Braids and tableaux for unipotent Hecke algebras

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Abstract

This talk describes a family of Hecke algebras that generalize the classical Iwahori-Hecke algebra. While many of the results extend to other groups of Lie type, this talk focuses on the case where the underlying group is the general linear group over a finite field. The main results include (a) an indexing of the basis elements in terms of row and column degree-sum matrices, (b) a set of braid-like relations for multiplying basis elements, and (c) a generalization of the RSK-correspondence that maps sets of monomial matrices to multi-tableaux.

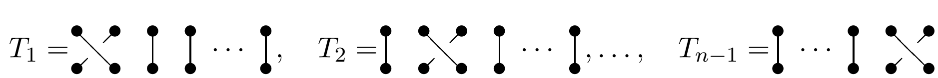
Cet exposé décrit une famille d'algèbres de Hecke qui généralisent l'algèbre classique d'Iwahori-Hecke. Tandis que plusieurs deces résultats se prolongent à d'autres groupes de type de Lie, cet exposé se concentre sur le cas où le groupe sous-jacent est un groupe linéaire général sur un corps fini. Les résultats principaux incluent (a) une indexation des éléments de la base par des colonnes et des rangées de matrices "degree-sum," (b) un ensemble de "braid-like" relations pour multiplier des éléments de la base, et (c) une généralisation de la correspondance RSK qui met en correspondance les ensembles de matrices de monôme avec les multi-tableaux.

1 Introduction

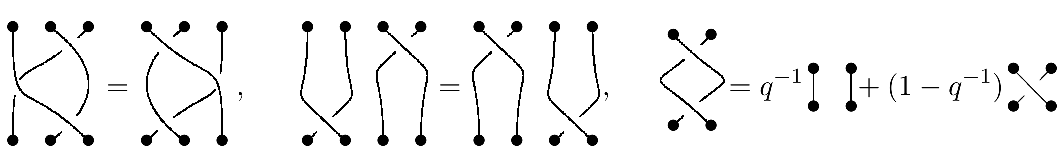
Iwahori [Iw] and Iwahori-Matsumoto [IM] introduced the Iwahori-Hecke algebra as a first step in classifying the irreducible representations of finite Chevalley groups and reductive p-adic Lie groups. Subsequent work (e.g. [Cu] [KL] [LV]) has established Hecke algebras as fundamental tools in the representation theory of Lie groups and Lie algebras, and advances on subfactors and quantum groups by Jones [Jo1], Jimbo [Ji], and Drinfeld [Dr] gave Hecke algebras a central role in knot theory [Jo2], statistical mechanics [Jo3], mathematical physics, and operator algebras. This paper considers a generalization of the classical Iwahori-Hecke algebra obtained by replacing the Borel subgroup B with a maximal unipotent subgroup U.

The Iwahori-Hecke algebra for the general linear group over a finite field ($GL_n(\mathbb{F}_q)$) has a presentation that generalizes the braid-like relations of the symmetric group S_n, given by

Generators.



Relations.



Such a presentation facilitates computations in the Iwahori Hecke algebra and leads to an explicit construction of its representation theory based on the combinatorics of the symmetric group.

This paper examines a family of Hecke algebras that both preserve more of the group structure of $GL_n(\mathbb{F}_q)$ in their representation theory, but also maintain the underlying braid structure of the symmetric group.

Let $G = GL_n(\mathbb{F}_q)$ be the general linear group over the finite field $\mathbb{F}_q$ with q elements. Define subgroups

$$\begin{gathered} T = \left\{ \begin{array}{c} \text{diagonal} \\ \text{matrices} \end{array} \right\}, \quad N = \left\{ \begin{array}{c} \text{monomial} \\ \text{matrices} \end{array} \right\}, \\ W = \left\{ \begin{array}{c} \text{permutation} \\ \text{matrices} \end{array} \right\}, \quad \text{and} \quad U = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\}, \end{gathered} \tag{1.1}$$

where a monomial matrix is a matrix with exactly one nonzero entry in each row and column.

