \omega_r$ are defined by $\langle \omega_p, \alpha_q^\vee \rangle = \delta_{pq}$.

We now define a ring R that contains the characters of all integrable highest weight modules for the corresponding Kac-Moody algebra. In the finite case, one may simply take R to be the group ring of Λ, but in general more care is required.

First, we choose a height function $\mathrm{ht} : V \to \mathbb{R}$, that is, a linear map assigning the value 1 to all simple roots. Second, for each $\lambda \in \Lambda$, let e^λ denote a formal exponential subject to the rules $e^\mu \cdot e^\nu = e^{\mu+\nu}$ for all $\mu, \nu \in \Lambda$. We now define the ring R to consist of all formal sums $\sum_{\lambda \in \Lambda} c_\lambda e^\lambda$ with $c_\lambda \in \mathbb{Z}$ satisfying the condition that there are only finitely many weights λ with $\mathrm{ht}(\lambda) > h$ and $c_\lambda \neq 0$, for all $h \in \mathbb{R}$.

For each $\lambda \in \Lambda^+$ with a finite W-stabilizer, we define

$$\Delta(\lambda) := \sum_{w \in W} \mathrm{sgn}(w) e^{w(\lambda)},$$

where $\mathrm{sgn}(w) = (-1)^{\ell(w)}$. It is not hard to check that $\Delta(\lambda)$ is a well-defined member of R. Since the scalar product is nondegenerate, we may select $\rho \in \Lambda^+$ so that $\langle \rho, \alpha_p^\vee \rangle = 1$ for all $p = 1, \ldots, r$. One can verify that $\Delta(\rho)$ is invertible in R. This given, for each $\lambda \in \Lambda^+$ we define

$$\chi(\lambda) := \frac{\Delta(\lambda + \rho)}{\Delta(\rho)} = \frac{\sum_{w \in W} \mathrm{sgn}(w) e^{w(\lambda+\rho)-\rho}}{\sum_{w \in W} \mathrm{sgn}(w) e^{w(\rho)-\rho}} \in R.$$

It is easy to show that $w(\rho) - \rho$, and hence $\chi(\lambda)$, do not depend on the choice of ρ. By the Kac-Weyl character formula [Kac], these are the characters of the irreducible highest weight modules for the corresponding Kac-Moody algebra.

3. Crystals

This section closely follows [Ste, Section 2]. We refer to this paper for more details.

Definition 3.1. (cf. [Ste]). A *crystal* is a 4-tuple $(X, \mu, \delta, \{F_1, \ldots, F_r\})$ satisfying Axioms (A1)-(A3) below, where

- X is a set whose elements are called objects;
- μ and δ are maps $X \to \Lambda$;
- F_p are bijections between two subsets of X.

For each $x \in X$, we call $\mu(x)$, $\delta(x)$, and $\varepsilon(x) := \mu(x) - \delta(x)$ the *weight*, *depth*, and *rise* of x.

(A1) $\delta(x) \in -\Lambda^+$, $\varepsilon(x) \in \Lambda^+$.

We define the depth and rise in the direction α_p by $\delta(x, p) := \langle \delta(x), \alpha_p^\vee \rangle$ and $\varepsilon(x, p) := \langle \varepsilon(x), \alpha_p^\vee \rangle$. In fact, we will develop the whole theory in terms of $\delta(x, p)$ and $\varepsilon(x, p)$ rather than $\delta(x)$ and $\varepsilon(x)$.

(A2) F_p is a bijection from $\{x \in X \mid \varepsilon(x, p) > 0\}$ to $\{x \in X \mid \delta(x, p) < 0\}$.

(A3) $\mu(F_p(x)) = \mu(x) - \alpha_p$, $\delta(F_p(x), p) = \delta(x, p) - 1$.

Hence, we also have $\varepsilon(F_p(x), p) = \varepsilon(x, p) - 1$. We let $E_p := F_p^{-1}$ denote the inverse map. The maps E_p and F_p act as raising and lowering operators that provide a partition of the objects into α_p-strings that are closed under the action of E_p and F_p. For example, the *α_p-string through* x is (by definition) $F_p^{\varepsilon}(x), \ldots, F_p(x), x, E_p(x), \ldots, E_p^{-\delta}(x)$, where $\delta = \delta(x, p)$ and $\varepsilon = \varepsilon(x, p)$. Let us now present some additional axioms.

(A4) For all real numbers h, there are only finitely many objects x such that $\mathrm{ht}(\mu(x)) > h$.

Axiom (A4) implies that the generating series $G_X := \sum_{x \in X} e^{\mu(x)}$ is a well-defined member of R.

We define a partial order on X by $x \preceq_p y$ if $x = F_p^k(y)$ for some $k \geq 0$. We call an object of X *maximal* if it is maximal with respect to all partial orders $\preceq_p$, for $p = 1, \ldots, r$.

(A5) X has a unique maximal object.

Stembridge defined *admissible systems* as crystals satisfying Axiom (A4) and an extra axiom, which is related to the existence of a certain map $(x, p) \mapsto t(x, p)$ on pairs (x, p) with $\delta(x, p) < 0$. This map is called *coherent timing pattern*, and is used to construct a certain sign-reversing involution allowing one to cancel the negative terms in the Kac-Weyl character formula. We call an admissible system a *semiperfect crystal* if it satisfies Axiom (A5).

Given $P \subseteq \{1, \ldots, r\}$, let Φ_P denote the root subsystem of Φ with simple roots $\{\alpha_p \mid p \in P\}$. Following [Ste], we let $W_P \subseteq W$, $\Lambda_P \supseteq \Lambda$, and R_P denote the corresponding Weyl group, weight lattice, and character ring. Given $\lambda \in \Lambda_P^+$, we let $\chi(\lambda; P) \in R_P$ denote the Weyl character (relative to Φ_P) corresponding to λ.

Finally, note that one can define on X the structure of a directed colored graph by constructing arrows $x \to y$ colored p for each $F_p(x) = y$.

Definition 3.2. A crystal $(X, \mu, \delta, \{F_1, \ldots, F_r\})$ is called a *perfect crystal* if the associated directed colored graph is isomorphic to the *crystal graph* of an irreducible representation of a quantum group.

