RIGIDITY THEORY FOR MATROIDS

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Abstract. Combinatorial rigidity theory seeks to describe the rigidity or flexibility of bar-joint frameworks in $\mathbb{R}^d$ in terms of the structure of the underlying graph G. The goal of this article is to broaden the foundations of combinatorial rigidity theory by replacing G with an arbitrary representable matroid M. The notions of rigidity independence and parallel independence, as well as Laman's and Recski's combinatorial characterizations of 2-dimensional rigidity for graphs, can naturally be extended to this wider setting. As we explain, many of these fundamental concepts really depend only on the matroid associated with G (or its Tutte polynomial), and have little to do with the special nature of graphic matroids or the field $\mathbb{R}$.

Our main result is a "nesting theorem" relating the various kinds of independence. Immediate corollaries include generalizations of Laman's Theorem, as well as the equality of 2-rigidity independence and 2-parallel independence. A key tool in our study is the *space of photos of M*, a natural algebraic variety whose irreducibility is closely related to the notions of rigidity independence and parallel independence. The number of points of this variety, when working over a finite field, turns out to be an interesting Tutte polynomial evaluation.

Resumé. Un des objectifs de la théorie combinatoire de rigidité est de décrire, utilisant la structure du graphe fondamental G, la rigidité ou la flexibilité des cadres des barres et joints dans $\mathbb{R}^d$. Le but de ce travail est d'élargir la théorie combinatoire de rigidité en remplaçant G par un matroïde arbitraire représentable M. Dans ce sens, les idées d'indépendance de rigidité et d'indépendance parallèle, les caractérisations combinatoires de Laman et de Recski de la rigidité 2-dimensionelle pour les graphes, peuvent naturellement être étendues. Comme nous le monterons, beaucoup de ces concepts fondamentaux dépendent seulement du matroïde associé à G (ou à son polynôme de Tutte), et ils sont très peu liés á la nature spéciale des matroïdes graphiques ou du champ $\mathbb{R}$.

Notre principal résultat est un "théorème d'emboîtement" relatif aux divers genres d'indépendance. Quelques conséquences directes de ce théorème sont les généralisations du théorème de Laman et l'équivalence de la propriété d'indépendance 2-rigidité avec celle 2-parallèle. Notre étude est fondamentalement basée sur *l'éspace des photos de M* représentant une variété algébrique naturelle dont l'irréductibilité est étroitement liée aux notions d'indépendance de rigidité et d'indépendance parallèle. Le cardinal de cette variété, en travaillant dans un champ fini, est en fait une évaluation intéressante de polynôme de Tutte.

1. Introduction

1.1. **A brief tour through rigidity theory.**

Combinatorial rigidity theory is concerned with frameworks built out of bars and joints in $\mathbb{R}^d$, representing the vertices V and edges E of an (undirected, finite) graph G. (For comprehensive treatments of the subject, see, e.g., [5, 18, 19].) The motivating problem is to determine how the combinatorics of G governs the rigidity or flexibility of its frameworks. Typically, one makes a generic choice of coordinates $p = \{p_v : v \in V\} \subset \mathbb{R}^d$ for the vertices of G and considers infinitesimal motions Δp of the vertices. The following two questions are pivotal:

(I.) What is the dimension of the space of infinitesimal motions Δp that preserve all squared edge lengths $Q(p_u - p_v)$, for $\{u, v\} \in E$, where $Q(x) = \sum_{i=1}^{d} x_i^2$?

(II.) What is the dimension of the space of infinitesimal motions Δp that preserve all edge directions $p_u - p_v$, up to scaling?

The answers to these questions are known to be determined by certain linear dependence matroids represented over transcendental extensions of $\mathbb{R}$, as we now explain.

Date: May 1, 2005.

1991 *Mathematics Subject Classification.* 52C25, 14N20.

Key words and phrases. matroid, combinatorial rigidity, parallel redrawing, Laman's Theorem, Tutte polynomial.

First author supported by the American Institute of Mathematics. Second author partially supported by an NSF Postdoctoral Fellowship. Third author partially supported by NSF grant DMS–0245379.

First, the *d-dimensional rigidity matroid* $\mathcal{R}^d(G)$ is represented by the vectors

$$\{(e_u - e_v) \otimes (p_u - p_v) : \ \{u,v\} \in E\} \ \subset \ \mathbb{R}^{|V|} \otimes \mathbb{R}(p)^d, \tag{1}$$

where $\mathbb{R}(p)$ is the extension of $\mathbb{R}$ by a collection of $d|V|$ transcendentals p, thought of as the coordinates of the vertices in a generic framework of G. The $|E| \times d|V|$ *rigidity matrix* $R^d(G)$ has as its rows the $|E|$ vectors in (1). Then the nullspace of $R^d(G)$ is precisely the space of infinitesimal motions of the vertices that preserve all edge distances (because $R^d(G)$ is $\frac{1}{2}$ times the Jacobian in the variables p of the vector of squared edge lengths $Q(p_u - p_v)$; cf. Remark 5.2 below). Since row rank equals column rank, knowing the matroid $\mathcal{R}^d(G)$ represented by the rows of $R^d(G)$ answers question (I).

Second, the *d-dimensional parallel matroid* $\mathcal{P}^d(G)$ is represented by the vectors

$$\{(e_u - e_v) \otimes \eta^{(j)}_{u,v} : \ \{u,v\} \in E, \ j = 1,2,\ldots,d-1\} \ \subset \ \mathbb{R}^{|V|} \otimes \mathbb{R}(p,\eta)^d, \tag{2}$$

where for each edge $\{u,v\} \in E$, the vectors $\eta^{(1)}_{u,v}, \ldots, \eta^{(d-1)}_{u,v}$ are generically chosen normals to $p_u - p_v$ in $\mathbb{R}^d$, and $\mathbb{R}(p,\eta)$ is an extension of $\mathbb{R}$ by $d|V|$ transcendentals p and $(d-1)|E|$ transcendentals η. In analogy to the preceding paragraph, the $|E| \times d|V|$ *parallel matrix* $P^d(G)$ has as its rows the $|E|$ vectors in (2), and its nullspace is the space of infinitesimal motions of the vertices that preserve all edge directions. Consequently, the matroid $\mathcal{P}^d(G)$ represented by the rows of $P^d(G)$ provides the answer to question (II).

For $d = 2$, the rigidity and parallel matroids coincide [18, Corollary 4.1.3]. The matroid $\mathcal{R}^2(G) = \mathcal{P}^2(G)$ has many equivalent combinatorial reformulations, of which the best known is *Laman's condition* [6]: an edge set $A \subset E$ is 2-rigidity-independent if and only if for every subset $A' \subset A$

$$2|V(A')| - 3 \geq |A'|, \qquad \text{or equivalently} \quad 2\left(|V(A')| - 1\right) > |A'| \tag{3}$$

where $V(A')$ denotes the set of vertices incident to at least one edge in A'. We refer to the triple equivalence between the 2-rigidity matroid, the 2-parallel matroid, and the matroid defined by Laman's condition as the *planar trinity.*

For $d > 2$, the parallel matroid has a simple combinatorial characterization that generalizes Laman's condition, while an analogous description for the rigidity matroid is not known.

1.2. **From graphs to matroids.** The purpose of this article is to broaden the scope of rigidity theory by replacing the graph G with a more general object: a matroid M equipped with a representation over a field $\mathbb{F}$. Indeed, the notions of rigidity and parallel independence, as well as Laman's combinatorial characterization, can be naturally generalized to the setting of matroids. In the process, we will see that many of the main results of do not depend on the special properties of graphs (or graphic matroids), nor on the field $\mathbb{R}$, but in fact remain valid for any matroid M and any field $\mathbb{F}$. In the process, we are led naturally to study an algebraic variety, the *space of k-plane-marked d-photos* of M, whose points play the role of "frameworks" of M embedded in $\mathbb{F}^d$.