Fix a composition $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$ of n and a nontrivial linear character $\psi : \mathbb{F}_q^+ \to \mathbb{C}^*$ of the additive group of the field. Place a 1 in the last box of every row in the Ferrer's diagram of μ and let all the other boxes contain 0. Let $\mu_{(1)}, \mu_{(2)}, \ldots, \mu_{(n)}$ be the sequence of 0's and 1's obtained by reading left to right, top to bottom. Then the map

$$\begin{array}{cccc} \psi_\mu : & U & \longrightarrow & \mathbb{C}^* \\ & \begin{pmatrix} 1 & t_1 & * & * \\ 0 & 1 & \ddots & * \\ \vdots & \ddots & \ddots & t_{n-1} \\ 0 & \cdots & 0 & 1 \end{pmatrix} & \mapsto & \psi\big(\mu_{(1)} t_1 + \mu_{(2)} t_2 + \cdots + \mu_{(n-1)} t_{n-1}\big) \end{array}$$

is a linear character of U. Let

$$e_\mu = \frac{1}{|U|} \sum_{u \in U} \psi_\mu(u^{-1}) u \qquad \in \mathbb{C}G$$

be the corresponding idempotent. Then the unipotent Hecke algebra $\mathcal{H}(G, U, \psi_\mu)$ is

$$\mathcal{H}_\mu = e_\mu \mathbb{C}G e_\mu \qquad \big(= \mathrm{End}_{\mathbb{C}G}(\mathrm{Ind}_U^G(\psi_\mu))\big),$$

with a natural "double-coset" basis given by

$$\{e_\mu v e_\mu \mid v \in N_\mu\}, \qquad \text{where} \quad N_\mu = \{v \in N \mid e_\mu v e_\mu \neq 0\}.$$

The main results

The main results of this paper are

Section 3 An enumeration of N_μ in terms of matrices with

(1) monic polynomials entries in $\mathbb{F}_q[X]$

(2) row degree-sums and column degree-sums equal to μ.

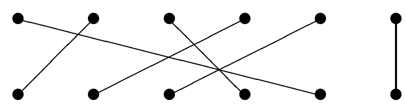
Section 4 A set of braid-like relations for multiplying elements of the double coset basis of $\mathcal{H}_\mu$.

Section 5 An RSK-correspondence between N_μ and column strict multi-tableaux.

This abstract focuses on results rather than proofs; for a more in depth analysis, see [Th1, Th2].

2 Skein model

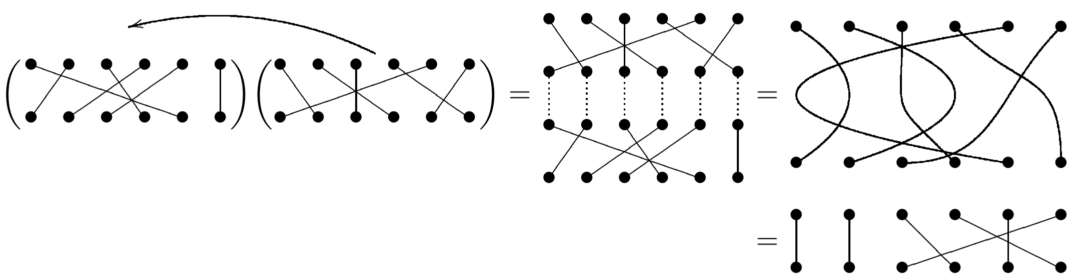
For the results that follow, it will be useful to view elements of $\mathbb{C}G$ as braid-like diagrams instead of matrices. The basic idea is to depict an $n \times n$ permutation matrix w as two rows of n vertices each, with an edge (called a strand) from the ith top vertex to the jth bottom vertex if $w(i) = j$. For example,



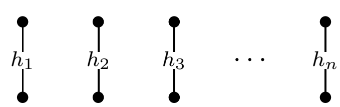
corresponds to

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Matrix multiplication corresponds to concatenation of diagrams, so

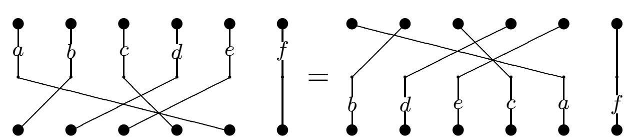


We generalize these diagrams to N by adding "beads" to these diagrams that slide along the strands. Thus, a diagonal matrix corresponds to the identity permutation with a bead on each strand, such as



corresponds to $\quad \mathrm{diag}(h_1, h_2, \ldots, h_n)$.

