Discrete surfaces and infinite smooth words*

Damien Jamet[1] and Geneviève Paquin[2]

[1] LIRMM, Université Montpellier II, 161 rue Ada, 34392 Montpellier Cedex 5 - FRANCE.
jamet@lirmm.fr
[2] LaCIM, Université du Québec à Montréal, Montréal (QC), CANADA
paquin@lacim.uqam.ca

May 1st, 2005

Abstract In the present paper, we study the $(1,1,1)$-discrete surfaces introduced in [Jam04]. In [Jam04], the $(1,1,1)$-discrete surfaces are not assumed to be connected. In this paper, we prove that assuming connectedness is not restrictive, in the sense that, any two-dimensional coding of a $(1,1,1)$-discrete surface is the two-dimensional coding of both connected and simply connected ones. In the second part of this paper, we investigate a particular class of discrete surfaces: those generated by infinite smooth words. We prove that the only smooth words generating such surfaces are $K_{(3,1)}$, $K_{(1,3)}$ and $2K_{(3,1)}$, where $K_{(a,b)}$ is the generalized Kolakoski's word over the two-letter alphabet $\{a,b\}$ with a as first letter.

Résumé Dans cet article, nous étudions les $(1,1,1)$-surfaces discrètes introduites dans [Jam04]. Dans l'article [Jam04], les surfaces ne sont pas supposées discrètes. Nous montrons dans cet article qu'il n'est pas restrictif de faire une telle supposition et que tout codage bi-dimensionel d'une $(1,1,1)$-surface discrète code à la fois une surface connexe et une surface simplement connexe. La seconde partie de cet article est consacrée à l'étude des surfaces discrètes engendrées par des mots lisses. Nous démontrons que les seuls mots lisses engendrant de telles surfaces sont les mots $K_{(3,1)}$, $K_{(1,3)}$ et $2K_{(3,1)}$, où $K_{(a,b)}$ est le mot de Kolakoski généralisé sur l'alphabet $\{a,b\}$ commençant par a.

1 Introduction

A wide literature has been devoted to the study of Sturmian words, that is, the infinite words over a binary alphabet which have $n+1$ factors of length n [Lot02]. These words are also equivalently defined as a discrete approximation of a line with irrational slope. Then, many attempts have been done to generalize this class of infinite words to two-dimensional words. For instance, in [Vui98,BV00,ABS04], it is shown that the orbit of an element $\mu \in [0,1[$ under the action of two rotations codes a discrete plane. Furthermore, the problem of one

* with the support of NSERC (Canada)

or two-dimensional words characterizing discrete lines or planes is investigated in [BV00,Lot02,ABS04,BT04,DGK03]. In [Jam04], the author introduces the $(1,1,1)$-discrete surfaces as a quite natural generalization of the discrete planes of [BV00,ABS04] and shows how to decide whether a given two-dimensional sequence over the three-letter alphabet $\{1,2,3\}$ codes a $(1,1,1)$-discrete surface.

In the present paper, we study the connectedness and the simply-connectedness of the $(1,1,1)$-discrete surfaces and we show that, given a two-dimensional sequence u over the three-letter alphabet $\{1,2,3\}$, then u codes a $(1,1,1)$-discrete surface if and only if u codes a connected one and a simply connected one. Secondly, we study the $(1,1,1)$-discrete surfaces associated with smooth words for the case of two-letter alphabets [BL02,BLL02,BBLP03] and for arbitrary alphabets [BBC04]. These surfaces have local geometric properties and we give an explicit description of the associated smooth words.

This paper is organized as follows. In Section 2, we recall the basic notions concerning $(1,1,1)$-discrete surfaces and the combinatorics on two-dimensional words over a finite alphabet. In Section 3, we prove the first main result of this paper, namely

Theorem. *Let $u \in \{1,2,3\}^{\mathbb{Z}^2}$ be a two-dimensional sequence. The following assertions are equivalent:*

i) the sequence u codes a $(1,1,1)$-discrete surface;
ii) the sequence u codes a connected and simply connected $(1,1,1)$-discrete surface.

We also prove that any connected surface coded by an element of $\{1,2,3\}^{\mathbb{Z}^2}$ is simply connected. Finally, in Section 4, after having recalled basic notions concerning smooth words, we demonstrate the second main result, that is:

Theorem. *Let w be a smooth word over the alphabet $\{1,2,3\}$. The tiling $T(w)$ associated to w is a piece of a discrete surface if and only if $w \in \{K_{(1,3)}, K_{(3,1)}, 2K_{(3,1)}\}$, where $K_{(a,b)}$ is the generalized Kolakoski's word over the two-letter alphabet $\{a,b\}$ with a as first letter.*

2 Basic notions

2.1 Discrete surfaces

In this section we recall the basic notions concerning $(1,1,1)$-discrete surfaces and discrete planes.