Whether or not the photo space is irreducible plays a key role in characterizing the matroidal analogues of rigidity independence and parallel independence. In turn, the question of irreducibility can be answered combinatorially. Furthermore, when the field $\mathbb{F}$ is finite, the number of photos of M is given by an evaluation of the Tutte polynomial using q-binomial coefficients. (Theorem 4.1).

In order to summarize our results, we define the main protagonists here. Recall that for a finite set E, an *(abstract) simplicial complex* on E is a collection $\mathcal{I}$ of subsets of E satisfying the following hereditary condition: if $I \in \mathcal{I}$ and $I' \subset I$, then $I' \in \mathcal{I}$. The independent sets of a matroid always form a simplicial complex. From here on we will make free use of standard terminology and notions from matroid theory; background and definitions may be found in standard texts such as [1, 12, 17].

Definition 1.1. Let E be a set of cardinality n, and let M be a (not necessarily representable) matroid on ground set E, with rank function r. Let m be a real number in the open interval $(1,\infty)_{\mathbb{R}}$. Then $A \subset E$ is called *m-Laman independent* if

$$m \cdot r(A') > |A'| \quad \text{for all nonempty subsets } A' \subseteq A. \tag{4}$$

The *m-Laman complex* $\mathcal{L}^m(M)$ is defined to be the abstract simplicial complex of all m-Laman independent subsets of E.

We will prove that if m is a positive integer, then $\mathcal{L}^m(M)$ is the collection of independent sets of a matroid. Moreover, $\mathcal{L}^m(M)$ has several alternate combinatorial descriptions: one of these generalizes Recski's Theorem characterizing rigidity-independent graphs; another is related to Edmonds' classic result on partitioning a matroid into independent subsets [4].

We now consider the case that M is a matroid represented by vectors $v_1, \ldots, v_n$ in $\mathbb{F}^r$, where $\mathbb{F}$ is a field. For notational convenience, we identify the ground set E with the numbers $[n] := \{1, 2, \ldots, n\}$. When $m > 1$ is a rational number, the Laman complex $\mathcal{L}^m(M)$ is closely related to a certain algebraic variety over $\mathbb{F}$, which we now describe. Denote by $\mathbb{G}r(k, d)$ the Grassmannian of k-planes in $\mathbb{F}^d$, regarded as a projective variety over $\mathbb{F}$ via the usual Plücker embedding.

Definition 1.2. The *space of k-plane-marked d-photos* (or just *(k, d)-photos*) of M is the algebraic set

$$X_{k,d}(M) := \{(\varphi, W) \in \mathrm{Hom}_{\mathbb{F}}(\mathbb{F}^r, \mathbb{F}^d) \times \mathbb{G}r(k, \mathbb{F}^d)^n : \ \varphi(v_i) \in W_i\}. \tag{5}$$

One may think of the map $\varphi \in \mathrm{Hom}_{\mathbb{F}}(\mathbb{F}^r, \mathbb{F}^d)$ as a camera taking a "snapshot" of M on photographic paper that looks like $\mathbb{F}^d$. The k-planes W_i are markings added later to highlight the image vectors $\varphi(v_i)$. Of course, whenever $\varphi(v_i) = 0$ (perhaps the camera φ caught v_i at a bad angle), the k-plane W_i is unconstrained.

The *non-annihilating cellule* of the photo space is defined as the Zariski open subset

$$X^{\varnothing}_{k,d}(M) := \{(\varphi, W) \in X_{k,d}(M) : \ \varphi(v_i) \neq 0 \text{ for } i = 1, 2, \ldots, n\}.$$

Its image under the projection map $\pi : \ \mathrm{Hom}(\mathbb{F}^r, \mathbb{F}^d) \times \mathbb{G}r(k, d)^n \to \mathbb{G}r(k, d)^n$ measures the constraints on the W_i when none of the v_i are mapped to zero. Accordingly, we make the following definition.

Definition 1.3. The matroid M is called *(k, d)-slope independent* if $\pi X^{\varnothing}_{k,d}(M)$ is Zariski dense in $\mathbb{G}r(k, d)^n$. The *(k, d)-slope complex* is defined as

$$\mathcal{S}^{k,d}(M) := \{A \subset E : M|_A \text{ is } (k, d) \text{ -slope independent}\}. \tag{6}$$

The third notion of matroidal rigidity generalizes the d-dimensional rigidity matroid $\mathcal{R}^d(G)$ of a graph G. Let φ be a $d \times r$ matrix of algebraically independent transcendentals, regarded as a generic linear transformation $\mathbb{F}^r \to \mathbb{F}^d$. Consider the pseudo-distance quadratic form $Q(x) := \sum_{i=1}^{d} x_i^2$ on $\mathbb{F}(\varphi)^d$. Provided that the field $\mathbb{F}$ has characteristic $\neq 2$, we wish to define a rigidity matrix $R^d(M)$ whose nullspace consists of the infinitesimal changes of φ that preserve the values $Q(\varphi(v_i))$.

Definition 1.4. The *d-dimensional (generic) rigidity matroid* is the matroid represented by the vectors

$$\{v_i \otimes \varphi(v_i)\}_{i=1}^n \subset \mathbb{F}^r \otimes_{\mathbb{F}} \mathbb{F}(\varphi)^d. \tag{7}$$

where $\mathbb{F}(\varphi)$ is the purely transcendental field extension of $\mathbb{F}$ by the dr entries of φ. The *d-rigidity complex* $\mathcal{R}^d(M)$ is the complex of independent sets of the d-dimensional rigidity matroid, and the *d-rigidity matrix* $R^d(M)$ is the $n \times dr$ matrix whose rows are given by the vectors (7).

In contrast, if we wish to extend the notion of graph rigidity that keeps track of edge *slopes* instead of edge *lengths* (see Question II above), then we need a matrix $P^d(M)$ whose nullspace consists of the infinitesimal changes $\Delta\varphi$ in the matrix φ which preserve the slopes of all the direction vectors $\varphi(v_i)$.

Definition 1.5. The *d-dimensional hyperplane-marking matroid* is the matroid represented by the vectors

$$\{v_i \otimes \eta_i\}_{i=1}^n \subset \mathbb{F}^r \otimes_{\mathbb{F}} \mathbb{F}(\varphi, \eta)^d$$

over the field $\mathbb{F}(\varphi, \eta)$, the extension of $\mathbb{F}$ by dr transcendentals φ_{ij} (the entries of the matrix φ) and $(d-1)n$ more transcendentals η_{ij}. The complex $\mathcal{H}^d(M)$ is defined to be the complex of independent sets of this matroid. The *d-dimensional parallel matroid* is defined as

$$\mathcal{P}^d(M) := \mathcal{H}^d((d-1)M),$$

where $(d-1)M$ is the matroid whose ground set consists of $d-1$ parallel copies of each element of E. The *d-parallel matrix* $P^d(M)$ is the $n \times dr$ matrix whose rows represent $\mathcal{P}^d(M)$.

These definitions generalize the ordinary definitions from the rigidity theory of graphs. Strikingly, the geometric constraints on the photo space can be categorized combinatorially: the identity

$$\mathcal{S}^{k,d}(M) = \mathcal{L}^{\frac{d}{d-k}}(M),$$

(Corollary 3.3) provides a geometric interpretation of $\mathcal{L}^m(M)$ for rational m.