The advantage of this approach is that it allows a visual shortcut to computing products (such as the permutations above) and commutations in N. For example, by simply pushing the beads of $h \in T$ along the strands of $w \in W$,



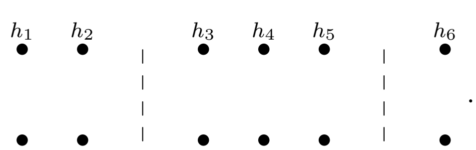
gives

$$s_4 s_3 s_4 s_2 s_3 s_1 \mathrm{diag}(a, b, c, d, e, f) = \mathrm{diag}(b, d, e, c, a, f) s_4 s_3 s_4 s_2 s_3 s_1,$$

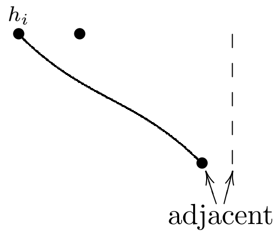
where s_i is the simple transposition $(i, i+1)$.

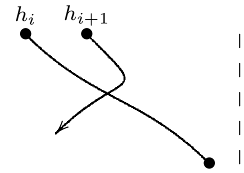
3 A natural basis for $\mathcal{H}_\mu$

We may characterize the elements of N_μ in the following fashion. Suppose $v \in N$. Partition the top vertices by μ; for example, $\mu = (2, 3, 1)$ gives



Then $e_\mu v e_\mu \neq 0$ if and only if the diagram for v satisfies

(1) if 

, then 

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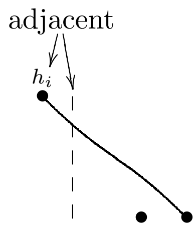
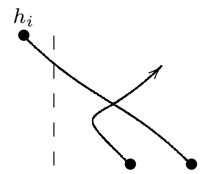
(2) if 

, then 

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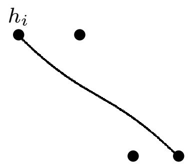
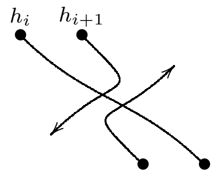
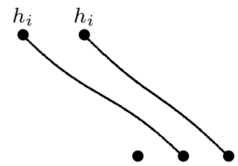
(3) if 

, then either 

or 

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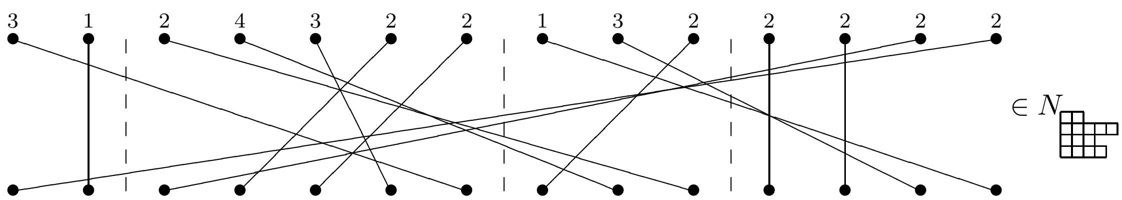
Example. The element



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Let $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$ be a composition of n, and define

$$M_\mu = \left\{ a = (a_{ij}) \in M_\ell(\mathbb{F}_q[X]) \;\middle|\; \begin{array}{l} a_{ij}(X) \text{ is monic with } a_{ij}(0) \neq 0 \\ \sum_{j=1}^{\ell} deg(a_{ij}) = \mu_i, \sum_{i=1}^{\ell} deg(a_{ij}) = \mu_j \end{array} \right\}$$

$$m_\mu = \left\{ a = (a_{ij}) \in M_\ell(\mathbb{Z}_{\geq 0}) \;\mid\; \sum_{j=1}^{\ell} a_{ij} = \mu_i, \sum_{i=1}^{\ell} a_{ij} = \mu_j \right\},$$

where the degree of a polynomial f is denoted by $deg(f)$.

Theorem 3.1. *Let μ be a composition of n. Then there is a bijection*

$$N_\mu \stackrel{1-1}{\longleftrightarrow} M_\mu.$$

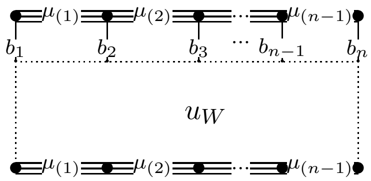
Corollary 1. *Let μ be a composition of n. Then*

$$\dim(\mathcal{H}_\mu) = \sum_{a \in m_\mu} (q-1)^{\ell(a)} q^{n-\ell(a)},$$

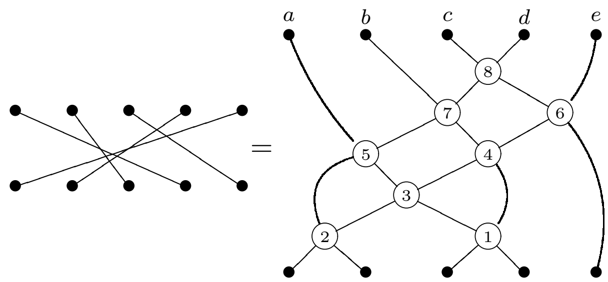
where $\ell(a) = |\{a_{ij} \neq 0 \;\mid\; 1 \leq i, j \leq \ell\}|$.