4. λ-Chains of Roots

Fix a dominant weight λ. Throughout this paper, we will use the term "sequence" for any map $i \mapsto a_i$ from a totally ordered set I to some other set; we will use the notation $\{a_i\}_{i \in I}$.

Definition 4.1. A λ-*chain (of roots)* is a sequence of positive roots $\{\beta_i\}_{i \in I}$ indexed by the elements of a totally ordered set I, which satisfies the following conditions:

(1) the number of occurrences of any positive root α is $\langle \lambda, \alpha^\vee \rangle$;
(2) for each triple of positive roots (α, β, γ) with $\gamma^\vee = \alpha^\vee + \beta^\vee$, the finite sequence $\{\beta_j\}_{j \in J}$, where $J := \{j \in I \mid \beta_j \in \{\alpha, \beta, \gamma\}\}$ has the induced total order, is a concatenation of pairs (α, γ) and (β, γ) (in any order).

Note that finding a λ-chain amounts to defining a total order on the set

$$I := \{(\alpha, k) \mid \alpha \in \Phi^+,\, 0 \leq k < \langle \lambda, \alpha^\vee \rangle\} \tag{4.1}$$

such that the second condition above holds, where $\beta_i = \alpha$ for any $i = (\alpha, k)$ in I. One particular example of such an order can be constructed as follows. Fix a total order on the set of simple roots $\alpha_1 < \alpha_2 < \ldots < \alpha_r$. For each $i = (\alpha, k)$ in I, let $\alpha^\vee = c_1 \alpha_1^\vee + \ldots + c_r \alpha_r^\vee$, and define the vector

$$v_i := \frac{1}{\langle \lambda, \alpha^\vee \rangle}(k, c_1, \ldots, c_r) \tag{4.2}$$

in $\mathbb{Q}^{r+1}$. The map $i \mapsto v_i$ is easily seen to be injective.

Proposition 4.2. *Consider the total order on the set I in* (4.1) *defined by $i < j$ iff $v_i < v_j$ in the lexicographic order on $\mathbb{Q}^{r+1}$. The sequence $\{\beta_i\}_{i \in I}$ given by $\beta_i = \alpha$ for $i = (\alpha, k)$ is a λ-chain.*

For the rest of our construction (Sections 5–7), let us fix a dominant integral weight λ and fix an arbitrary λ-chain $\{\beta_i\}_{i\in I}$. We will use the notation r_i for the reflection s_{β_i}.

5. Folding Chains of Roots

We start by associating to our fixed λ-chain the closely related object $\Gamma(\emptyset) := (\{(\beta_i,\beta_i)\}_{i\in I},\rho)$, where ρ is a fixed dominant weight satisfying $\langle\rho,\alpha_p^\vee\rangle = 1$ for all $p = 1,\ldots,r$. Here, as well as throughout this article, we let ∞ be greater than all elements in I. We use operators called *folding operators* to construct from $\Gamma(\emptyset)$ new objects of the form $\Gamma = (\{(\gamma_i,\gamma_i')\}_{i\in I},\gamma_\infty)$; here (γ_i,γ_i') are pairs of roots with $\gamma_i' = \pm\gamma_i$, any given root appears only finitely many times in Γ, and γ_∞ is in the W-orbit of ρ. More precisely, given Γ as above and i in I, we let $t_i := s_{\gamma_i}$ and we define

$$\phi_i(\Gamma) := (\{(\delta_j,\delta_j')\}_{j\in I}, t_i(\gamma_\infty))\,, \qquad \text{where} \quad (\delta_j,\delta_j') := \begin{cases} (\gamma_j,\gamma_j') & \text{if } j<i \\ (\gamma_j, t_i(\gamma_j')) & \text{if } j=i \\ (t_i(\gamma_j), t_i(\gamma_j')) & \text{if } j>i\,. \end{cases}$$

Let us now consider the set of all Γ that are obtained from $\Gamma(\emptyset)$ by applying folding operators; we call these objects the *foldings* of $\Gamma(\emptyset)$. Clearly, ϕ_i is an involution on the set of foldings of $\Gamma(\emptyset)$. In order to describe this set, let us note that the folding operators commute. This means that every folding Γ of $\Gamma(\emptyset)$ is determined by the set $J := \{j \mid \gamma_j' = -\gamma_j\}$. More precisely, if $J = \{j_1 < j_2 < \ldots < j_s\}$, then $\Gamma = \phi_{j_1}\ldots\phi_{j_s}(\Gamma(\emptyset))$. We call the elements of J the *folding positions* of Γ, and write $\Gamma = \Gamma(J)$.

Throughout this paper, we will use J and $\Gamma = \Gamma(J)$ interchangeably. For instance, according to the above discussion, we have $\phi_i(\Gamma(J)) = \Gamma(J\triangle\{i\})$, where $\triangle$ denotes the symmetric difference of sets. Hence, it makes sense to define the folding operator ϕ_i on J (compatibly with the action of ϕ_i on $\Gamma(J)$) by $\phi_i\,:\,J\mapsto J\triangle\{i\}$.

Remark 5.1. Although a folding Γ of $\Gamma(\emptyset)$ is an infinite sequence if the root system is infinite, we are, in fact, always working with finite objects. Indeed, we are examining Γ by considering only one root at a time.

Given a folding Γ of $\Gamma(\emptyset)$, we associate to each pair of roots (or the corresponding index i in I) an integer l_i, which we call *level*; the sequence $L = L(\Gamma) := \{l_i\}_{i\in I}$ will be called the *level sequence* of Γ. The definition is as follows:

$$l_i := \varepsilon + \sum_{j<i,\,\gamma_j=\gamma_j'=\pm\gamma_i} \operatorname{sgn}(\gamma_j)\,, \qquad \text{where} \quad \varepsilon := \begin{cases} 0 & \text{if } \gamma_i>0 \\ -1 & \text{otherwise}\,. \end{cases} \tag{5.1}$$

We make the convention that the sum is 0 if it contains no terms. The definition makes sense since the sum is always finite. In particular, we have the level sequence $L_\emptyset = L(\Gamma(\emptyset)) := \{l_i^\emptyset\}_{i\in I}$ of $\Gamma(\emptyset)$. Given a root α, we will use the following notation:

$$I_\alpha = I_\alpha(\Gamma) := \{i\in I \mid \gamma_i = \pm\alpha\}\,, \quad L_\alpha = L_\alpha(\Gamma) := \{l_i \mid i\in I_\alpha\}\,.$$