The slope complex $\mathcal{S}^{k,d}(M)$ is closely related to the rigidity and parallel matroids. The precise relationship is given by the *Nesting Theorem* (Theorem 5.4):

$$\mathcal{S}^{1,d}(M) \subseteq \mathcal{R}^d(M) \subseteq \mathcal{L}^d(M) = \mathcal{H}^d(M) = \mathcal{S}^{d-1,d}(M)$$

for all integers $d \geq 2$. In particular, when $d = 2$,

$$\mathcal{H}^2(M) = \mathcal{S}^{1,2}(M) = \mathcal{R}^2(M) = \mathcal{L}^2(M). \tag{8}$$

Thus matroid rigidity theory leads to a proof of the planar trinity (the second and third inequalities in (8)).

For $d \geq 3$, the d-rigidity matroid $\mathcal{R}^d(M)$ is the hardest of these objects to understand (as it is for graphic matroids). One fundamental question is whether $\mathcal{R}^d(M)$ depends on the choice of representation of M. It is invariant for $d = 2$ and up to projective equivalence of representations but the problem remains open in general. We also study the behavior of the d-rigidity matroid as $d \to \infty$: it turns out that $R^d(M)$ stabilizes when $d \geq r(M)$.

In this extended abstract, we omit or merely sketch the proofs of many of our results. The complete proofs can be found in the full-length article [3].

2. Laman independence

The main result of this section, Theorem 2.1, states that the generalized Laman's condition (4) always gives a matroid when m is an integer. The proof is completely combinatorial; that is, it is a statement about abstract matroids, not represented matroids. In addition, we describe some useful equivalent characterizations of d-Laman independence: one uses the Tutte polynomial, another is reminiscent of Recski's Theorem, and another is related to Edmonds' theorem on decomposing a matroid into independent sets.

Theorem 2.1.
(i) *Let d be a positive integer and let M be any matroid. Then the simplicial complex $\mathcal{L}^d(M)$ is a matroid complex.*
(ii) *Let $m \in (1,\infty)_{\mathbb{R}}$ be a real number which is not an integer. Then there exists a represented matroid M for which $\mathcal{L}^m(M)$ is not a matroid complex.*

We omit the proof, which is technical but not difficult. The difference between the two cases makes itself felt in the following way. If C and C' are distinct minimal d-Laman-dependent sets, then $C \cap C'$ is d-Laman-independent; that is,

$$|C \cap C'| < d \cdot r(C \cap C'), \tag{9a}$$

where r is the rank function of M. If d is an integer, then (9a) implies the logically stronger

$$|C \cap C'| \leq d \cdot r(C \cap C') - 1, \tag{9b}$$

from which it eventually follows that the minimal nonmembers of $\mathcal{L}^d(M)$ satisfy the matroid circuit axioms [1, p. 264, eq. 6.13]. On the other hand, if $d \notin \mathbb{Z}$, then (9b) does not follow from (9a), and one can exploit this to write down a matroid M whose minimal d-Laman-dependent sets fail the circuit axioms.

One of the equivalent phrasings of m-Laman independence involves the *Tutte polynomial* $T_M(x, y)$ of M, a fundamental isomorphism invariant of the matroid M. For background on the Tutte polynomial, see the excellent survey article by Brylawski and Oxley [2]. Given a subset A of the ground set E, denote by $\overline{A}$ the *matroid closure* or *span* of A. If $A = \overline{A}$, then A is called a *flat* of M.

Proposition 2.2. *Let M be a matroid on ground set E with rank function r, and fix $m \in (1,\infty)_{\mathbb{R}}$. Then the following are equivalent:*
(i) *E is m-Laman independent, that is, $\mathcal{L}^m(M) = 2^E$ (the power set of E).*
(ii) *$m \cdot r(\overline{A}) > |\overline{A}|$ for every nonempty subset $A \subset E$. (Equivalently, $m \cdot r(F) > |F|$ for every flat F of M.)*
(iii) *The Tutte polynomial specialization $T_M(q^{m-1}, q)$ is monic of degree $(m-1)r(M)$.*

Sketch of proof. The equivalence of (i) and (ii) is clear from the definition of m-Laman independence since $r(\overline{A}) = r(A)$ and $|\overline{A}| \geq |A|$ for any $A \subset E$. The equivalence of (i) and (iii) arises from expanding $T_M(q^{m-1}, q)$ as a polynomial in q using the Whitney corank-nullity formula [2, eq. 6.13]. $\square$

Note that in (iii) we must allow (non-integral) real number exponents for a "polynomial" in q, but the notions of "degree" and "monic" for such polynomials should still be clear. The connection between the Tutte polynomial and rigidity of graphs was observed by the second author in [8].

Suppose that $m = d$ is a positive integer, so that $\mathcal{L}^d(M)$ is a matroid complex. Here d-Laman independence has two more equivalent formulations, one of which extends a classical result in the rigidity theory of graphs.

Recski's Theorem [13]. *Let $G = (V, E)$ be a graph, and let E' be a spanning set of edges of size $2|V| - 3$. Then E' is a 2-rigidity basis if and only if for any $e \in E'$, we can partition the multiset $E' \cup \{e\}$ (that is, adding an extra copy of e to E') into two disjoint spanning trees of G.*

This notion can be naturally extended to arbitrary matroids and dimensions.

Definition 2.3. Let M be a matroid on E. We say that E is *d-Recski independent* if for any element $e \in E$, the multiset $E \cup \{e\}$ can be partitioned into d disjoint independent sets for M.

We wish to show that this purely matroidal condition is equivalent to d-Laman independence. To prove this, we use a powerful classic result of Edmonds.

Edmonds' Decomposition Theorem [4, Theorem 1]. *Let M be a matroid of rank r on ground set E. Then E has a decomposition into d disjoint independent sets $I_1, \ldots, I_d$ if and only if $d \cdot r(A) \geq |A|$ for every subset $A \subset E$.*

Definition 2.4. Let M be a matroid on E. A *d-Edmonds decomposition* of M is a family of independent sets $I_1, \ldots, I_d$ whose disjoint union is E, with the following property: given subsets $I'_1 \subset I_1, \ldots, I'_d \subset I_d$ with not all I'_i empty, then it is not the case that $\overline{I'_1} = \overline{I'_2} = \cdots = \overline{I'_d}$.

Theorem 2.5. *Let M be a matroid on ground set E, and let d be a positive integer. Then the following are equivalent:*

(1) *E has a d-Edmonds decomposition;*
(2) *E is d-Laman independent;*
(3) *E is d-Recski independent.*

Again, the proof is purely technical.

As we have seen in Theorem 2.1 (ii), when m is not an integer, the Laman complex $\mathcal{L}^m(M)$ need not form the collection of independent sets of a matroid. However, $\mathcal{L}^m(M)$ is related to a more general (and less well-known) object called a *polymatroid* [17, chapter 18], as we now explain. (We will not consider polymatroids in the remainder of the paper.)

Definition 2.6. Fix a ground set $E = [n]$. A function $\rho : 2^E \to \mathbb{R}_{\geq 0}$ is the *ground set rank function of a polymatroid* on E if

- $\rho(A) \leq \rho(B)$ whenever $A \subset B \subset E$ (*monotonicity*);
- $\rho(A \cup B) + \rho(A \cap B) \leq \rho(A) + \rho(B)$ for all $A, B \subset E$ (*submodularity*); and
- $\rho(\varnothing) = 0$ (*normalization*).

The *polymatroid* associated with ρ is the convex polytope

$$P_\rho := \{x \in \mathbb{R}^n_{\geq 0} : \sum_{a \in A} x_a \leq \rho(A) \text{ for all } A \subseteq E\},$$

also called the set of *independent vectors* of the polymatroid.