Let $\{\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}\}$ denotes the canonical basis of the Euclidean space $\mathbb{R}^3$. An element of $\mathbb{Z}^3$ is called a *voxel*. The *fundamental unit cube* $\mathcal{C}$ is the set defined by:

$$\mathcal{C} = \left\{x_1\overrightarrow{e_1} + x_2\overrightarrow{e_2} + x_3\overrightarrow{e_3} \mid (x_1,x_2,x_3) \in [0,1]^3\right\}.$$

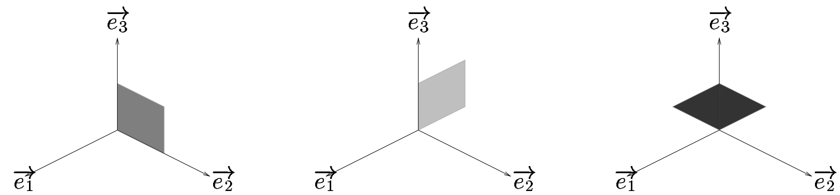
Let $\overrightarrow{x} \in \mathbb{Z}^3$. The set $\overrightarrow{x} + \mathcal{C}$ is called the *unit cube pointed* by $\overrightarrow{x}$.

Let $\mathcal{P}$ be the plane with equation $(\overrightarrow{v}, \overrightarrow{x}) = \mu$ with $\overrightarrow{v} \in \mathbb{R}^3_+$, $\mu \in \mathbb{R}$ and $(\overrightarrow{v}, \overrightarrow{x}) = v_1x_1 + v_2x_2 + v_2x_3$ denoting the usual scalar product of the vectors

$\overrightarrow{v}$ and $\overrightarrow{x}$. Let $\mathcal{C}_{\mathcal{P}}$ be the union of the unit cubes pointed by a voxel $x \in \mathbb{Z}^3$ and intersecting the open half-space $(\overrightarrow{v}, \overrightarrow{x}) < \mu$. We call *discrete plane* associated to $\mathcal{P}$ the set $\mathfrak{P}_{\mathcal{P}} = \overline{\mathcal{C}_{\mathcal{P}}} \setminus \overset{\circ}{\mathcal{C}_{\mathcal{P}}}$, where $\overline{\mathcal{C}_{\mathcal{P}}}$ (resp. $\overset{\circ}{\mathcal{C}_{\mathcal{P}}}$) is the closure (resp. the interior) of the set $\mathcal{C}_{\mathcal{P}}$ in $\mathbb{R}^3$, provided with its usual topology.

Let us now define the three *fundamental faces* (see Fig. 1):

$$\begin{aligned}
E_1 &= \{x_2\overrightarrow{e_2} + x_3\overrightarrow{e_3} \mid (x_2, x_3) \in [0,1[^2\},\\
E_2 &= \{-x_1\overrightarrow{e_1} + x_3\overrightarrow{e_3} \mid (x_1, x_3) \in [0,1[^2\},\\
E_3 &= \{-x_1\overrightarrow{e_1} - x_2\overrightarrow{e_2} \mid (x_1, x_2) \in [0,1[^2\}.
\end{aligned}$$



(a) Face of type 1.

(b) Face of type 2

(c) Face of type 3

Fig. 1. The three fundamental faces.

Let $\overrightarrow{x} \in \mathbb{Z}^3$ and $k \in \{1, 2, 3\}$. The set $\overrightarrow{x} + E_k$ is called a *pointed face of type* k. The vector $\overrightarrow{x}$ is called the *distinguished vertex* of $\overrightarrow{x} + E_k$.

Let $\pi : \mathbb{R}^3 \longrightarrow \{\overrightarrow{x} \in \mathbb{R}^3 \mid (\overrightarrow{e_1} + \overrightarrow{e_2} + \overrightarrow{e_3}, \overrightarrow{x}) = 0\}$ be the orthogonal projection map onto the plane $\mathcal{P}_0$ with equation $(\overrightarrow{e_1} + \overrightarrow{e_2} + \overrightarrow{e_3}, \overrightarrow{x}) = 0$.

One has:

Theorem 1. [ABS04] *Let $\mathfrak{P}_{\mathcal{P}}$ be a discrete plane and let $\mathcal{V}_{\mathcal{P}} = \mathfrak{P} \cap \mathbb{Z}^3$ be the set of vertices of $\mathfrak{P}_{\mathcal{P}}$. We suppose that $\mathcal{P}$ admits a normal vector $\overrightarrow{v} \in \mathbb{R}^3_+$.*

1. *The set $\mathfrak{P}_{\mathcal{P}}$ is partitioned by pointed faces.*
2. *The restriction maps $\pi_{|\mathcal{V}_{\mathcal{P}}} : \mathcal{V}_{\mathcal{P}} \longrightarrow \pi\left(\mathbb{Z}^3\right)$ and $\pi_{|\mathfrak{P}_{\mathcal{P}}} : \mathfrak{P}_{\mathcal{P}} \longrightarrow \left\{\overrightarrow{x} \in \mathbb{R}^3 \mid (\overrightarrow{e_1} + \overrightarrow{e_2} + \overrightarrow{e_3}, \overrightarrow{x}) = 0\right\}$ are bijective.*

Let us now define the $(1,1,1)$-discrete surfaces as follows:

Definition 1. [Jam04] *A disjoint union $\mathfrak{S} \subseteq \mathbb{R}^3$ of pointed faces is a $(1,1,1)$-*discrete surface *if the map*

$$\begin{aligned}
\pi_{|\mathfrak{S}} : \mathfrak{S} &\longrightarrow \mathcal{P}_0\\
\overrightarrow{x} &\mapsto \pi(\overrightarrow{x})
\end{aligned}$$

is a bijection (see Fig. 2). The set $\mathcal{V}_{\mathfrak{S}} = \mathfrak{S} \cap \mathbb{Z}^3$ is called the set of vertices *of $\mathfrak{S}$.*