Remark 5.2. It is often useful to use the following graphical representation. Let $I_\alpha = \{i_1 < i_2 < \ldots < i_n\}$, and let us define the continuous piecewise-linear function $g_\alpha\,:\,[0,n]\to\mathbb{R}$ by

$$g_\alpha(0) = -\frac{1}{2}\,,\;\; g_\alpha'(x) = \begin{cases} \operatorname{sgn}(\gamma_{i_k}) & \text{if } x\in(k-1,k-\frac{1}{2}) \\ \operatorname{sgn}(\gamma_{i_k}') & \text{if } x\in(k-\frac{1}{2},k)\,, \end{cases}$$

for $k = 1,\ldots,n$. Then $l_{i_k} = g_\alpha(k-\frac{1}{2})$. For instance, assume that the entries of Γ indexed by the elements of I_α are $(\alpha,-\alpha)$, $(-\alpha,-\alpha)$, (α,α), (α,α), $(\alpha,-\alpha)$, $(-\alpha,-\alpha)$, $(\alpha,-\alpha)$, (α,α), in this order. The graph of g_α is shown on Figure 1.

We will now consider certain affine reflections corresponding to foldings Γ of $\Gamma(\emptyset)$. Let $\widehat{t}_i := s_{|\gamma_i|,l_i}$; recall that the latter is the reflection in the affine hyperplane $H_{|\gamma_i|,l_i}$. In particular, we have the affine reflections $\widehat{r}_i := s_{\beta_i,l_i^\emptyset}$ corresponding to $\Gamma(\emptyset)$.

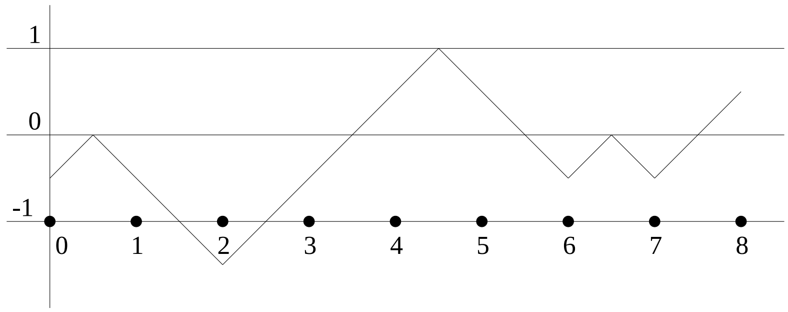

FIGURE 1

Definition 5.3. Given $J = \{j_1 < j_2 < \ldots < j_s\} \subseteq I$ and $\Gamma = \Gamma(J)$, we let

$$\mu = \mu(\Gamma) = \mu(J) := \widehat{r}_{j_1} \ldots \widehat{r}_{j_s}(\lambda)\,,$$

and call μ the *weight* of Γ (respectively J). We also use the notation $w(J) = w(\Gamma) := r_{j_1} \ldots r_{j_s}$ (recall that $r_i := s_{\beta_i}$), and

$$\widehat{I}_\alpha = \widehat{I}_\alpha(\Gamma) := I_\alpha \cup \{\infty\}\,, \quad \widehat{L}_\alpha = \widehat{L}_\alpha(\Gamma) := L_\alpha \cup \{l_\alpha^\infty\}\,, \qquad \text{where} \quad l_\alpha^\infty := \langle \mu(\Gamma), \alpha^\vee \rangle.$$

The following proposition is our main technical result, which relies heavily on the defining properties of λ-chains.

Proposition 5.4. *Let $\Gamma = \Gamma(J)$ for some $J = \{j_1 < j_2 < \ldots < j_s\} \subseteq I$, and let $j_p < j \leq j_{p+1}$ (the first or the second inequality is dropped if $p = 0$ or $p = s$, respectively). Using the notation above, we have*

$$H_{|\gamma_j|, l_j} = \widehat{r}_{j_1} \ldots \widehat{r}_{j_p}(H_{\beta_j, l_j^\emptyset})\,.$$

Furthermore, if $\Gamma' = \phi_i(\Gamma)$, then $\mu(\Gamma') = \widehat{t}_i(\mu(\Gamma))$.

The next proposition shows that all inner products of $\mu(\Gamma)$ with positive roots can be easily read off from the level sequence $L(\Gamma) = (l_i)_{i \in I}$. Recall that, given $\Gamma = (\{(\gamma_i, \gamma_i')\}_{i \in I}, \gamma_\infty)$, we defined $t_i := s_{\gamma_i}$.

Proposition 5.5. *Given a positive root α, let $m := \max I_\alpha(\Gamma)$, assuming that $I_\alpha(\Gamma) \neq \emptyset$. Then we have*

$$\langle \mu(\Gamma), \alpha^\vee \rangle = \begin{cases} l_m + 1 & \text{if } \gamma_m' > 0 \text{ and } t_{j_1} \ldots t_{j_s}(\alpha) > 0 \\ l_m - 1 & \text{if } \gamma_m' < 0 \text{ and } t_{j_1} \ldots t_{j_s}(\alpha) < 0 \\ l_m & \text{otherwise}. \end{cases}$$

On the other hand, if $I_\alpha(\Gamma) = \emptyset$, then we have

$$\langle \mu(\Gamma), \alpha^\vee \rangle = \begin{cases} 0 & \text{if } t_{j_1} \ldots t_{j_s}(\alpha) > 0 \\ -1 & \text{if } t_{j_1} \ldots t_{j_s}(\alpha) < 0\,. \end{cases}$$

Remark 5.6. If $\widehat{I}_\alpha = \{i_1 < i_2 < \ldots < i_n = m < i_{n+1} = \infty\}$, we can extend the definition of the function g_α in Remark 5.2 to the interval $[0, n + \frac{1}{2}]$ in order to express $l_\alpha^\infty := \langle \mu(\Gamma), \alpha^\vee \rangle$, as given by Proposition 5.5. More precisely, letting $g_\alpha'(x) = \mathrm{sgn}(\langle \gamma_\infty, \alpha^\vee \rangle)$ for $x \in (n, n + \frac{1}{2})$, we have $l_\alpha^\infty = g_\alpha(n + \frac{1}{2})$.