4 Multiplication relations in $\mathcal{H}_\mu$

Suppose $u \in N_\mu$ and write $u = u_W u_T$ where $u_W \in W$ and $u_T = \mathrm{diag}(b_1, b_2, \ldots, b_n) \in T$. Depict the corresponding unipotent Hecke algebra element as

$e_\mu u e_\mu =$  .

It will be necessary to select a specify a minimal decomposition of u_W in W. Depict this choice, by numbering the crossings from 1 to the length of u_W. For example, if $u_W = s_3 s_1 s_2 s_3 s_1 s_4 s_2 s_3$, then write

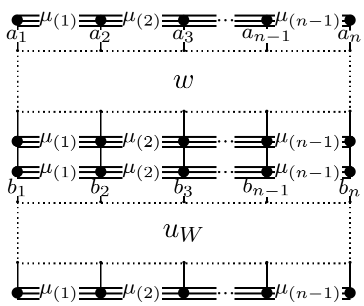
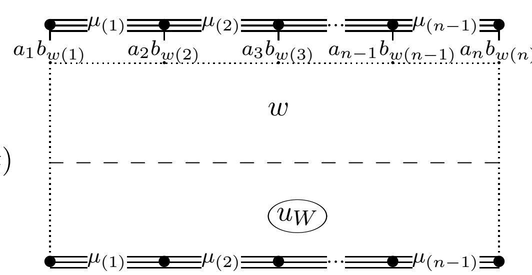


For $1 \le i < j \le \ell$, let

$$\mu_{ij} = \begin{cases} \mu_{(i)}, & \text{if } j = i+1, \\ 0, & \text{otherwise.} \end{cases}$$

Suppose $v \in N_\mu$ such that $v = w\mathrm{diag}(a_1, a_2, \ldots, a_n)$ with $w \in W$. Then

Relation 0

 $= \dfrac{1}{q^r} \displaystyle\sum_{t \in \mathbb{F}_q^r} (\psi \circ f_u)(t)$ 

where $\textcircled{u_W} = s_{i_1} s_{i_2} \ldots, s_{i_r}$ according to some choice of minimal decomposition in W, and $f_u \in \mathbb{F}_q[y_1, y_2, \ldots, y_r]$ is given by

$$f_u(y_1, y_2, \ldots, y_r) = -\mu_{i_1 j_1} b_{i_1}^{-1} b_{j_1} y_1 - \mu_{i_2 j_2} b_{i_2}^{-1} b_{j_2} y_2 - \cdots - \mu_{i_r j_r} b_{i_r}^{-1} b_{j_r} y_r, \tag{4.1}$$

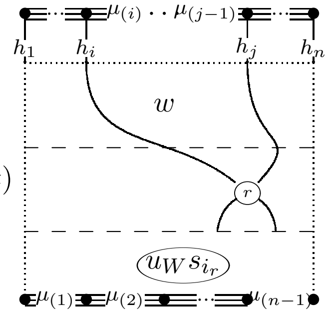
where $(i_k, j_k) = (l, m)$, if the kth crossing in u crosses the strands coming from the lth and mth top vertices in u.

To simplify the concatenated product, apply one of the following two relations to crossing $\textcircled{r}$ in $\textcircled{u_W}$.

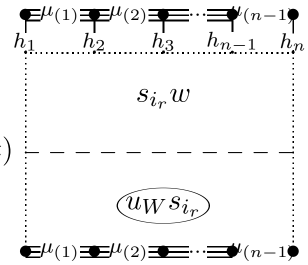
Relation 1

If the strands that cross at ⓡ do not cross in w, then for any $f \in \mathbb{F}_q[y_1^{\pm 1}, \dots, y^{\pm r}]$,

$$\frac{1}{q^r} \sum_{t \in \mathbb{F}_q^r} (\psi \circ f)(t)$$



$$= \frac{1}{q^r} \sum_{t \in \mathbb{F}_q^r} (\psi \circ f^{(+0)})(t)$$



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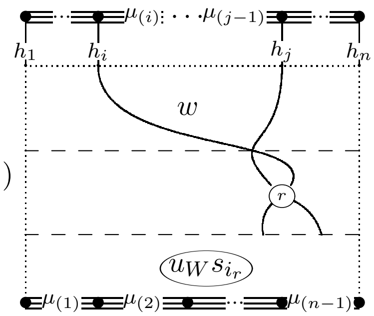
where $f^{(+0)} = f + \mu_{ij} h_i^{-1} h_j y_r$. Note that $f^{(+0)} = f$ unless $j = i + 1$.