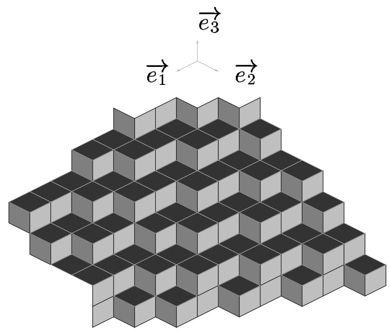


Fig. 2. A piece of a discrete surface

From now on, for clarity issue, we refer to $(1, 1, 1)$-discrete surface as discrete surface.

Before associating a two-dimensional coding over the three-letter alphabet $\{1, 2, 3\}$ to any discrete surface $\mathfrak{S}$, let us recall a technical lemma.

Lemma 1. *Let $\mathfrak{S}$ be a discrete surface. The following properties hold:*

i) The map

$$\begin{array}{rcl} \pi_{|\mathcal{V}_\mathcal{S}} : \mathfrak{S} & \longrightarrow & \pi\left(\mathbb{Z}^3\right) \\ \overrightarrow{x} & \mapsto & \pi(\overrightarrow{x}) \end{array}$$

is a bijection.

ii) Each vertex $\overrightarrow{x}$ of $\mathcal{V}_\mathcal{P}$ is the distinguished vertex of one and only one pointed face.

We can now associate a two-dimensional coding over the three letter-alphabet $\{1, 2, 3\}$ to any discrete surface $\mathfrak{S}$ as follows: let $\Gamma = \pi(\mathbb{Z}^3) = \mathbb{Z}\pi(\overrightarrow{e_1}) \oplus \mathbb{Z}\pi(\overrightarrow{e_2})$. We identify Γ and $\mathbb{Z}^2$ by the lattice isomorphism

$$\begin{array}{rrcl} \Phi : & \mathbb{Z}^2 & \longrightarrow & \Gamma \\ & (m, n) & \mapsto & m\pi(\overrightarrow{e_1}) + n\pi(\overrightarrow{e_2}). \end{array} \tag{1}$$

To any discrete surface $\mathfrak{S}$, we associate the two-dimensional coding $u \in \{1, 2, 3\}^{\mathbb{Z}^2}$ defined by: $\forall (m, n) \in \mathbb{Z}^2$, $\forall k \in \{1, 2, 3\}$,

$$u_{m,n} = k \text{ if and only if } \pi^{-1}_{|\mathcal{V}_\mathcal{S}}(m\pi(\overrightarrow{e_1}) + n\pi(\overrightarrow{e_2})) \text{ is of type } k \text{ in } \mathfrak{S}.$$

In other words, $u_{m,n} = k$ if the pre-image of the points $m\pi(\overrightarrow{e_1}) + n\pi(\overrightarrow{e_2})$ is of type k in $\mathfrak{S}$.

Hence it becomes natural to wonder whether a given two-dimensional sequence $u \in \{1, 2, 3\}^{\mathbb{Z}^2}$ codes a discrete surface. In order to investigate this problem, we need several notions about combinatorics on two-dimensional words over a finite alphabet.

2.2 Basic notions on two-dimensional sequences over a finite alphabet

In this section, we recall some basic notions of combinatorics on two-dimensional words over a finite alphabet (see for instance [GR97]).

Let Σ be a finite alphabet. Let Ω be a finite subset of $\mathbb{Z}^2$. A function $w : \Omega \longrightarrow \Sigma$ is called a *finite pointed pattern over the alphabet* Σ.

A *shape* $\overline{\Omega}$ of $\mathbb{Z}^2$ is the equivalence class of $\Omega \subseteq \mathbb{Z}^2$ for the following equivalence relation:

$$\Omega \sim \Omega' \iff \exists (v_1, v_2) \in \mathbb{Z}^2,\ \Omega_1 = \Omega_2 + (v_1, v_2).$$

One defines an equivalence relation on the set of finite pointed patterns as follows. Let $w : \Omega \to \Sigma$ and $w' : \Omega' \to \Sigma$ be two finite pointed patterns. Then ω is said to be equivalent to ω' if and only if:

$$\exists (v_1, v_2) \in \mathbb{Z}^2,\ \Omega = \Omega' + (v_1, v_2),\ \forall (m, n) \in \Omega,\ w_{m,n} = w_{m+v_1, n+v_2}. \quad (2)$$

One can easily convince himself that the previous relation is an equivalence one, where the equivalence classes are called *finite patterns of shape* Ω.

For short, we denote the finite patterns w instead of $\overline{w}$ and we will denote the shapes Ω instead of $\overline{\Omega}$.