We will now define some special foldings $\Gamma(J)$ of $\Gamma(\emptyset)$.

Definition 5.7. An *admissible subset* is a finite subset of I (possibly empty), that is, $J = \{j_1 < j_2 < \ldots < j_s\}$, such that we have the following saturated chain in the Bruhat order on W:

$$1 \lessdot r_{j_1} \lessdot r_{j_1} r_{j_2} \lessdot \ldots \lessdot r_{j_1} r_{j_2} \ldots r_{j_s}\,.$$

If J is an admissible subset, we will call $\Gamma = \Gamma(J)$ an *admissible folding* (of $\Gamma(\emptyset)$). We denote by $\mathcal{A}$ the collection of all admissible subsets corresponding to our fixed λ-chain.

Admissible foldings have many nice combinatorial properties, such as the ones below.

Proposition 5.8. *Any pair of roots (γ_i, γ_i') in an admissible folding has one of the following forms: (α, α), $(-\alpha, -\alpha)$, or $(\alpha, -\alpha)$, for some positive root α. The first occurence after (α, α) of a pair containing a simple root α or its negative is of the form $(\alpha, \pm\alpha)$ (assuming that such a pair exists). The same is true for the very first occurence of such a pair, if any. If (α, α) is the last occurence of a pair containing a simple root α or its negative, then $\langle \gamma_\infty, \alpha^\vee \rangle > 0$. The same is true if there are no occurences of $(\pm\alpha, \pm\alpha)$.*

6. Root Operators

We will now define *root operators* on the collection $\mathcal{A}$ of admissible subsets corresponding to our fixed λ-chain. Let J be such an admissible subset, let Γ be the associated admissible folding, and $L(\Gamma) = (l_i)_{i \in I}$ its level sequence, denoted as in Section 5.

We will first define a partial operator F_p on admissible subsets J for each p in $\{1, \ldots, r\}$, that is, for each simple root α_p. Let p in $\{1, \ldots, r\}$ be fixed throughout this section. Let $M = M(\Gamma) = M(\Gamma, p) = M(J, p)$ be the maximum of the finite set of integers $\widehat{L}_{\alpha_p}(\Gamma)$. Proposition 5.8 implies that $M \geq 0$. Assume that $M > 0$. Let $m = m_F(\Gamma) = m_F(\Gamma, p)$ be the minimum index i in $I_{\alpha_p}(\Gamma)$ for which we have $l_i = M$. If no such index exists, then $M = \langle \mu(\Gamma), \alpha_p^\vee \rangle$; in this case, we let $m = m_F(\Gamma) = m_F(\Gamma, p) := \infty$. Now let $k = k_F(\Gamma) = k_F(\Gamma, p)$ be the predecessor of m in $\widehat{I}_{\alpha_p}(\Gamma)$. Proposition 5.8 implies that this always exists and we have $l_k = M - 1 \geq 0$.

Let us now define

$$F_p(J) := \phi_k \phi_m(J)\,, \tag{6.1}$$

where ϕ_∞ is the identity map. Note that the folding of $\Gamma(\emptyset)$ associated to $F_p(J)$, which will be denoted by $F_p(\Gamma) = (\{(\delta_i, \delta_i')\}_{i \in I}, \delta_\infty)$, is defined by a similar formula. More precisely, we have

$$(\delta_i, \delta_i') = \begin{cases} (\gamma_i, \gamma_i') & \text{if } i < k \text{ or } i > m \\ (\gamma_i, s_p(\gamma_i')) & \text{if } i = k \\ (s_p(\gamma_i), s_p(\gamma_i')) & \text{if } k < i < m \\ (s_p(\gamma_i), \gamma_i') & \text{if } i = m\,, \end{cases} \qquad \text{and} \qquad \delta_\infty = \begin{cases} \gamma_\infty & \text{if } m \neq \infty \\ s_p(\gamma_\infty) & \text{if } m = \infty\,. \end{cases}$$

We can say that applying the root operator F_p amounts to performing a "folding" in position k, and, if $m \neq \infty$, an "unfolding" in position m.

We now define a partial inverse E_p to F_p. Assume that $M > \langle \mu(\Gamma), \alpha_p^\vee \rangle$. Let $k = k_E(\Gamma) = k_E(\Gamma, p)$ be the maximum index i in $I_{\alpha_p}(\Gamma)$ for which we have $l_i = M$. Proposition 5.8 implies that such indices always exist. Now let $m = m_E(\Gamma) = m_E(\Gamma, p)$ be the successor of k in $\widehat{I}_{\alpha_p}(\Gamma)$. By invoking Proposition 5.8 again, we can see that, if $m = \infty$, then we have $\langle \mu(\Gamma), \alpha_p^\vee \rangle = M - 1$, while, otherwise, we have $l_m = M - 1$. Finally, we define $E_p(J)$ by the same formula as $F_p(J)$, namely (6.1). Hence, the folding of $\Gamma(\emptyset)$ associated to $E_p(J)$ is also defined in the same way as above.

Let us now define

$$\varepsilon(J, p) = \varepsilon(\Gamma, p) := M(J, p)\,, \quad \delta(J, p) = \delta(\Gamma, p) := \langle \mu(J), \alpha_p^\vee \rangle - M(J, p)\,.$$

Proposition 6.1. *If $F_p(J)$ is defined, then it is also an admissible subset. Similarly for $E_p(J)$. Furthermore, the operators F_p and E_p satisfy Axioms* (A2) *and* (A3).

7. Main Results

Recall that $\mathcal{A}$ is the collection of all admissible subsets corresponding to our fixed λ.

Theorem 7.1. *The collection $\mathcal{A}$ of admissible subsets together with the root operators form a semiperfect crystal. Thus we have the following character formula:*

$$\chi(\lambda) = \sum_{J \in \mathcal{A}} e^{\mu(J)}\,.$$

The two corollaries below follow from a general result about admissible systems (Theorem 2.4 in [Ste]).

Corollary 7.2. *(Littlewood-Richardson rule). We have*

$$\chi(\lambda) \cdot \chi(\nu) = \sum \chi(\nu + \mu(J))\,,$$

where the summation is over all J in $\mathcal{A}$ satisfying $\langle \nu + \mu(J), \alpha_p^\vee \rangle \geq M(J,p)$ for all $p = 1, \ldots, r$.