The connection between Laman independence and polymatroids is as follows.

Proposition 2.7. *For every loopless matroid M on ground set $E = [n]$, and every real number $m \in (1, \infty)_{\mathbb{R}}$, there is a polymatroid rank function ρ on E with the following property: $A \subset E$ is m-Laman independent if and only if its characteristic vector is independent for ρ.*

3. Slope independence and the space of photos

Throughout this section, we work with a matroid M with rank function r, represented over a field $\mathbb{F}$ by nonzero[1] vectors $v_1, \dots, v_n \in \mathbb{F}^r$. In addition, fix positive integers $k < d$, and let $m = \frac{d}{d-k}$.

Recall (Definition 1.2) that the *space of* (k,d)*-photos* of M is

$$\left\{ (\varphi, W) \in \operatorname{Hom}_{\mathbb{F}}(\mathbb{F}^r, \mathbb{F}^d) \times \mathbb{G}r(k, \mathbb{F}^d)^n \ : \ \varphi(v_i) \in W_i \text{ for all } 1 \leq i \leq n \right\},$$

an algebraic subset of $\operatorname{Hom}_{\mathbb{F}}(\mathbb{F}^r, \mathbb{F}^d) \times \mathbb{G}r(k, \mathbb{F}^d)^n$, hence a scheme over $\mathbb{F}$. The symbol $X_{k,d}(M)$ is a slight abuse of notation; as defined, the photo space depends on the representation $\{v_i\}$, and it is not at all clear to what extent it depends only on the structure of M as an abstract matroid. (We will return to this question later.)

For each photo (φ, W), $\ker \varphi$ is a linear subspace of $\mathbb{F}^r$, hence intersects E in some flat F of M. It is useful to classify photos according to what this flat is. Accordingly, for a flat $F \subset E$, we define the corresponding *cellule* as

$$X^F_{k,d}(M) = \{(\varphi, W) \in X_{k,d}(M) \ : \ \ker \varphi \cap E = F\}.$$

Each photo belongs to exactly one cellule; that is, $X_{k,d}(M)$ decomposes as a disjoint union of its cellules.

The cellule $X^{\varnothing}_{k,d}(M)$ corresponding to the empty flat $\varnothing$ is called the *non-annihilating cellule.* It is a Zariski open subset of $X_{k,d}(M)$, defined by the open conditions $\varphi(v_i) \neq 0$ for $i = 1, \dots, n$. At the other extreme, the cellule $X^E_{k,d}(M)$ corresponding to the improper flat E is called the *degenerate cellule.* It is precisely $\{0\} \times \mathbb{G}r(k, \mathbb{F}^d)^n$, where 0 is the zero map $\mathbb{F}^r \to \mathbb{F}^d$.

The following facts are easy consequences of the preceding discussion.

Proposition 3.1. *Let M and $X_{k,d}(M)$ be as above. Then:*

(i) *The natural projection map $X^{\varnothing}_{k,d}(M) \to \operatorname{Hom}_{\mathbb{F}}(\mathbb{F}^r, \mathbb{F}^d)$ makes $X^{\varnothing}_{k,d}(M)$ into a bundle with fiber $\mathbb{G}r(k-1, \mathbb{F}^{d-1})$ and base the Zariski open subset of $\operatorname{Hom}_{\mathbb{F}}(\mathbb{F}^r, \mathbb{F}^d)$ defined by $\varphi(v_i) \neq 0$ for $i = 1, \dots, n$.*

(ii) *For each flat F, $X^F_{k,d}(M) \cong X^{\varnothing}_{k,d}(M/F) \times \mathbb{G}r(k, \mathbb{F}^d)^F$. In particular, $X^F_{k,d}(M)$ is irreducible, and*

$$\dim X^F_{k,d}(M) = d(r - r(F)) + (n - |F|)(k-1)(d-k) + |F|k(d-k). \tag{10}$$

Let π denote the projection map

$$\operatorname{Hom}_{\mathbb{F}}(\mathbb{F}^r, \mathbb{F}^d) \times \mathbb{G}r(k, \mathbb{F}^d)^n \ \xrightarrow{\pi} \ \mathbb{G}r(k, \mathbb{F}^d)^n, \tag{11}$$

and define M to be (k,d)*-slope independent* if $\pi X^{\varnothing}_{k,d}(M)$ is Zariski dense in $\mathbb{G}r(k,d)^n$.

Theorem 3.2. *The following are equivalent:*

(i) *M is (k,d)-slope independent, i.e., $\pi X^{\varnothing}_{k,d}(M)$ is dense in $\mathbb{G}r(k,d)^n$.*
(ii) *M is m-Laman independent, i.e., $m \cdot r(F) > |F|$ for every nonempty flat F of M.*
(iii) *$\dim X^F_{k,d}(M) < \dim X^{\varnothing}_{k,d}(M)$ for every nonempty flat F of M.*
(iv) *The photo space $X_{k,d}(M)$ is irreducible.*
(v) *The photo space $X_{k,d}(M)$ coincides with the Zariski closure of its non-annihilating cellule.*

The result is analogous to Theorem 4.5 of [7], and the proof uses the cellule decomposition in a similar way. In particular, the equivalence of (i) and (ii) immediately gives the following equality between the slope and Laman complexes.

Corollary 3.3. *Let $m \in \mathbb{Q} \cap (1, \infty)_{\mathbb{R}}$. Write m as $\frac{d}{d-k}$, where $0 < k < d$ are integers.*
Then $\mathcal{S}^{k,d}(M) = \mathcal{L}^m(M)$.

Remark 3.4. The condition $d \geq 2$ is implicit in Corollary 3.3. However, there is a sense in which the result is still valid for $d = 1$. When $k = 1$, the result asserts that $\mathcal{S}^{1,d}(M) = \mathcal{L}^{\frac{d}{d-1}}(M)$. Now, if one establishes conventions properly, this equality remains valid as d approaches 1, so that $m = \frac{d}{d-1}$ approaches infinity. That is, $\mathcal{S}^{1,1}(M) = \mathcal{L}^{\infty}(M) = 2^E$. Indeed, the full simplex 2^E is logically equal to $\mathcal{S}^{1,1}(M)$: there is only one possible line through any point in $\mathbb{F}^1$, so the projection map π is dense. Meanwhile, it is easy to see that $\mathcal{L}^{\infty}(M) = 2^E$, where $\mathcal{L}^{\infty}(M) := \lim_{m \to \infty} \mathcal{L}^m(M)$.

[1]That is, we assume that M contains no loops. Our results still hold—with trivial but notationally annoying modifications—when loops are present.

Remark 3.5. For a given matroid M and *irrational* number m, it is not hard to see that there exists a rational number $\tilde{m}$, chosen sufficiently close to m, such that $\mathcal{L}^{\tilde{m}}(M) = \mathcal{L}^m(M)$. Therefore, Corollary 3.3 actually gives a geometric interpretation for *every* instance of Laman independence.

Remark 3.6. Another surprising consequence of Corollary 3.3 is that (k, d)-slope-independence is invariant under simultaneously scaling k and d. That is, $\mathcal{S}^{k,d}(M) = \mathcal{S}^{ak,ad}(M)$ for every integer $a > 0$. Moreover, if d is divisible by k, then $m = \frac{d}{d-k}$ is an integer, and in fact $\mathcal{S}^{k,d}(M) = \mathcal{L}^m(M)$ is a matroid by Theorem 2.1 (i). It is far from clear what the geometry is behind these phenomena.

A natural question is to determine the singularities of the photo space. While we cannot do this in general, we can at least say exactly for which matroids $X_{k,d}(M)$ is smooth. The result and its proof are akin to [8, Proposition 15], and do not depend on the parameters k and d.