Let $u \in \Sigma^{\mathbb{Z}^2}$ be a two-dimensional sequence and let $w : \Omega \longrightarrow \Sigma$ be a finite pointed pattern. We say that w occurs in u if there exists $(m_0, n_0) \in \mathbb{Z}^2$ such that for all $(m, n) \in \Omega$, $w_{m,n} = u_{m_0+m, n_0+n}$. Such a couple (m_0, n_0) is called an *occurrence* of w. We say that a finite pattern $\overline{w}$ occurs in u if one of its pointed represents occurs in u. The set $\mathcal{L}(u)$ (resp. $\mathcal{L}_\Omega(u)$) of finite patterns (resp. of finite patterns of shape Ω) occurring in u is called the *language* (resp. the Ω-*language*) of u. Let Ω be a shape. The Ω-*complexity map of* u is the function $p_\Omega : \Sigma^{\mathbb{Z}^2} \longrightarrow \mathbb{N} \cup \{\infty\}$ defined as follows:

$$\begin{array}{rcl} p_\Omega : \Sigma^{\mathbb{Z}^2} & \longrightarrow & \mathbb{N} \cup \{\infty\} \\ u & \mapsto & |\mathcal{L}_\Omega(u)|, \end{array}$$

where $|\mathcal{L}_\Omega(u)|$ is the cardinality of the set $\mathcal{L}_\Omega(u)$.

2.3 Recognition of discrete surfaces

Let $u \in \{1, 2, 3\}^{\mathbb{Z}^2}$. In this section we investigate the following question: does u code a discrete surface? Let us now introduce the pointed hooks and the hook shape of u.

Definition 2. *The* hook shape *is the equivalence class of the sets* $\{(m, n); (m, n+1); (m+1, n+1)\}$, *for* $(m, n) \in \mathbb{Z}^2$, *for the relation* $\sim$ *defined in Section 2.2. (see Fig. 3)*

The following theorem holds.

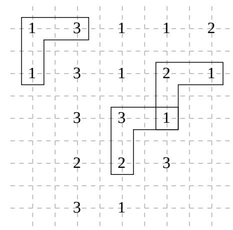


Fig. 3. Examples of hook words occurring in a sequence $u \in \{1,2,3\}^{\mathbb{Z}^2}$.

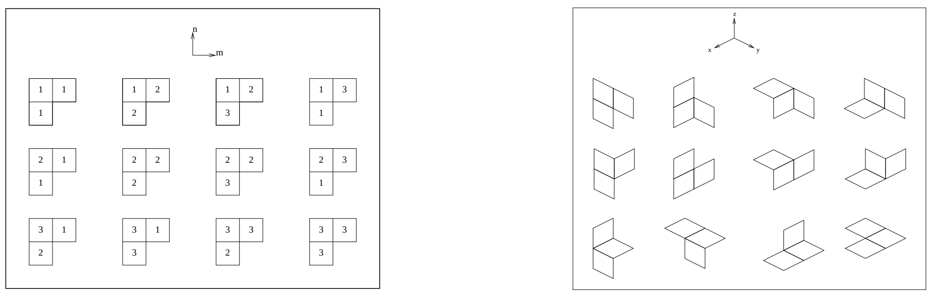


Fig. 4. Left: The permitted hook-words. Right: The 3-dimensional representation of the permitted hook-words.

Theorem 2. [Jam04] *Let $u \in \{1,2,3\}^{\mathbb{Z}^2}$. Then u codes a discrete surface if and only if the hook-language of u is included in the following set of patterns (see Fig. 4).*

A drawback of the previous definition of discrete surface is to be non-intuitive. For instance, let us consider a discrete plane $\mathfrak{P}_{\mathcal{P}}$. By construction, $\mathfrak{P}_{\mathcal{P}}$ is connected and simply connected, that is, it does not contain any hole. Let $\overrightarrow{x} \in \mathcal{V}_{\mathcal{P}}$ be a vertex of $\mathfrak{P}_{\mathcal{P}}$ of type k. Then, $\mathfrak{P} \setminus \{\overrightarrow{x} + E_k\} \cup \{(\overrightarrow{x} + \overrightarrow{e_1} + \overrightarrow{e_2} + \overrightarrow{e_3}) + E_k\}$ is still a discrete surface.

In the following section, we show that assuming the discrete surfaces to be connected is not a restriction. More precisely, we prove that any sequence $u \in \{1,2,3\}^{\mathbb{Z}^2}$ coding a discrete surface also codes a connected and a simply-connected one.

3 The connected discrete surfaces

In this section, we investigate the discrete surfaces introduced in [Jam04] and prove that, for any discrete surface, there exists a connected one with the same two-dimensional coding.

Theorem 3. *Let $u \in \{1,2,3\}^{\mathbb{Z}^2}$ be a two-dimensional sequence. The following assertions are equivalent:*

i) the sequence u codes a discrete surface;
ii) the sequence u codes a connected discrete surface.

Let $x = m\pi(\overrightarrow{e_1}) + n\pi(\overrightarrow{e_2})$, with $(m,n) \in \mathbb{Z}^2$. Using the previously introduced identification of Γ as $\mathbb{Z}^2$ (see (1)), we denote $u_{\overrightarrow{x}}$ instead of $u_{m,n}$.