Corollary 7.3. *(Branching rule). Given $P \subseteq \{1, \ldots, r\}$, we have the following rule for decomposing $\chi(\lambda)$ as a sum of Weyl characters relative to Φ_P:*

$$\chi(\lambda) = \sum \chi(\mu(J); P)\,,$$

where the summation is over all J in $\mathcal{A}$ satisfying $\langle \mu(J), \alpha_p^\vee \rangle = M(J,p)$ for all $p \in P$.

8. Lakshmibai-Seshadri Chains

In this section, we explain the connection between our model and LS chains. We start with the relevant definitions.

The Bruhat order on the orbit $W\lambda$ of a dominant or antidominant weight is defined by

$$s_\alpha(\mu) < \mu \quad \text{if} \quad \langle \mu, \alpha^\vee \rangle > 0 \quad (\mu \in W\lambda,\, \alpha \in \Phi^+)\,.$$

As usual, we write $\nu \lessdot \mu$ to indicate that μ covers ν. Given $\pm\lambda \in \Lambda^+$ and a fixed real number b, one defines the b-*Bruhat order* $<_b$ as the transitive closure of the relations

$$s_\alpha(\mu) <_b \mu \quad \text{if} \quad s_\alpha(\mu) \lessdot \mu \;\; \text{and} \;\; b\langle \mu, \alpha^\vee \rangle \in \mathbb{Z} \quad (\mu \in W\lambda,\, \alpha \in \Phi^+)\,.$$

Definition 8.1. Given $\pm\lambda \in \Lambda^+$, we say that a pair consisting of a chain $\mu_0 < \mu_1 < \ldots < \mu_l$ in the W-orbit of λ and an increasing sequence of rational numbers $0 < b_1 < \ldots < b_l < 1$ is a *Lakshmibai-Seshadri chain* (LS chain) if $\mu_0 <_{b_1} \mu_1 <_{b_2} \ldots <_{b_l} \mu_l$.

Following [Ste], we identify an LS chain (denoted as above) with the map $\gamma : (0,1] \to W\lambda$ given by $\gamma(t) := \mu_k$ for $b_k < t \leq b_{k+1}$, where $k = 0, \ldots, l$ and $b_0 := 0$, $b_{l+1} := 1$. To each LS chain γ, we associate the continuous piecewise-linear path $\pi : [0,1] \to \mathfrak{h}_\mathbb{R}^*$ given by

$$\pi(t) := \int_0^t \gamma(s)\, ds\,.$$

Let us fix λ in Λ^+. Recall the set I in (4.1), and the λ-chain $(\beta_i)_{i \in I}$ given by Proposition 4.2, which depends on a total order on the set of simple roots $\alpha_1 < \cdots < \alpha_r$. We will now describe a bijection between the corresponding admissible subsets (cf. Definition 5.7) and the LS chains in the W-orbit of the antidominant weight $-\lambda$.

Given an index $i = (\alpha, k)$, we let $\beta_i := \alpha$ and $t_i := k/\langle \lambda, \alpha^\vee \rangle$. Recall the notation $r_i := s_{\beta_i}$ and $\widehat{r}_i := s_{\beta_i, l_i^\emptyset}$. Consider an admissible subset $J = \{j_1 < j_2 < \ldots < j_s\}$ and let

$$\{0 = a_0 < a_1 < \ldots < a_l\} := \{t_{j_1} \leq t_{j_2} \leq \ldots \leq t_{j_s}\} \cup \{0\}\,.$$

Let $0 = n_0 \leq n_1 < \ldots < n_{l+1} = s$ be such that $t_{j_h} = a_k$ if and only if $n_k < h \leq n_{k+1}$, for $k = 0, \ldots, l$. Define Weyl group elements u_h for $h = 0, \ldots, s$ and w_k for $k = 0, \ldots, l$ by $u_0 := 1$, $u_h := r_{j_1} \ldots r_{j_h}$, and $w_k := u_{n_{k+1}}$. Let also $\mu_k := w_k(\lambda)$. For any $k = 1, \ldots, l$, we have the following saturated chain in Bruhat order of minimum (left) coset representatives modulo W_λ (the stabilizer of the weight λ):

$$w_{k-1} = u_{n_k} \lessdot u_{n_k+1} \lessdot \ldots \lessdot u_{n_{k+1}} = w_k\,;$$

indeed, none of the reflections $r_{j_1}, \ldots, r_{j_s}$ lies in W_λ, since $\langle \lambda, \beta_i^\vee \rangle \neq 0$ for all $i \in I$. The above chain gives rise to a saturated increasing chain from $-\mu_{k-1}$ to $-\mu_k$ in the Bruhat order on $-W\lambda$. It is not hard to show that this chain is, in fact, a chain in a_k-Bruhat order. Hence $-\mu_0 <_{a_1} -\mu_1 <_{a_2} \ldots <_{a_l} -\mu_l$ is an LS chain in the W-orbit of $-\lambda$. We denote it by $\gamma(J)$, and the associated continuous piecewise-linear path by $\pi(J)$.

Theorem 8.2. *The map $J \mapsto \gamma(J)$ is a bijection between the admissible subsets considered above and the LS chains in the W-orbit of the antidominant weight $-\lambda$. Moreover, we have*

$$\pi(J)(1) = -\mu(J)\,, \quad E_p(\pi(J)) = \pi(F_p(J))$$

for all admissible subsets J (here E_p is the root operator on Littelmann paths as defined in [Li1, Li2]*, while F_p in the one defined in Section* 6*).*

Remarks 8.3. (1) The proof of Theorem 8.2 contains the justification of the fact that the minima of the paths associated to LS chains are integers. This justification is based only on the combinatorics in Section 5. Note that the same fact was proved by Littelmann in [Li1] using different methods.

(2) The proof of Theorem 8.2 shows that LS chains can be viewed as a limiting case of a special case of our construction. The special choices of λ-chains that lead to LS chains represent a very small fraction of all possible choices.

Based on the independent results of Kashiwara [Kas], Lakshmibai [La], and Joseph [Jos], we deduce the following corollary.