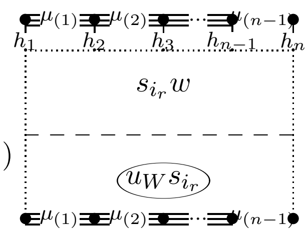
Relation 2

If the strands that cross at ⓡ cross in w, then

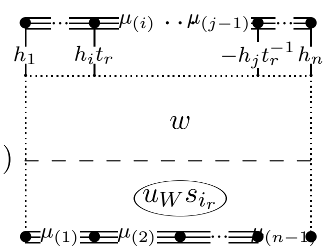
$$\frac{1}{q^r} \sum_{\substack{t' \in \mathbb{F}_q^{r-1} \\ t_r \in \mathbb{F}_q}} (\psi \circ f)(t', t_r)$$



$$= \frac{1}{q^r} \sum_{\substack{t' \in \mathbb{F}_q^{r-1} \\ t_r = 0}} (\psi \circ f)(t', t_r)$$



$$+ \frac{1}{q^r} \sum_{\substack{t' \in \mathbb{F}_q^{r-1} \\ t_r \in \mathbb{F}_q^*}} (\psi \circ f^{(1)})(t', t_r)$$

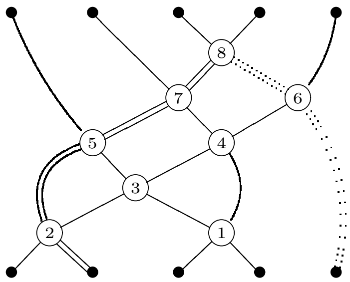


where $f^{(1)} = \varphi_r(f) + \mu_{ij} h_j h_i^{-1} y_r^{-1}$, and $\varphi_r : \mathbb{F}_q[y_1^{\pm 1}, \dots, y_r^{\pm 1}] \to \mathbb{F}_q[y_1^{\pm 1}, \dots, y_r^{\pm 1}]$ is computed in Lemma 4.1 below. Note that we could have applied these steps for any f, u, and v, so we can iterate the process with each sum until we have applied either Relation 1 or Relation 2 to to every numbered crossing in u_W.

Relation $2'$: A combinatorial way to compute $\varphi_k(f)$.

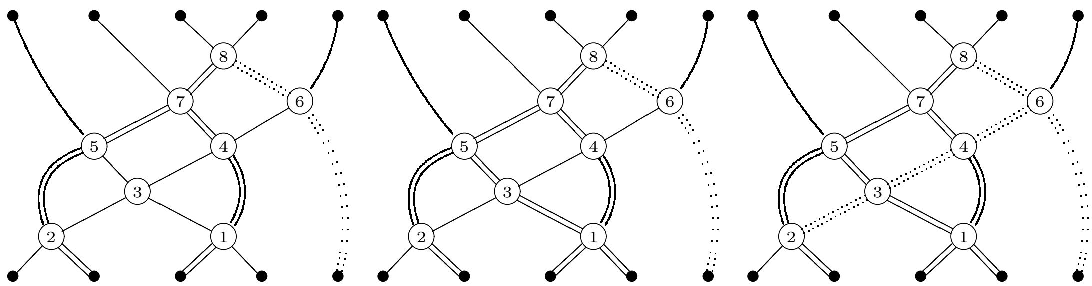
Paint the strands below crossing ⓚ in u_W. Suppose $u_W = s_{i_1} s_{i_2} \cdots s_{i_k} \in W$ is a minimal expression. Each step is illustrated with the example $u_W = s_3 s_1 s_2 s_3 s_1 s_4 s_2 s_3$.

(1) Paint the left [respectively right] strand exiting $\textcircled{k}$ below red [blue] all the way to the bottom of the diagram.



where red is ═══, blue is ┈┈┈, and $\textcircled{k}$ is $\textcircled{8}$.