Let $u \in \{1,2,3\}^{\mathbb{Z}^2}$ be the two-dimensional coding of a discrete surface, that is,

$$\bigcup_{\overrightarrow{x} \in \Gamma} \left(\overrightarrow{x} + \pi(E_{u_{\overrightarrow{x}}})\right),$$

where E_i is a fundamental face ($i \in \{1,2,3\}$), is a partition of the plane $\mathcal{P}_0$ (see [Jam04]). We define a partial order relation $\xrightarrow{u}$ over Γ as follows:

$$\forall (\overrightarrow{x}, \overrightarrow{y}) \in \Gamma^2,\ \overrightarrow{x} \xrightarrow{u} \overrightarrow{y} \iff \overrightarrow{y} \in \overline{\overrightarrow{x} + \pi(E_{u_{\overrightarrow{x}}})} \setminus \overset{\circ}{\overrightarrow{x} + \pi(E_{u_{\overrightarrow{x}}})}.$$

Lemma 2. *Let $\overrightarrow{x} \in \Gamma$. Then $\overrightarrow{x} \xrightarrow{u} \overrightarrow{x} - \pi(\overrightarrow{e_1})$ and $\overrightarrow{x} \xrightarrow{u} \overrightarrow{x} - \pi(\overrightarrow{e_1}) - \pi(\overrightarrow{e_2})$ (see Fig. 5).*

Proof. This is immediately deduced from $\pi(\overrightarrow{e_3}) = -(\pi(\overrightarrow{e_1}) + \pi(\overrightarrow{e_2}))$ and from the definitions of E_1, E_2 and E_3 and the projection map π. ■

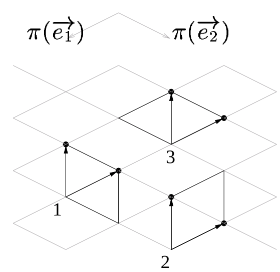


Fig. 5. Lattice representation of the partial order relation $\xrightarrow{u}$.

Finally, Theorem 3 is a direct consequence of the following lemma.

Lemma 3. *Let $\overrightarrow{x} \in \mathbb{Z}^3$ and let $\overrightarrow{l} = \pi(\overrightarrow{x}) - \pi(\overrightarrow{e_1}) - \pi(\overrightarrow{e_2}) \in \Gamma$ (resp. $\overrightarrow{r} = \pi(\overrightarrow{x}) - \pi(\overrightarrow{e_1}) \in \Gamma$) be the left (resp. the right) targets of the arrows whose source is $\pi(\overrightarrow{x})$ in the graph of Fig. 5. Then, the set*

$$\left(\overrightarrow{x} + E_{u_{\pi(\overrightarrow{x})}}\right) + \left(\overrightarrow{y} + E_{u_{\overrightarrow{l}}}\right) + \left(\overrightarrow{z} + E_{u_{\overrightarrow{r}}}\right),$$

with

$$\begin{aligned} \overrightarrow{y} = \overrightarrow{x} + \overrightarrow{e_3} \text{ and } \overrightarrow{z} = \overrightarrow{x} + \overrightarrow{e_2} + \overrightarrow{e_3} &\text{ if } u_{\pi(\overrightarrow{x})} = 1,\\ \overrightarrow{y} = \overrightarrow{x} + \overrightarrow{e_3} \text{ and } \overrightarrow{z} = \overrightarrow{x} - \overrightarrow{e_1} &\text{ if } u_{\pi(\overrightarrow{x})} = 2,\\ \overrightarrow{y} = \overrightarrow{x} - \overrightarrow{e_1} - \overrightarrow{e_2} \text{ and } \overrightarrow{z} = \overrightarrow{x} - \overrightarrow{e_1} &\text{ if } u_{\pi(\overrightarrow{x})} = 3. \end{aligned}$$

is connected and, in each previous case, $\pi(\overrightarrow{y}) = \overrightarrow{l}$ and $\pi(\overrightarrow{z}) = \overrightarrow{r}$.

Proof. In each case, one can verify that $\{\overrightarrow{y}, \overrightarrow{z}\} \subseteq \overline{\left(\overrightarrow{x} + E_{u_{\pi(\overrightarrow{x})}}\right)}$. For instance, let us suppose that $u_{\pi(\overrightarrow{x})} = 1$. Then,

$$\overline{\left(\overrightarrow{x} + E_{u_{\pi(\overrightarrow{x})}}\right)} = \overrightarrow{x} + \left\{x_2\overrightarrow{e_2} + x_3\overrightarrow{e_3} \mid (x_2, x_3) \in [0,1]^2\right\},$$

and $\overrightarrow{y} = \overrightarrow{x} + \overrightarrow{e_3} \in \overline{\left(\overrightarrow{x} + E_{u_{\pi(\overrightarrow{x})}}\right)}$ (see Fig. 6). Idem for $\overrightarrow{z} = \overrightarrow{x} + \overrightarrow{e_2} + \overrightarrow{e_3}$. Finally, $\pi(\overrightarrow{y}) = \pi(\overrightarrow{x} + \overrightarrow{e_3}) = \pi(\overrightarrow{x}) - \pi(\overrightarrow{e_1}) - \pi(\overrightarrow{e_2}) = \pi(\overrightarrow{l})$. ■

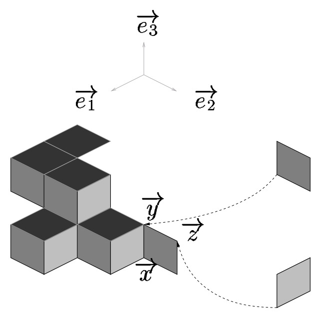


Fig. 6. Computing a connected surface by induction.