Corollary 8.4. *Given a complex symmetrizable Kac-Moody algebra $\mathfrak{g}$, consider the colored directed graph defined by the action of root operators (cf. Section* 6*) on the admissible subsets corresponding to the special choice of a λ-chain above. This graph is isomorphic to the crystal graph of the irreducible representation with highest weight λ of the associated quantum group $U_q(\mathfrak{g})$.*

We make the following conjecture, which is the analog of a result due to Littelmann [Li2].

Conjecture 8.5. The colored directed graph defined by the action of root operators on the admissible subsets corresponding to any λ-chain does not depend on the choice of this chain.

This conjecture would imply that any choice of a λ-chain leads to a perfect crystal.

9. The Finite Case

In this section, we discuss the way in which the model in this paper specializes to the one in [LP1] in the case of finite irreducible root systems.

Let Φ be the root system of a simple Lie algebra. Let W_{aff} be the *affine Weyl group* for $\Phi^\vee$, that is, the group generated by the affine reflections $s_{\alpha,k}$ (defined in Section 2). The corresponding affine hyperplanes $H_{\alpha,k}$ divide the real vector space $\mathfrak{h}^*_{\mathbb{R}}$ into open regions, called *alcoves.* The *fundamental alcove* $A_\circ$ is given by

$$A_\circ := \{\lambda \in \mathfrak{h}^*_{\mathbb{R}} \mid 0 < \langle \lambda, \alpha^\vee \rangle < 1 \text{ for all } \alpha \in \Phi^+\}.$$

We say that two alcoves are adjacent if they are distinct and have a common wall. For a pair of adjacent alcoves, let us write $A \xrightarrow{\alpha} B$ if the common wall of A and B is of the form $H_{\alpha,k}$ and the root $\alpha \in \Phi$ points in the direction from A to B.

Definition 9.1. An *alcove path* is a sequence of alcoves $(A_0, A_1, \ldots, A_l)$ such that A_{j-1} and A_j are adjacent, for $j = 1, \ldots, l$. We say that an alcove path is *reduced* if it has minimal length l among all alcove paths from A_0 to A_l.

Let $A_\lambda = A_\circ + \lambda$ be the alcove obtained via the affine translation of the fundamental alcove $A_\circ$ by a weight λ. The reduced alcove paths from $A_\circ$ to A_λ are in bijection with the reduced decompositions of the element v_λ in W_{aff} defined by $v_\lambda(A_\circ) = A_\lambda$; see [LP1]. Let us fix a dominant weight λ.

Proposition 9.2. *The sequence of roots $\{\beta_i\}_{i\in I}$ with $I = \{1, \ldots, l\}$ is a λ-chain (cf. Definition* 4.1*) if and only if there exists a reduced alcove path $A_0 = A_\circ \xrightarrow{-\beta_1} \cdots \xrightarrow{-\beta_l} A_l = A_{-\lambda}$.*

Note that, in [LP1], (reduced) λ-chains were defined as chains of roots determined by a reduced alcove path. As we have seen, the mentioned definition is equivalent to the one in this paper.

Definition 9.3. A *gallery* is a sequence $\gamma = (F_0, A_0, F_1, A_1, F_2, \dots, F_l, A_l, F_{l+1})$ such that $A_0, \dots, A_l$ are alcoves; F_j is a codimension one common face of the alcoves A_{j-1} and A_j, for $j = 1, \dots, l$; F_0 is a vertex of the first alcove A_0; and F_{l+1} is a vertex of the last alcove A_l. Furthermore, we require that $F_0 = \{0\}$, $A_0 = A_\circ$, and $F_{l+1} = \{\mu\}$ for some weight $\mu \in \Lambda$, which is called the *weight* of the gallery. The folding operator ϕ_j is the operator which acts on a gallery by leaving its initial segment from A_0 to A_{j-1} intact and by reflecting the remaining tail in the affine hyperplane containing the face F_j. In other words, we define

$$\phi_j(\gamma) := (F_0, A_0, F_1, A_1, \dots, A_{j-1}, F'_j = F_j, A'_j, F'_{j+1}, A'_{j+1}, \dots, A'_l, F'_{l+1}),$$

where $F_j \subset H_{\alpha,k}$, $A'_i := s_{\alpha,k}(A_i)$, and $F'_i := s_{\alpha,k}(F_i)$, for $i = j, \dots, l+1$.

The galleries defined above are special cases of the generalized galleries in [GL].

Let us fix a reduced alcove path $A_0 = A_\circ \xrightarrow{-\beta_1} \cdots \xrightarrow{-\beta_l} A_l = A_{-\lambda}$, which determines the λ-chain $\{\beta_i\}_{i\in I}$ with $I := \{1, \dots, l\}$. The alcove path also determines an obvious gallery

$$\gamma(\emptyset) = (F_0, A_0, F_1, \dots, F_l, A_l, F_{l+1})$$

of weight $-\lambda$. We use the same notation as in Sections 4-6. For instance, $r_i := s_{\beta_i}$ and $\widehat{r}_i := s_{\beta_i, l_i^\emptyset}$.

Definition 9.4. Given an admissible subset $J = \{j_1 < \cdots < j_s\} \subseteq I$ (cf. Definition 5.7), we define the gallery $\gamma(J)$ as $\phi_{j_1} \cdots \phi_{j_s}(\gamma(\emptyset))$, and call it an *admissible folding* of $\gamma(\emptyset)$.

It is easy to see that the weight of the gallery $\gamma(J)$ is $-\mu(J)$ (cf. Definition 5.3).

Since we assumed that Φ is irreducible, there is a unique *highest coroot* $\theta^\vee \in \Phi^\vee$, i.e., a unique coroot that has maximal height. The *dual Coxeter number* of $\Phi^\vee$ is $h^\vee := \langle \rho, \theta^\vee \rangle + 1$ (in the finite case, the dominant weight ρ considered at the end of Section 2 is unique, and is given by $\frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha$). Let Z be the set of the elements of the lattice $\Lambda/h^\vee$ that do not belong to any affine hyperplane $H_{\alpha,k}$. Each alcove A contains precisely one element ζ_A of the set Z (cf. [Kos, LP1]); this will be called the *central point* of A. In particular, $\zeta_{A_\circ} = \rho/h^\vee$. For a pair of adjacent alcoves $A \xrightarrow{\alpha} B$, we have $\zeta_B - \zeta_A = \alpha/h^\vee$.