Proposition 3.7. *Let M be a loopless matroid equipped with a representation $\{v_1, \ldots, v_n\}$ as above. Then, for all integers $0 < k < d$, the photo space $X_{k,d}(M)$ is smooth if and only if each ground set element is either a loop or a coloop.*

Sketch of proof. If M consists solely of loops and coloops, then its photo space has the structure of an iterated fiber bundle over a point, in which every fiber is smooth (in fact, a copy of a projective space). Otherwise, one can explicitly describe the tangent space to $X_{k,d}(M)$ at a point in the degenerate cellule, and show that its dimension exceeds that of the photo space. □

4. Counting photos

Although it will not be needed in the sequel, we digress to prove an enumerative result, possibly of independent interest: when the field of representation of M is finite, the cardinality of the photo space $X_{k,d}(M)$ is an evaluation of the Tutte polynomial $T(M) = T_M(x, y)$. We refer the reader to [2] for details on the Tutte polynomial; roughly, it is the most general matroid isomorphism invariant satisfying the *deletion-contraction recurrence*

$$T(M) = T(M\backslash v) + T(M/v)$$

for every ground set element v that is neither a loop nor a coloop.

For $n \in \mathbb{N}$, define the q-analogues of n and $n!$ by

$$[n]_q := \frac{1 - q^n}{1 - q} = 1 + q + q^2 + \cdots + q^{n-1}, \qquad [n]!_q := [n]_q [n-1]_q \cdots [2]_q [1]_q,$$

and define the *q-binomial coefficient*

$$\begin{bmatrix} d \\ k \end{bmatrix}_q := \frac{[d]!_q}{[k]!_q [d-k]!_q}. \tag{12}$$

Theorem 4.1. *Let $\mathbb{F}$ be the finite field with q elements. Let M be a matroid of rank r, represented over $\mathbb{F}$ by vectors $v_1, \ldots, v_n$ spanning $\mathbb{F}^r$, and let $d \geq 2$. Then the number of (k, d)-photos of M is*

$$|X_{k,d}(M)| \;=\; \begin{bmatrix} d-1 \\ k-1 \end{bmatrix}_q^{r(M^\perp)} \left(q^k \begin{bmatrix} d-1 \\ k \end{bmatrix}_q \right)^{r(M)} T_M \left(\frac{\begin{bmatrix} d \\ k \end{bmatrix}_q}{\begin{bmatrix} d-1 \\ k \end{bmatrix}_q}, \frac{\begin{bmatrix} d \\ k \end{bmatrix}_q}{\begin{bmatrix} d-1 \\ k-1 \end{bmatrix}_q} \right).$$

Here $M^\perp$ denotes the *dual* or *orthogonal* matroid to M, defined combinatorially as the matroid on E whose bases are the complements of the bases of M.

The proof uses the commutative diagram

$$\begin{array}{ccc} X_{k,d}(M/e) \times \mathbb{G}r(k, \mathbb{F}^d) & \hookrightarrow & X_{k,d}(M) \\ \tilde{\pi} \big\downarrow & & \big\downarrow \pi \\ X_{k,d}(M/e) & \hookrightarrow & X_{k,d}(M - v) \end{array} \tag{13}$$

to give a deletion-contraction recurrence for $|X_{k,d}(M)|$. This recurrence can then be translated into a Tutte polynomial evaluation. The q-binomial coefficient (12) arises as the cardinality of the Grassmannian of k-planes in $\mathbb{F}^n$; see [14, Proposition 1.3.18]. The argument resembles that of [8, Theorem 1], in which the second author used a similar commutative diagram to express the Poincaré series of the picture space of a graph (over $\mathbb{C}$) as an analogous Tutte polynomial evaluation. (In contrast, when $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, the topology of the photo space is much simpler: there is a deformation retraction of $X_{k,d}(M)$ onto its degenerate cellule, which is homeomorphic to $\mathbb{G}r(k, \mathbb{F}^d)^n$.)

Since the Tutte polynomial of M does not depend on the choice of representation, neither does the number of photos. Moreover, there is a curious symmetry between the number of photos of a matroid M and of its dual $M^\perp$. Since $T_{M^\perp}(x, y) = T_M(y, x)$ [2, Prop. 6.2.4] and $\left[{d \atop k}\right]_q = \left[{d \atop d-k}\right]_q$, we have

$$q^{d \cdot r(M)}|X_{d-k,d}(M^\perp)| = q^{(d-k)n}|X_{k,d}(M)|. \tag{14}$$

A direct combinatorial explanation of this equality would be of interest.

5. Rigidity and parallel independence

In this section, we examine more closely the special cases $k = 1$ and $k = d-1$ of (k, d)-slope independence for a represented matroid M. It turns out that they are intimately related to the d-dimensional generic rigidity matroid $\mathcal{R}^d(M)$ and the d-dimensional generic hyperplane-marking matroid $\mathcal{H}^d(M)$. As before, let M be a matroid represented by vectors $E = \{v_1, \ldots, v_n\}$ spanning $\mathbb{F}^r$, and let $d > 0$ be an integer.

5.1. **Interpreting $\mathcal{R}^d(M)$ and $\mathcal{H}^d(M)$.** Recall (Definition 1.4) that the *d-dimensional rigidity matroid* is represented over $\mathbb{F}(\varphi)$ by the vectors $\{v_i \otimes \varphi(v_i)\}_{i=1}^n \subset \mathbb{F}^r \otimes_{\mathbb{F}} \mathbb{F}(\varphi)^d$. where $\mathbb{F}(\varphi)$ is the extension of $\mathbb{F}$ by dr transcendentals (the entries of the matrix $\varphi : \mathbb{F}^r \to \mathbb{F}(\varphi)^d$). The complex $\mathcal{R}^d(M)$ is defined to be the complex of independent sets of this matroid. The *d-rigidity matrix* $R^d(M)$ is the $n \times dr$ matrix whose rows represent $\mathcal{R}^d(M)$.

Recall also (Definition 1.5) that the *d-dimensional hyperplane-marking matroid* is represented over $\mathbb{F}(\varphi, n)$ by the vectors $\{v_i \otimes \eta_i\}_{i=1}^n \subset \mathbb{F}^r \otimes_{\mathbb{F}} \mathbb{F}(\varphi, \eta)^d$. where $\mathbb{F}(\varphi)$ is the extension of $\mathbb{F}$ by $dr + (d-1)n$ transcendentals (the dr entries of the matrix φ, and the $(d-1)n$ coordinates of the normal vectors η_i to $\varphi(v_i)$). The complex $\mathcal{H}^d(M)$ is defined to be the complex of independent sets of this matroid. Denote by $H^d(M)$ the $n \times dr$ matrix whose rows represent $\mathcal{H}^d(M)$.