We can now prove Theorem 3:

Sketch of the proof. For all $\overrightarrow{x}_0 \in \Gamma$ and $r \in \mathbb{R}_+$, let us denote $B_\Gamma(\overrightarrow{x}_0, r) = \{m\pi(\overrightarrow{e_1}) + n\pi(\overrightarrow{e_2}) \in \Gamma \mid \max\{|m|, |n|\} < r\}$.

Let $\mathfrak{S}_1 = E_{u_{\overrightarrow{0}}}$. Then, the set $\mathfrak{S}_1$ is connected, $\pi(\mathfrak{S}_1 \cap \mathbb{Z}^3) = B_\Gamma(\overrightarrow{0}, 1)$ and for all $\overrightarrow{x} \in \mathbb{Z}^3 \cap \mathfrak{S}_1$, $\overrightarrow{x}$ is of type $u_{\pi(\overrightarrow{x})}$. Let $r \in \mathbb{N}^\star$ and let us suppose that $\mathfrak{S}_r$ is a connected disjoint union of pointed faces such that $\pi(\mathfrak{S}_r) \cap \mathbb{Z}^3 = B_\Gamma(\overrightarrow{0}, r)$ and for all $\overrightarrow{x} \in \mathbb{Z}^3 \cap \mathfrak{S}_r$, $\overrightarrow{x}$ is of type $u_{\pi(\overrightarrow{x})}$. Then, with Lemma 3 and the connectedness of the relation $\xrightarrow{u}$ (see Fig.7), that is the connectedness of the corresponding graph, one can be convinced that it is possible to build, by induction on the sets $\left(B_\Gamma(\overrightarrow{0}, r)\right)_{r \in \mathbb{N}^\star}$, a connected union $\mathfrak{S}_{r+1}$ of pointed faces such that $\pi(\mathfrak{S}_{r+1} \cap \mathbb{Z}^3) = B_\Gamma(\overrightarrow{0}, r+1)$. Indeed, for any element $\overrightarrow{y} \in B_\Gamma(\overrightarrow{0}, r+1) \setminus B_\Gamma(\overrightarrow{0}, r)$, there exists an element $\overrightarrow{x} \in B_\Gamma(\overrightarrow{0}, r)$ with $\overrightarrow{x} \xrightarrow{u} \overrightarrow{y}$. We thus obtain an increasing sequence $(\mathfrak{S}_r)_{r \in \mathbb{N}^\star}$ of connected unions of pointed faces such that, for all $r \in \mathbb{N}^\star$, $\pi_{|\mathfrak{S}_r} : \mathfrak{S}_r \longrightarrow \mathcal{P}$ is 1-1 (remind that we assume u to code a discrete surface). Let

$$\mathfrak{S} = \bigcup_{r \in \mathbb{N}^\star} \mathfrak{S}_r.$$

The set $\mathfrak{S}$ is connected (it is an increasing sequence of connected sets) and $\pi : \mathfrak{S} \longrightarrow \mathcal{P}$ is 1-1. Finally, since $\pi(\mathfrak{S} \cap \mathbb{Z}^3) = \Gamma$ and u codes a discrete surface,

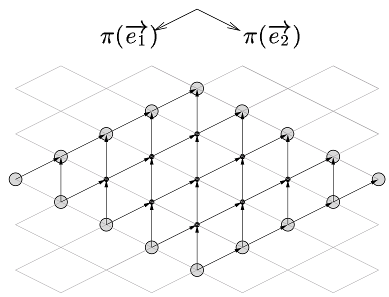


Fig. 7. A part of the graph of the relation $\xrightarrow{u}$.

that is,

$$\bigcup_{\overrightarrow{x} \in \mathfrak{S} \cap \mathbb{Z}^3} \Big(\pi(\overrightarrow{x}) + \pi(E_{u_{\pi(\overrightarrow{x})}})\Big) = \mathcal{P}_0,$$

we conclude that $\pi : \mathfrak{S} \longrightarrow \mathcal{P}_0$ is onto. ■

An other interesting property of two-dimensional sequences coding discrete surfaces is:

Theorem 4. *Let $\mathfrak{S}$ be a connected discrete surface coded by $u \in \{1,2,3\}^{\mathbb{Z}^2}$. Then $\mathfrak{S}$ is simply-connected, that is, it admits no hole.*

Sketch of the proof. Let $(\mathfrak{S}_r)_{r \in \mathbb{N}^\star}$ be the sequence computed in the proof of Theorem 3. Then the following assertion holds:

$$\forall r \in \mathbb{N}^\star,\ \overline{\mathfrak{S}_r} \subseteq \mathfrak{S}_{r+2}.$$