Let us now associate to the gallery $\gamma(\emptyset)$ a continuous piecewise-linear path. Consider the points $\eta_0 := 0$, $\eta_{2i+1} := \zeta_{A_i}$ for $i = 0, \dots, l$, $\eta_{2i} := \frac{1}{2}(\eta_{2i-1} + \eta_{2i+1})$ for $i = 1, \dots, l$, and $\eta_{2l+2} := -\lambda$. Note that η_{2i} lies on F_i for $i = 0, \dots, l+1$. Let $\pi(\emptyset)$ be the piecewise-linear path obtained by joining $\eta_0, \eta_1, \dots, \eta_{2l+2}$. Given an admissible subset J, let $\eta'_0 = 0$, $\eta'_1 = \rho/h^\vee$, $\eta'_2, \dots, \eta'_{2l+2} = -\mu(J)$ be the points on the faces of the gallery $\gamma(J)$ that are obtained (in the obvious way) from $\eta_0, \eta_1, \eta_2, \dots, \eta_{2l+2}$ in the process of constructing $\gamma(J)$ from $\gamma(\emptyset)$ via folding operators. Clearly, η'_{2i+1} are the central points of the corresponding alcoves in $\gamma(J)$, for $i = 0, \dots, l$. By joining $\eta'_0, \eta'_1, \dots, \eta'_{2l+2}$, we obtain a piecewise-linear path that we call $\pi(J)$. Note that $\pi(J)$ can be described using folding operators, once these operators are appropriately defined. The maps $J \mapsto \gamma(J)$ and $J \mapsto \pi(J)$ are one-to-one.

Proposition 9.5. *Let* $\Gamma(J) = (\{(\gamma_i, \gamma'_i)\}_{i\in I}, \gamma_\infty)$. *Then, for all* $i \in I$, *we have*

$$\eta'_{2i-1} - \eta'_{2i} = \frac{\gamma_i}{2h^\vee}, \qquad \eta'_{2i} - \eta'_{2i+1} = \frac{\gamma'_i}{2h^\vee}, \qquad \eta'_{2l+1} - \eta'_{2l+2} = \frac{\gamma_\infty}{h^\vee}.$$

It turns out that, in general, the collection of paths $\pi(J)$, for J ranging over admissible subsets, does *not* coincide with the collection of Littelmann paths obtained from $\pi(\emptyset)$ by applying the root operators E_p. Indeed, it is not true in general that $E_p(\pi(J)) = \pi(F_p(J))$, as was the case with the paths corresponding to LS chains (cf. Theorem 8.2). The reason is that the root operators E_p and F_p might act on a Littelmann path $\pi : [0,1] \to \mathfrak{h}^*_\mathbb{R}$ by applying the reflection s_p to the direction $\pi'(t)$ of the path for t in more than one subinterval of $[0,1]$; by contrast, the root operators on admissible foldings always apply the reflection s_p to the pairs of roots in an admissible folding corresponding to a single interval of the totally ordered index set I. The situation is the same if we define $\pi(\emptyset)$ by joining the centers of the faces F_i, or the centers of both the alcoves A_i and the faces F_i (in the order they appear in $\gamma(\emptyset)$).

Example 9.6. Suppose that the root system Φ is of type G_2. The positive roots are $\gamma_1 = \alpha_1,\ \gamma_2 = 3\alpha_1 + \alpha_2,\ \gamma_3 = 2\alpha_1 + \alpha_2,\ \gamma_4 = 3\alpha_1 + 2\alpha_2,\ \gamma_5 = \alpha_1 + \alpha_2,\ \gamma_6 = \alpha_2$. The corresponding coroots are $\gamma_1^\vee = \alpha_1^\vee,\ \gamma_2^\vee = \alpha_1^\vee + \alpha_2^\vee,\ \gamma_3^\vee = 2\alpha_1^\vee + 3\alpha_2^\vee,\ \gamma_4^\vee = \alpha_1^\vee + 2\alpha_2^\vee,\ \gamma_5^\vee = \alpha_1^\vee + 3\alpha_2^\vee,\ \gamma_6^\vee = \alpha_2^\vee$.

Suppose that $\lambda = \omega_2$. Proposition 4.2 gives the following ω_2-chain:

$$(\beta_1, \ldots, \beta_{10}) = (\gamma_6, \gamma_5, \gamma_4, \gamma_3, \gamma_2, \gamma_5, \gamma_3, \gamma_4, \gamma_5, \gamma_3)\,.$$

Thus, we have $\widehat{r}_1 = s_{\gamma_6,0},\ \widehat{r}_2 = s_{\gamma_5,0},\ \widehat{r}_3 = s_{\gamma_4,0},\ \widehat{r}_4 = s_{\gamma_3,0},\ \widehat{r}_5 = s_{\gamma_2,0},\ \widehat{r}_6 = s_{\gamma_5,1},\ \widehat{r}_7 = s_{\gamma_3,1},\ \widehat{r}_8 = s_{\gamma_4,1},\ \widehat{r}_9 = s_{\gamma_5,2},\ \widehat{r}_{10} = s_{\gamma_3,2}$. There are six saturated chains in the Bruhat order (starting at the identity) on the corresponding Weyl group that can be retrieved as subchains of the ω_2-chain. We indicate each such chain and the corresponding admissible subsets in $\{1, \ldots, 10\}$.

(1) 1: $\{\}$;
(2) $1 < s_{\gamma_6}$: $\{1\}$;
(3) $1 < s_{\gamma_6} < s_{\gamma_6}s_{\gamma_5}$: $\{1,2\}$, $\{1,6\}$, $\{1,9\}$;
(4) $1 < s_{\gamma_6} < s_{\gamma_6}s_{\gamma_5} < s_{\gamma_6}s_{\gamma_5}s_{\gamma_4}$: $\{1,2,3\}$, $\{1,2,8\}$, $\{1,6,8\}$;
(5) $1 < s_{\gamma_6} < s_{\gamma_6}s_{\gamma_5} < s_{\gamma_6}s_{\gamma_5}s_{\gamma_4} < s_{\gamma_6}s_{\gamma_5}s_{\gamma_4}s_{\gamma_3}$: $\{1,2,3,4\}$, $\{1,2,3,7\}$, $\{1,2,3,10\}$, $\{1,2,8,10\}$, $\{1,6,8,10\}$;
(6) $1 < s_{\gamma_6} < s_{\gamma_6}s_{\gamma_5} < s_{\gamma_6}s_{\gamma_5}s_{\gamma_4} < s_{\gamma_6}s_{\gamma_5}s_{\gamma_4}s_{\gamma_3} < s_{\gamma_6}s_{\gamma_5}s_{\gamma_4}s_{\gamma_3}s_{\gamma_2}$: $\{1,2,3,4,5\}$.