(2) For each crossing that the red [blue] strand passes through, paint the right [left] strand (if possible) red [blue] until that strand either reaches the bottom or crosses the blue [red] strand of (1).

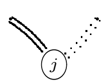


(4.2)

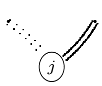
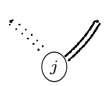
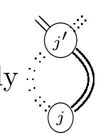
(3) Set

$$\textcircled{u_W}^{\textcircled{k}} = \text{ the diagram } \textcircled{u_W} \text{ painted according to (1) and (2).} \tag{4.3}$$

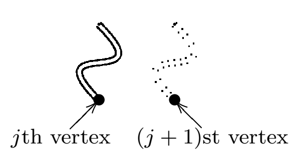
Sinks. The diagram $\textcircled{u_W}^{\textcircled{k}}$ has a *crossed sink at* $\textcircled{j}$ if $\textcircled{j}$ is a crossing between a red strand and a blue one, or



Note that since u_W is decomposed according to a minimal expression in W, there will be no crossings of the form

 (since  would imply  for some $j' \geq j$.)

The diagram $\textcircled{u_W}^{\textcircled{k}}$ has a *bottom sink at* j if a red strand enters jth bottom vertex *and* a blue strand enters the $(j+1)$st bottom vertex, or



Example (continued) In the running example above $\textcircled{u_W}^{\textcircled{8}}$ has crossed sinks at $\textcircled{2}$, $\textcircled{3}$, and $\textcircled{4}$, and a bottom sink at 4. Note that $\textcircled{1}$ is not a crossed sink since both strands are red.

Paths. A *red [respectively blue] path* p from a sink s (either crossed or bottom) in $\textcircled{u_W}^{\textcircled{k}}$ is an increasing sequence

$$j_1 < j_2 < \cdots < j_l = k,$$

such that in $\textcircled{u_W}^{\textcircled{k}}$

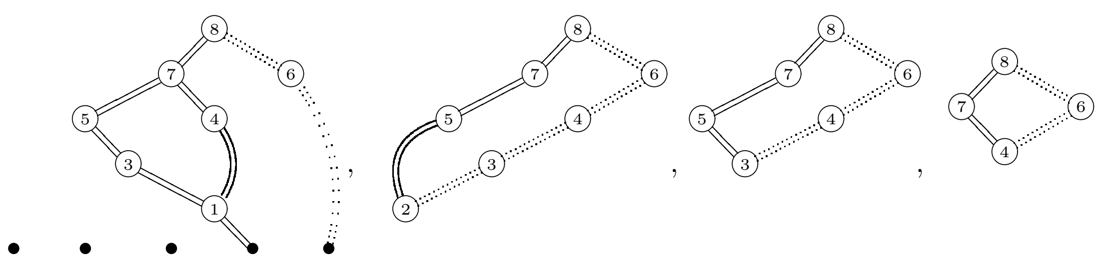
(a) $\textcircled{j_m}$ is directly connected (no intervening crossings) to $\textcircled{j_{m+1}}$ by a red [blue] strand,

(b) if s is a crossed sink, then $\textcircled{j_1} = s$,

(b′) if s is a bottom sink, then

- in a red path, the sth bottom vertex connects to the crossing $\textcircled{j_1}$ with a red strand.
- in a blue path, the $(s+1)$st bottom vertex connects to the crossing $\textcircled{j_1}$ with a blue strand.

Example (continued). The sinks with their corresponding paths for $\textcircled{u_W}^{\textcircled{8}}$ are



Let

$$P_{=}(\textcircled{u_W}^{\textcircled{k}}, s) = \begin{Bmatrix} \text{red paths from} \\ s \text{ in } \textcircled{u_W}^{\textcircled{k}} \end{Bmatrix} \quad \text{and} \quad P_{:}(\textcircled{u_W}^{\textcircled{k}}, s) = \begin{Bmatrix} \text{blue paths from} \\ s \text{ in } \textcircled{u_W}^{\textcircled{k}} \end{Bmatrix} \tag{4.4}$$

The *weight* of a path p is

$$\mathrm{wt}(p) = \begin{cases} \displaystyle\prod_{\substack{p \text{ switches} \\ \text{strands at } \textcircled{i}}} y_i, & \text{if } p \in P_{=}(\textcircled{u_W}^{\textcircled{k}}, s), \\ \displaystyle\prod_{\substack{p \text{ switches} \\ \text{strands at } \textcircled{i}}} (-y_i), & \text{if } p \in P_{:}(\textcircled{u_W}^{\textcircled{k}}, s). \end{cases} \tag{4.5}$$