Then,

$$\mathfrak{S} = \bigcup_{r \in \mathbb{N}^\star} \overline{\mathfrak{S}_r}$$

and $\mathfrak{S}$ is a union of closed sets. On can notice that each set $B(\overrightarrow{x}, r) = \{\overrightarrow{y} \in \mathbb{R}^3 \mid ||\overrightarrow{x} - \overrightarrow{y}||_\infty < r\}$, with $r \in \mathbb{R}_+^\star$, intersects at most a finite number of closed pointed faces. Hence, $\mathfrak{S}$ is closed. Since $\pi : \mathbb{R}^3 \longrightarrow \mathcal{P}_0$ is continuous, it follows that $\pi_{|\mathfrak{S}} : \mathfrak{S} \longrightarrow \mathcal{P}_0$ is continuous. It remains to show that $\pi_{|\mathfrak{S}}^{-1} : \mathcal{P}_0 \longrightarrow \mathfrak{S}$ is continuous. Indeed, $\pi : \mathbb{R}^3 \longrightarrow \mathcal{P}_0$ is a closed map. Finally, we have proved that $\pi_{|\mathfrak{S}} : \mathfrak{S} \longrightarrow \mathcal{P}$ is an homeomorphism. Hence, since Γ is simply-connected,we deduce that so is $\mathfrak{S}$.

4 Discrete surfaces generated by smooth words

In this section, we first recall some notions of combinatorics on words over arbitrary alphabets, as defined in [BBC04]. Then, we study the discrete surfaces generated by a specific class of words, the right infinite smooth words over the alphabet {1,2,3}. We prove that there are only three such discrete surfaces.

Let us consider a finite *alphabet* Σ of *letters*. A *right infinite word* is a sequence $w \in \Sigma^{\mathbb{N}}$. Every word $w \in \Sigma^{\mathbb{N}}$ can be uniquely written as a product of factors as follows:

$$w = {\alpha_1}^{e_1} {\alpha_2}^{e_2} {\alpha_3}^{e_3} \dots$$

with $e_j \in \mathbb{N}^\star$ and $\alpha_i \neq \alpha_{i+1}$. Hence, the *run-length encoding* defined by:

$$\begin{array}{rccc} \Delta : & \Sigma^{\mathbb{N}} & \longrightarrow & \mathbb{N}^{\mathbb{N}} \\ & w = {\alpha_1}^{e_1} {\alpha_2}^{e_2} {\alpha_3}^{e_3} \dots & \mapsto & e_1 e_2 e_3 \dots , \end{array}$$

is well defined on $\Sigma^{\mathbb{N}}$.

Example 1. If $\Sigma = \{1, 2\}$, the operator Δ as two fixpoints, namely

$$\Delta(K_{(1,2)}) = K_{(1,2)}, \qquad \Delta(K_{(2,1)}) = K_{(2,1)},$$

where $K_{(2,1)}$ is the well-known Kolakoski word [Kol66], whose first terms are

$$K_{(2,1)} = 2211212212211211221211212211211212212211212212112112 2\dots$$

and $K_{(1,2)} = 1K_{(2,1)}$.

A right infinite word is said to be smooth if its alphabet is invariant under Δ. More precisely, the set $\mathcal{K}_\Sigma$ of the right infinite smooth words over Σ is:

$$\mathcal{K}_\Sigma = \left\{ w \in \Sigma^{\mathbb{N}} \mid \forall k \in \mathbb{N}, \Delta^k(w) \in \Sigma^{\mathbb{N}} \right\}.$$

Given a smooth words w over a finite alphabet Σ, we define the *tiling associated to* w (see [BBLP03]) as the two-dimensional sequence $(T(w)_{m,n})_{(m,n)\in\mathbb{N}^2}$ as follows:

$$\forall m \in \mathbb{N},\ (T(w)_{m,\bullet}) = \Delta^m(w).$$

In other words, for any $m \in \mathbb{N}$, the m-th line of $(T(w)_{m,n})_{(m,n)\in\mathbb{N}^2}$ is the right infinite word $\Delta^m(w)$.

Let us now state the main result of this section:

Theorem 5. *Let w be a smooth word over the alphabet $\{1, 2, 3\}$. The tiling $T(w)$ associated to w is a piece of a discrete surface if and only if $w \in \{K_{(1,3)}, K_{(3,1)}, 2K_{(3,1)}\}$.*

Proof. Using the permitted hook-words of Fig. 4 and the smoothness condition, that is,

$$\forall m \in \mathbb{N},\ T(w)_{m+1,\bullet} = \Delta\left(T(w)_{m,\bullet}\right),$$

an exhaustive inspection gives that $T(w)$ must start by one of the patterns of Fig.8.

Clearly, the other 5 patterns are excluded because they do not respect the smoothness condition. We proceed by exhaustive inspection. Let us for instance investigate the first case (see Fig. 9). In the two first extensions of the initial word, the smoothness condition does not hold. In the third extension, the smoothness condition provides a forbidden pattern. In the last extension, we obtain the tiling associated to the word $K_{(1,3)}$.

The other cases can be treated in the same way. For instance, we obtain the tiling $T\left(2K_{(1,3)}\right)$ in the second case, and the tiling $T\left(K_{(3,1)}\right)$ in the third case. None of the other cases leads to a discrete surface. ■

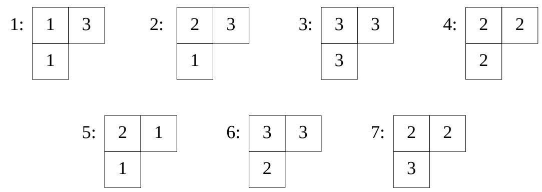


Fig. 8. The possible starting patterns.