The weight of each admissible subset is now easy to compute (by applying the corresponding affine reflections above to ω_2, cf. Definition 5.3). This leads to the expression for the character $\chi(\omega_2)$ as the following sum over admissible subsets:

$$\chi(\omega_2) = e^{\omega_2} + e^{\widehat{r}_1(\omega_2)} + e^{\widehat{r}_1\,\widehat{r}_2(\omega_2)} + e^{\widehat{r}_1\,\widehat{r}_6(\omega_2)} + e^{\widehat{r}_1\,\widehat{r}_9(\omega_2)} + \cdots + e^{\widehat{r}_1\,\widehat{r}_6\,\widehat{r}_8\,\widehat{r}_{10}(\omega_2)} + e^{\widehat{r}_1\,\widehat{r}_2\,\widehat{r}_3\,\widehat{r}_4\,\widehat{r}_5(\omega_2)}.$$

Figure 2 displays the galleries $\gamma(J)$ corresponding to the admissible subsets J indicated above, the associated paths $\pi(J)$, as well as the action of the root operators F_p on J. For each path, we shade the fundamental alcove, mark the origin by a white dot "$\circ$", and mark the endpoint of a black dot "$\bullet$". Since some linear steps in $\pi(J)$ might coincide, we display slight deformations of these paths, so that no information is lost in their graphical representations. As discussed above, the weights of the irreducible representation V_{ω_2} are obtained by changing the signs of the endpoints of the paths $\pi(J)$ (marked by black dots). The roots in the corresponding admissible foldings $\Gamma(J)$ can also be read off; see Proposition 9.5. At each step, a path $\pi(J)$ either crosses a wall of the affine Coxeter arrangement or bounces off a wall. The associated admissible subset J is the set of indices of bouncing steps in the path.

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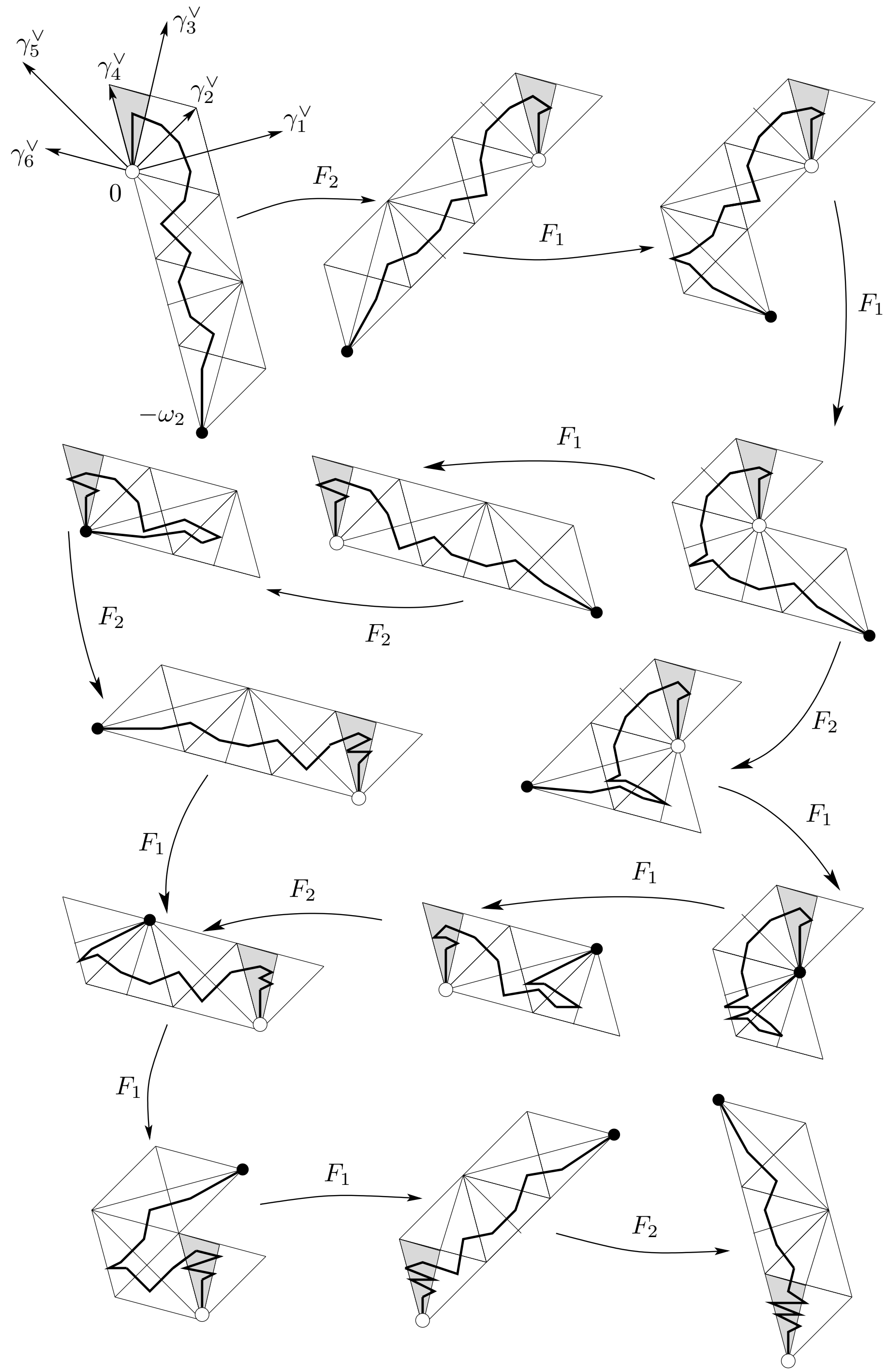

FIGURE 2. The crystal for the fundamental weight ω_2 for type G_2.

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