Each sink s in $\textcircled{u_W}^{\textcircled{k}}$ (either crossed $\textcircled{j}$ or bottom j) has an associated polynomial $g_s \in \mathbb{F}_q[y_1, y_2, \ldots, y_{k-1}, y_k^{-1}]$ given by

$$g_s = \sum_{p \in P_{=}(\textcircled{u_W}^{\textcircled{k}}, s)} \; \sum_{p' \in P_{:}(\textcircled{u_W}^{\textcircled{k}}, s)} \mathrm{wt}(p) y_k^{-1} \mathrm{wt}(p'). \tag{4.6}$$

Example (continued). Consider the weights of the above paths,

Sink	4	4	4	②	②
Path	$1 < 3 < 5 < 7 < 8$	$1 < 4 < 7 < 8$	$6 < 8$	$2 < 5 < 7 < 8$	$2 < 3 < 4 < 6 < 8$
Weight	y_5	$y_1 y_7$	1	1	$-y_6$

Sink	③	③	④	④
Path	$3<5<7<8$	$3<4<6<8$	$4<7<8$	$4<6<8$
Weight	y_5	$-y_6$	y_7	$-y_6$

The corresponding polynomials are

$$g_4 = y_5y_8^{-1} + y_1y_7y_8^{-1}, \quad g_{②} = -y_8^{-1}y_6, \quad g_{③} = -y_5y_8^{-1}y_6, \quad g_{④} = -y_7y_8^{-1}y_6. \tag{4.7}$$

Lemma 4.1. *Let u_W and φ_r be as in Relation 2; suppose $\textcircled{u_W}^{\,ⓡ}$ is painted as above. Then*

$$\varphi_r(f) = f\Big|_{\{y_j\mapsto y_j - g_{ⓙ} \ \mid\ ⓙ \text{ a crossed sink}\}} + \sum_{\substack{j \text{ a bottom}\\ \text{sink}}} \mu_{(j)}g_j.$$

For example (see (4.7)),

$$\varphi_8(f) = f\Big|_{\substack{y_4\mapsto y_4 - g_{④}\\ y_3\mapsto y_3 - g_{③}\\ y_2\mapsto y_2 - g_{②}}} + \mu_4 g_4 = f\Big|_{\substack{y_4\mapsto y_4 + y_7y_8^{-1}y_6\\ y_3\mapsto y_3 + y_5y_8^{-1}y_6\\ y_2\mapsto y_2 + y_8^{-1}y_6}} + \mu_4(y_5y_8^{-1} + y_1y_7y_8^{-1}).$$

5 A generalized RSK-correspondence

Let

$$\Phi = \left\{ f\in\mathbb{C}[t] \ : \ \begin{array}{c} f \text{ is monic, irre-} \\ \text{ducible and } f(0)\neq 0 \end{array} \right\}. \tag{5.1}$$

A Φ-*partition* $\lambda = (\lambda^{(f_1)}, \lambda^{(f_2)}, \ldots)$ is a sequence of partitions indexed by Φ. The *size* of λ is

$$|\lambda| = \sum_{f\in\Phi} deg(f)|\lambda^{(f)}|.$$

A *column strict tableau* $P = (P^{(f_1)}, P^{(f_2)}, \ldots)$ *of shape* λ is a column strict filling of λ by positive integers. That is, $P^{(f)}$ is a column strict tableau of shape $\lambda^{(f)}$. Write $\mathrm{sh}(P) = \lambda$. The *weight of* P is the composition $\mathrm{wt}(P) = (\mathrm{wt}(P)_1, \mathrm{wt}(P)_2, \ldots)$ given by

$$\mathrm{wt}(P)_i = \sum_{f\in\Phi} deg(f) \begin{pmatrix} \text{number of} \\ i \text{ in } P^{(f)} \end{pmatrix}.$$

If λ is a Φ-partition and μ is a composition, then let

$$\hat{\mathcal{H}}_\mu^\lambda = \{\text{column strict tableaux } P \ : \ \mathrm{sh}(P) = \lambda, \mathrm{wt}(P) = \mu\} \tag{5.2}$$

and

$$\hat{\mathcal{H}}_\mu = \{\lambda \text{ a } \Phi\text{-partition} \ : \ \hat{\mathcal{H}}_\mu^\lambda \text{ is not empty}\}. \tag{5.3}$$

The following theorem is a consequence of double centralizer theory and [Ze, Theorem 5.5].