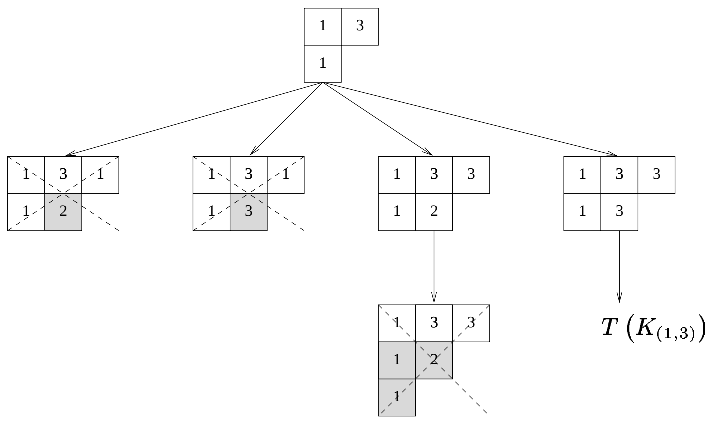


Fig. 9. The different extensions in the first case.

5 Concluding remarks

Since the identification of the three fundamental faces to the letters 1, 2 and 3 is arbitrary, a natural question arises: what is the action of permutation on the coding alphabet? By an exhaustive inspection of the 5 possible permutations, it can be shown that the only smooth tilings describing a discrete surface are generated by a generalized Kolakoski's word. It would be interesting to find a general proof showing which smooth words generate a discrete surface, for an arbitrary permutation on the coding alphabet. The next table gives the results.

Permutation	Smooth words generating discrete surfaces
(123)	$K_{(1,2)}$, $K_{(2,1)}$
(132)	$K_{(2,3)}$, $K_{(3,2)}$
(12)	$K_{(2,3)}$, $K_{(3,2)}$
(13)	$K_{(1,3)}$, $K_{(3,1)}$
(23)	$K_{(1,2)}$, $K_{(2,1)}$, $3K_{(2,1)}$

6 Acknowledgments

The authors wish to thank Valérie Berthé and Srečko Brlek for their useful comments and their support.

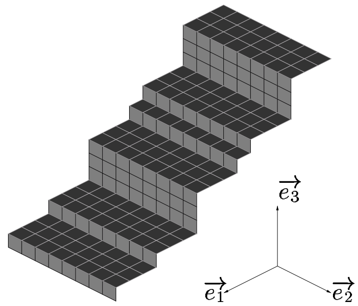


Fig. 10. A connected discrete surface associated to the word $K_{(1,3)}$.

References

[ABS04] P. Arnoux, V. Berthé, and A. Siegel. Two-dimensional iterated morphisms and discrete planes. *Theoret. Comput. Sci.*, 319(1-3):145–176, 2004.

[BBC04] V. Berthé, S. Brlek and P. Choquette. Smooth words over arbitrary alphabets, *Theoret. Comput. Sci.* (2004) submitted.

[BBLP03] A. Bergeron-Brlek, S. Brlek, A. Lacasse, X. Provençal, Patterns in Smooth Tilings, Proc. WORDS'03, 4-th International Conference on Words Turku (Finland) September 10-13, 2003, TUCS General Publication No 27, August (2003) 370-381.

[BLL02] S. Brlek, A. del Lungo, A. Ladouceur. The class of smooth tilings, Journées Montoises d'informatique théorique (Montpellier, France) September 9-11, 2002.

[BL02] S. Brlek, A. Ladouceur. Tiling with a new class of bi-infinite words, Proc. Gian-Carlo Rota Memorial Conference, Barasciano (AQ) Italie, April 25-27, 2002, Università di l'Aquila (2002) 35–46.

[BMP04] S. Brlek, G. Melançon and G. Paquin, Properties of the extremal infinite smooth words, Proc. the Journées Montoises d'informatique théorique (Liège, Belgique) September 8-11, 2004, 291–303

[BT04] V. Berthé and R. Tijdeman. Lattices and multi-dimensional words. *Theoret. Comput. Sci.*, 319(1-3):177–202, 2004.

[BV00] V. Berthé and L. Vuillon. Tilings and rotations on the torus: a two-dimensional generalization of sturmian sequences. *Discrete Math.*, 223(1-3):27–53, 2000.

[DGK03] F. Durand, A. Guerziz, M. Koskas. Words and morphisms with Sturmian erasures. Bulletin of the Belgian Mathematical Society (accepted), 11 pages.

[GR97] D. Giammarresi and A. Restivo. Two-dimensional languages. pages 215–267, 1997.

[Jam04] D. Jamet. On the Language of Discrete Planes and Surfaces. In *Proceedings of the Tenth International Workshop on Combinatorial Image Analysis*, pages 227-241. Springer-Verlag, 2004.

[Kol66] W. Kolakoski, Self Generating Runs, Problem 5304, *American Math. Monthly*, **72** (1965) 674. Solution: *American Math. Monthly* **73** (1966) 681–682.

[Lot02] Lothaire. *Algebraic Combinatorics on Words.* Cambridge University Press, 2002.

[Vui98] L. Vuillon. Combinatoire des motifs d'une suite sturmienne bidimensionelle. *Theoret. Comput. Sci.*, 209(1-2):261–285, 1998.