To interpret $R^d(M)$ and $H^d(M)$, we study their (right) nullspaces. Both matrices have row vectors in $\mathbb{F}^r \otimes_{\mathbb{F}} \mathbb{F}^d$, so their nullvectors live in the same space. It will be convenient to freely use the identifications

$$\mathbb{F}^r \otimes_{\mathbb{F}} \mathbb{F}^d \;\cong\; (\mathbb{F}^r)^* \otimes_{\mathbb{F}} \mathbb{F}^d \;\cong\; \mathrm{Hom}_{\mathbb{F}}(\mathbb{F}^r, \mathbb{F}^d).$$

The second of these isomorphisms is canonical; the first comes from identifying $\mathbb{F}^r$ and $(\mathbb{F}^r)^*$ by the standard bilinear form $\langle x, y\rangle = \sum_{i=1}^r x_i y_i$ on $\mathbb{F}^r$, whose associated quadratic form is $Q(x) = \langle x, x\rangle = \sum_{i=1}^r x_i^2$. With these identifications, one has

$$\langle v \otimes x, \psi\rangle = \langle x, \psi(v)\rangle.$$

for every $\psi \in \mathrm{Hom}_{\mathbb{F}}(\mathbb{F}^r, \mathbb{F}^d)$, $v \in \mathbb{F}^r$, and $x \in \mathbb{F}^d$. Using this fact, one can prove the following:

Proposition 5.1. *Let M be a matroid represented by E as above, and let $\psi \in \mathbb{F}^r \otimes_{\mathbb{F}} \mathbb{F}^d \cong \mathrm{Hom}_{\mathbb{F}}(\mathbb{F}^r, \mathbb{F}^d)$.*

(i) *The vector ψ lies in $\ker H^d(M)$ if and only if $(\varphi + \psi)(v_i)$ is normal to η_i for every $i = 1, 2, \ldots, n$.*

(ii) *Provided that $\mathbb{F}$ does not have characteristic 2, the vector ψ lies in $\ker R^d(M)$ if and only if*

$$Q\big((\varphi + \epsilon\psi)(v_i)\big) \equiv Q\big(\varphi(v_i)\big) \mod \epsilon^2$$

for every $i = 1, 2, \ldots, n$.

Remark 5.2. Part (i) of Proposition 5.1 says that the nullspace of $H^d(M)$ is the space of directions in which one can perturb the map φ while keeping every image $\varphi(v_i)$ in the same hyperplane normal to η_i.

In contrast, part (ii) of Proposition 5.1 says that the nullspace of $R^d(M)$ is the space of infinitesimal changes that can be made to φ while keeping $Q(\varphi(v_i))$ constant (up to first order) for every i. (This is a rephrasing of a familiar fact from rigidity theory: the rigidity matrix $R^d(M)$ is just the Jacobian matrix (after scaling by $\frac{1}{2}$) of the map $\mathrm{Hom}_{\mathbb{F}}(\mathbb{F}^r, \mathbb{F}^d) \to \mathbb{F}^n$ sending φ to $Q(\varphi(v_i))_{i=1}^n$.)

Denote by $(d-1)M$ the matroid whose ground set consists of $d-1$ copies of each vector in E. The *d-parallel matrix* of M is defined as $H^d((d-1)M)$, and the matroid represented by its rows is the *d-dimensional parallel matroid* $\mathcal{P}^d(M) := \mathcal{H}^d((d-1)M)$. Part (ii) of Proposition 5.1 leads to an interpretation of the geometric meaning carried by the d-parallel matrix:

Corollary 5.3. *Let $\psi \in \mathbb{F}^r \otimes_{\mathbb{F}} \mathbb{F}^d \cong \mathrm{Hom}_{\mathbb{F}}(\mathbb{F}^r, \mathbb{F}^d)$. Then $\psi \in \ker P^d(M)$ if and only if $(\varphi+\psi)(v_i)$ is parallel to $\varphi(v_i)$ for all $i = 1, 2, \ldots, n$.*

Proof. Since there are $d-1$ copies of the vector v_i in $(d-1)M$, there will be $(d-1)$ accompanying normal vectors to $\varphi(v_i)$. Because these normals are chosen with generic coordinates, the only vectors normal to all $d-1$ of them are those parallel to $\varphi(v_i)$. Now apply Proposition 5.1. □

5.2. **The Nesting Theorem.** We now give one of our main results, the *Nesting Theorem*, which describes the relationship between the various independence systems associated to an arbitrary representable matroid M. In the special case that M is graphic and the ambient dimension d is 2, the Nesting Theorem gives what we have called the *planar trinity* (Corollary 5.5 below).

Theorem 5.4 (The Nesting Theorem)**.** *Let M be a matroid represented by vectors $E = \{v_1, \ldots, v_n\} \subset \mathbb{F}^r$, and let $d > 1$ be an integer. Then*

$$\mathcal{S}^{1,d}(M) \subseteq \mathcal{R}^d(M) \subseteq \mathcal{L}^d(M) = \mathcal{H}^d(M) \quad (= \mathcal{S}^{d-1,d}(M)).$$

Sketch of proof. To prove that $\mathcal{R}^d(M) \subseteq \mathcal{L}^d(M)$, it suffices to show that whenever $d \cdot r(M) \leq n$, there is an $\mathbb{F}(\varphi)$-linear dependence among the n rows of $R^d(M)$. The construction of $R^d(M)$ implies that these rows lie in a $\mathbb{F}(\varphi)$-vector space of dimension $d \cdot r(M)$. Thus if $d \cdot r(M) < n$, then the desired linear dependence is immediate, while if $d \cdot r(M) = n$, then the form of $R^d(M)$ allows us to exhibit an explicit nullvector. The proof that $\mathcal{H}^d(M) \subseteq \mathcal{L}^d(M)$ is analogous.

To prove that $\mathcal{S}^{1,d}(M) \subseteq \mathcal{R}^d(M)$, we assume that the rows of $R^d(M)$ are dependent and show that M is (k, d)-slope dependent for $k = 1$. Note that $\mathcal{S}^{k,d}(M) = \mathcal{L}^{\frac{d}{d-k}}(M) \subset \mathcal{L}^d(M)$. The equality is Corollary 3.3, and the inclusion follows from the definition of $\mathcal{L}^m(M)$ (because $\frac{d}{d-k} \leq d$). In particular, if M is d-Laman dependent then M is automatically (k, d)-slope dependent; we may therefore assume that M is d-Laman independent. Without loss of generality, $d \cdot r(M) \geq n$, so the dependence of the rows of $R^d(M)$ implies the vanishing of *every* one of its $n \times n$ minors. Moreover, by Theorem 2.5, M admits a d-Edmonds decomposition (see Definition 2.4).

Using the combinatorial properties of an Edmonds decomposition, we construct an $n \times n$ minor ξ of $R^d(M)$ that is a nonzero multihomogeneous polynomial in the coordinates of the vectors $\varphi(v_i)$. If ξ vanishes on the non-annihilating cellule $X^{\varnothing}_{k,d}(M)$ of the photo space, then the projection on $X^{\varnothing}_{k,d}(M) \to \mathbb{G}r(k, \mathbb{F}^d)$ is not Zariski dense, because the homogeneous coordinates of the $\varphi(v_i)$ are in fact the Plücker coordinates on $\mathbb{G}r(k, \mathbb{F}^d)$. This observation, together with Theorem 3.2, implies that $\mathcal{S}^{1,d}(M) \subseteq \mathcal{R}^d(M)$.

Replacing $R^d(M)$ with $H^d(M)$, $k = 1$ with $k = d-1$, and $\varphi(v_i)$ with η_i throughout, the same argument shows that $\mathcal{S}^{d-1,d} \subset \mathcal{H}^d(M)$. This completes the proof, since $\mathcal{S}^{d-1,d}(M) = \mathcal{L}^d(M)$ by Corollary 3.3. □

The case $d = 2$ is very special. Recall that $\mathcal{P}^d(M) = \mathcal{H}^d((d-1)M)$, so $\mathcal{P}^2(M) = \mathcal{H}^2(M)$. Indeed, setting $d = 2$ in the Nesting Theorem gives the following equalities:

Corollary 5.5. $\mathcal{S}^{1,2}(M) = \mathcal{R}^2(M) = \mathcal{L}^2(M) = \mathcal{H}^2(M) = \mathcal{P}^2(M)$.