Theorem 5.1. *The set $\hat{\mathcal{H}}_\mu$ indexes the irreducible $\mathcal{H}_\mu$-modules $\mathcal{H}_\mu^\lambda$ and*

$$\dim(\mathcal{H}_\mu^\lambda) = |\hat{\mathcal{H}}_\mu^\lambda|.$$

The $(\mathcal{H}_\mu, \mathcal{H}_\mu)$-bimodule decomposition

$$\mathcal{H}_\mu \cong \bigoplus_{\lambda \in \hat{\mathcal{H}}_\mu} \mathcal{H}_\mu^\lambda \otimes \mathcal{H}_\mu^\lambda \quad \text{implies} \quad |N_\mu| = \dim(\mathcal{H}_\mu) = \bigoplus_{\lambda \in \hat{\mathcal{H}}_\mu} \dim(\mathcal{H}_\mu^\lambda)^2 = \sum_{\lambda \in \hat{\mathcal{H}}_\mu} |\hat{\mathcal{H}}_\mu^\lambda|^2.$$

Theorem 5.2, below, gives a combinatorial proof of this identity.

Encode each matrix $a \in M_\mu$ as a Φ-sequence

$$(a^{(f_1)}, a^{(f_2)}, \ldots), \qquad f_i \in \Phi,$$

where $a^{(f)} \in M_{\ell(\mu)}(\mathbb{Z}_{\geq 0})$ is given by

$$a_{ij}^{(f)} = \text{highest power of } f \text{ dividing } a_{ij}.$$

Note that this is an entry by entry "factorization" of a such that

$$a_{ij} = \prod_{f \in \Phi} f^{a_{ij}^{(f)}}.$$

Let the classical RSK correspondence be given by

$$\begin{array}{ccc} M_\ell(\mathbb{Z}_{\geq 0}) & \longrightarrow & \left\{ \begin{array}{l} \text{Pairs } (P,Q) \text{ of column strict} \\ \text{tableaux of the same shape} \end{array} \right\} \\ b & \mapsto & (P(b), Q(b)). \end{array}$$

Theorem 5.2. *For $a \in M_\mu$, let $P(a)$ and $Q(a)$ be the Φ-column strict tableaux given by*

$$P(a) = (P(a^{(f_1)}), P(a^{(f_2)}), \ldots) \qquad \text{and} \qquad Q(a) = (Q(a^{(f_1)}), Q(a^{(f_2)}), \ldots) \qquad \text{for } f_i \in \Phi.$$

Then the map

$$\begin{array}{ccccc} N_\mu & \longrightarrow & M_\mu & \longrightarrow & \left\{ \begin{array}{l} \textit{Pairs } (P,Q) \textit{ of } \Phi\textit{-column} \\ \textit{strict tableaux of the same} \\ \textit{shape and weight } \mu \end{array} \right\} \\ v & \mapsto & a_v & \mapsto & (P(a_v), Q(a_v)), \end{array}$$

is a bijection, where the first map is the inverse of the bijection of Theorem 3.1.

By the construction above, the map is well-defined and since all the steps are invertible, the map is a bijection.

For example, suppose $\mu = (7,5,3,2)$ and $f, g, h \in \Phi$ are such that $deg(f) = 1$, $deg(g) = 2$, and $deg(h) = 3$. Then

$$a_v = \begin{pmatrix} g & f^2h & 1 & 1 \\ h & 1 & g & 1 \\ 1 & 1 & f & f^2 \\ g & 1 & 1 & 1 \end{pmatrix} \in M_{\ydiagram{7,5,3,2}}$$

corresponds to the sequence

$$(a_v^{(f_1)}, a_v^{(f_2)}, \ldots) = \left(\begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}^{(f)}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}^{(g)}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^{(h)} \right)$$

and

$$(P(a_v), Q(a_v)) = \left(\begin{ytableau} 2 & 2 & 4 \\ 3 & 4 \end{ytableau}^{(f)}, \begin{ytableau} 1 & 1 \\ 3 \end{ytableau}^{(g)}, \begin{ytableau} 1 & 2 \end{ytableau}^{(h)} \right) \left(\begin{ytableau} 1 & 1 & 3 \\ 3 & 3 \end{ytableau}^{(f)}, \begin{ytableau} 1 & 4 \\ 2 \end{ytableau}^{(g)}, \begin{ytableau} 1 & 2 \end{ytableau}^{(h)} \right).$$

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