When $d \geq 3$, the inclusion $\mathcal{R}^d(M) \subset \mathcal{L}^d(M)$ is typically strict. The nullspace of $R^d(M)$ contains the $\binom{d}{2}$-dimensional space of all vectors of the form $\sigma \circ \varphi$, as σ ranges over all skew-symmetric matrices in $\mathbb{F}^{d \times d}$. Consequently, every d-rigidity-independent subset $A \subset E$ must satisfy $|A| \leq d \cdot r(A) - \binom{d}{2}$. On the other hand, there may exist d-Laman independent sets A of cardinality up to $d \cdot r(A) - 1$.

6. Examples: Uniform matroids

Let E be a ground set with n elements. The *uniform matroid* of rank r on E is defined to be the matroid $U_{r,n}$ with independent sets $\{F \subset E : |F| \leq r\}$. $U_{r,n}$ may be regarded as the matroid represented by n generically chosen vectors in $\mathbb{F}^r$, where $\mathbb{F}$ is a sufficiently large field.

Predictably, the d-Laman and (k,d)-slope independence complexes on $U_{r,n}$ are also uniform matroids:

$$\mathcal{L}^d(U_{r,n}) = U_{s,n}, \quad \mathcal{S}^{k,d}(U_{r,n}) = U_{t,n} \qquad \text{where} \quad s = \min(\lceil dr-1\rceil, n),\ t = \min(\lceil \tfrac{dr}{d-k} - 1\rceil, n). \tag{15}$$

More striking is that d-Laman independence carries nontrivial geometric information about sets of n generic vectors in r-space: coplanarity for $U_{2,3}$ and the cross-ratio for $U_{2,4}$.

Example 6.1 ($U_{2,3}$)**.** Let $\mathbb{F}$ be any field, and let e_1, e_2 be the standard basis vectors in $\mathbb{F}^2$. The matroid $M = U_{2,3}$ is represented by the vectors $\{e_1, e_1+e_2, e_2\} \subset \mathbb{F}^2$; this representation is unique up to the action of the projective general linear group. By (15),

$$\mathcal{L}^d(U_{2,3}) = \begin{cases} U_{2,3} & \text{if } d \in (1, \frac{3}{2}]_{\mathbb{R}} \\ U_{3,3} & \text{if } d \in (\frac{3}{2}, \infty)_{\mathbb{R}} \end{cases} \qquad \text{and} \qquad \mathcal{S}^{1,d}(U_{2,3}) = \begin{cases} U_{3,3} & \text{if } d = 2 \\ U_{2,3} & \text{if } d \in \{3,4,\ldots\}. \end{cases}$$

We now consider what these equalities mean in terms of slopes. Let $\varphi : \mathbb{F}^2 \to \mathbb{F}^d$ be a linear transformation. If $d = 2$, then the images $\varphi(e_1), \varphi(e_1+e_2), \varphi(e_2)$ can have arbitrary slopes as φ varies. This is why $\mathcal{S}^{1,2}(U_{2,3}) = U_{3,3}$. On the other hand, when $d \geq 3$, those three vectors must be coplanar. This imposes a nontrivial constraint on the homogeneous coordinates for the lines spanned by the three images, and explains why $\mathcal{S}^{1,d}(U_{2,3}) = U_{2,3}$. By direct calculation, the inclusions $\mathcal{R}^d(M) \subseteq \mathcal{L}^d(M)$ given by Theorem 5.4 turn out to be equalities.

Example 6.2 ($U_{2,4}$)**.** Let $\mathbb{F}$ be a field of cardinality > 2, let $\mu \in \mathbb{F} \setminus \{0,1\}$, and let e_1, e_2 be the standard basis vectors in $\mathbb{F}^2$. The four vectors $\{e_1, e_1+e_2, e_2, e_1+\mu e_2\}$ represent $M = U_{2,4}$ over $\mathbb{F}$. Again, this representation is unique up to projective equivalence. By (15),

$$\mathcal{L}^d(U_{2,4}) = \begin{cases} U_{2,4} & \text{if } d \in (1, \frac{3}{2}]_{\mathbb{R}} \\ U_{3,4} & \text{if } d \in (\frac{3}{2}, 2]_{\mathbb{R}} \\ U_{4,4} & \text{if } d \in (2, \infty)_{\mathbb{R}} \end{cases} \qquad \text{and} \qquad \mathcal{S}^{1,d}(U_{2,4}) = \begin{cases} U_{3,4} & \text{if } d = 2 \\ U_{2,4} & \text{if } d \in \{3,4,\ldots\}. \end{cases}$$

Why is this correct from the point of view of slopes? From Example 6.1, we know that when $d \geq 3$, the lines spanned by the images of any three of the four vectors must be coplanar, so there is an algebraic dependence among the homogeneous coordinates for these three lines. For $d = 2$, this does not happen; the slopes of the images of any triple can be made arbitrary. However, applying a linear transformation to the representing vectors does not change their *cross-ratio* (in this case μ), so the fourth image vector is determined by the first three. This is the geometric interpretation of the combinatorial identity $\mathcal{S}^{1,2}(U_{2,4}) = U_{3,4}$.

Direct calculation shows that

$$\mathcal{R}^d(U_{2,4}) = \begin{cases} U_{2,4} & \text{if } d = 1, \\ U_{3,4} & \text{if } d \in \{2,3,\ldots\}. \end{cases}$$

This calculation is independent of the particular coordinates chosen for the representing vectors, even up to projective equivalence (that is, up to the choice of the parameter μ): that is, $\mathcal{R}^d(U_{2,4})$ is a *combinatorial* invariant. On the other hand, unlike the situation for $U_{2,3}$, the inclusions $\mathcal{R}^d(M) \subset \mathcal{L}^d(M)$ given by Theorem 5.4 are strict. (This behavior deviates from the case of graphic matroids; see below.)

7. More about $\mathcal{R}^d(M)$: invariance and stabilization

The examples in the previous section raise some natural questions. Clearly $\mathcal{L}^m(M)$ is a combinatorial invariant of M (that is, it does not depend on the choice of representation), so by Corollary 3.3 the same is true for $\mathcal{S}^{k,d}(M)$ (and in particular $\mathcal{H}^d(M)$ and $\mathcal{P}^d(M)$). But what about $\mathcal{R}^d(M)$? Note that this issue does not arise in classical rigidity theory, where the graphic matroid $M(G)$ is always represented by the vectors $\{e_i - e_j : \{i,j\} \in E(G)\}$, where e_i is the i^{th} standard basis vector in $\mathbb{R}^{|V(G)|}$.

Question 7.1. Is $\mathcal{R}^d(M)$ a combinatorial invariant of M, or does it depend on the choice of representation $\{v_1, \ldots, v_n\}$?

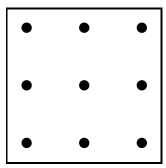
In the special case $d = 2$, the Nesting Theorem implies that $\mathcal{R}^d(M)$ is indeed a combinatorial invariant.

Call two sets of vectors $E = \{v_1, \ldots, v_n\}$, $E' = \{v'_1, \ldots, v'_n\} \subset \mathbb{F}^r$ *projectively equivalent* if there are nonzero scalars $c_1, \ldots, c_n \in \mathbb{F}^\times$ and an invertible linear transformation $g \in GL_r(\mathbb{F})$, such that $v'_i = g(c_i v_i)$

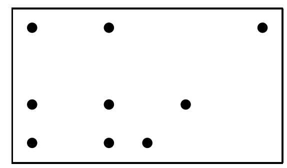
for every i. Then the matroids represented by E and E' are combinatorially identical. It is not hard to prove that their corresponding rigidity matroids are also identical. Unfortunately, this fact provides little insight into Question 7.1, because projective equivalence is a very strong condition.

On the other hand, we have not found a counterexample. We have seen that $\mathcal{R}^d(M)$ is indeed a combinatorial invariant for all d when $M = U_{2,3}$ or $U_{2,4}$. As another example, consider the following two sets of nine coplanar vectors in $\mathbb{R}^3$:

$$E = \{(1,0,0),(1,0,1),(1,0,2),(1,1,0),(1,1,1),(1,1,2),(1,2,0),(1,2,1),(1,2,2)\},$$
$$E' = \{(1,0,0),(1,0,1),(1,0,3),(1,2,0),(1,2,1),(1,2,3),(1,3,0),(1,4,1),(1,6,3)\}.$$



E



E'

Let M, M' be the matroids represented by E, E' respectively. These matroids are combinatorially isomorphic but projectively inequivalent. On the other hand, computations using `Mathematica` have shown that $\mathcal{R}^2(M) = \mathcal{R}^2(M')$ and $\mathcal{R}^3(M) = \mathcal{R}^3(M')$.

It is not hard to show that $\mathcal{R}^d(M) \subset \mathcal{R}^{d+1}(M)$ for all represented matroids M and integers d. Since there are only finitely many simplicial complexes on E, the tower $M = \mathcal{R}^1(M) \subseteq \mathcal{R}^2(M) \subseteq \mathcal{R}^3(M) \subseteq \cdots$ must eventually stabilize. Using standard facts about the transcendence degree of field extensions, we prove that this stabilization occurs no later than the rank $r = r(M)$: that is,

$$\mathcal{R}^d(M) = \mathcal{R}^r(M) \qquad \text{for all } d \geq r.$$

Moreover, if M is the graphic matroid for a graph $G = (V, E)$, equipped with the standard representation $\{e_i - e_j : ij \in E\}$ over an arbitrary field, then $\mathcal{R}^r(M)$ is the Boolean matroid on E.

This observation begs the question of whether $\mathcal{R}^d(M(G))$ depends on the field *before* d reaches the stable range. For an arbitrary representable matroid M, it is not true in general that $\mathcal{R}^\infty(M)$ is Boolean. We have already seen one example for which this fails, namely $U_{2,4}$. Another example is the well-known *Fano matroid* F, represented over the two-element field $\mathbb{F}_2$ by the seven nonzero elements of $\mathbb{F}_2^3$. It is not hard to show that $\mathcal{L}^d(F)$ is Boolean for $d > \frac{7}{3}$. On the other hand, computation with `Mathematica` indicates that $\mathcal{R}^2(F) = U_{5,7}$, but $\mathcal{R}^d(F) = U_{6,7}$ for all integers $d \geq 3$.

8. Open problems

The foregoing results raise many questions that we think are worthy of further study. Some of these have been mentioned earlier in the paper. In this final section, we restate the open problems and add a few more.

Problem 1. Determine the singular locus of the (k,d)-photo space $X_{k,d}(M)$ (perhaps by calculating the dimension of its various tangent spaces, as in Proposition 3.7).

Problem 2. Give a direct combinatorial explanation for the identity (14) (presumably by identifying some natural relationship between photos of M and of $M^\perp$).

Problem 3. Explain the "scaling phenomenon" of Remark 3.6 geometrically.

Problem 4. Determine whether or not the d-rigidity matroid $\mathcal{R}^d(M)$ is a combinatorial invariant of M (Question 7.1). If not, determine which matroids have this property, and to what extent $\mathcal{R}^d(M)$ depends on the field over which M is represented.

Problem 5. Let $M(G)$ be a graphic matroid equipped with the standard representation $\{e_i - e_j : \{i,j\} \in E(G)\}$. Is the rigidity matroid $\mathcal{R}^d(M)$ independent of the ambient field $\mathbb{F}$ for all d?

Problem 6. Generalize other aspects of classical (graph) rigidity theory to non-graphic matroids. One example is Crapo's "$(d+1)\mathbf{T}d$" characterization of the hyperplane-marking matroid of a graph [18, Theorem 8.2.2]. Another is Henneberg's construction of the bases of the 2-rigidity matroid [18, Theorem 2.2.3].

Our last open problem is similar in spirit to the results of [7] and [9], describing the algebraic and combinatorial structure of the equations defining the slope variety of a graph. It is motivated also by the appearance of the cross-ratio in Example 6.2.

Problem 7. Describe explicitly the defining equations (in Plücker coordinates on $\mathbb{G}r(k, \mathbb{F}^d)^n$) for $\overline{\pi X^{\varnothing}_{k,d}(M)}$, where π is the projection map of (11).

Acknowledgments

A substantial portion of this work was completed in July 2004 at the Park City Mathematical Institute, sponsored by the Institute for Advanced Study. The authors thank Neil White for helpful discussions while at PCMI, Walter Whiteley for sharing with them some of his manuscripts on classical rigidity theory, and Ezra Miller for discussions on algebraic geometry.

References

[1] M. Aigner, *Combinatorial Theory.* Springer-Verlag, Berlin, 1997.
[2] T. Brylawski and J. Oxley, *The Tutte polynomial and its applications*, Matroid applications, 123–225, Encyclopedia Math. Appl. **40**, Cambridge Univ. Press, Cambridge, 1992.
[3] M. Develin, J.L. Martin and V. Reiner, *Rigidity theory for matroids*, preprint, `arXiv:math.CO/0503050`.
[4] J. Edmonds, *Minimum partition of a matroid into independent subsets*, J. Res. Nat. Bur. Standards Sect. B **69B** (1965), 69–72.
[5] J. Graver, B. Servatius, and H. Servatius, *Combinatorial rigidity.* Graduate Studies in Mathematics, 2. American Mathematical Society, Providence, RI, 1993.
[6] G. Laman, *On graphs and rigidity of plane skeletal structures*, J. Engrg. Math. **4** (1970), 331–340.
[7] J.L. Martin, *Geometry of graph varieties*, Trans. Amer. Math. Soc. **355** (2003), 4151-4169.
[8] J.L. Martin, *On the topology of graph picture spaces*, Adv. Math. **191**, no. 2 (2005), 312–338.
[9] J.L. Martin, *The slopes determined by n points in the plane*, Duke Math. J., to appear; `arXiV:math.AG/0302106`.
[10] C.St.J.A. Nash-Williams, *Edge-disjoint spanning trees of finite graphs*, J. London Math. Soc. **36** (1961), 445–450.
[11] C.St.J.A. Nash-Williams, *Decompositions of finite graphs into forests*, J. London Math. Soc. **39** (1964), 12.
[12] J.G. Oxley, *Matroid theory.* Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1992.
[13] A. Recski, *A network theory approach to the rigidity of skeletal structures. II. Laman's theorem and topological formulae.* Discrete Appl. Math. **8** (1984), no. 1, 63–68.
[14] R.P. Stanley, *Enumerative combinatorics, Vol. I*, Cambridge Studies in Advanced Mathematics **49**, Cambridge University Press, Cambridge, 1997.
[15] T.-S. Tay, *A new proof of Laman's theorem*, Graphs Combin. **9** (1993), 365–370.
[16] W.T. Tutte, *On the problem of decomposing a graph into n connected factors.* J. London Math. Soc. **36** (1961), 221–230.
[17] D.J.A. Welsh, *Matroid theory.* London Math. Soc. Monographs **8**, Academic Press [Harcourt Brace Jovanovich, Publishers], London–New York, 1976.
[18] W. Whiteley, *Some matroids from discrete applied geometry*, Contemp. Math. **197** (1996), 171–311.
[19] W. Whiteley, *Rigidity and scene analysis*, Handbook of discrete and computational geometry, 893–916, CRC Press Ser. Discrete Math. Appl., CRC, Boca Raton, FL, 1997.

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