ZONAL POLYNOMIALS FOR WREATH PRODUCTS

HIROSHI MIZUKAWA

Abstract. The pair of groups, symmetric group S_{2n} and hyperoctohedral group H_n, is a Gelfand pair. The image of zonal spherical functions of this pair under the characteristic map are a family of symmetric functions called zonal polynomials. In the meaning of wreath products, a generalization of this Gelfand pair is considered in this abstract. Its zonal spherical functions are mapped to products of symmetric functions by characteristic map.

Résumé. La paire des groupes, du groupe symétrique S_{2n} et du groupe hyperoctohedral H_n, est une paire de Gelfand. L'image des fonctions sphériques zonales de cette paire sous characteristic map sont une famille des fonctions symétriques appelées les polynômes zonaux. Dans la signification de produits en couronne, une généralisation de cette paire de Gelfand est considérée dans cet abstrait. Ses fonctions sphériques zonales sont tracées aux produits des fonctions symétriques par characteristic map.

Key Words: zonal polynomials, Schur functions, Jack symmetric polynomials, Gelfand pairs of finite groups, zonal spherical functions

1. Introduction

The characteristic map can explain the relation of characters of symmetric groups and symmetric functions. We denote by $R(S_n)$ a complex vector space spanned by the irreducible characters of S_n. An element f of $R(S_n)$ can be identified an element $f = \sum_{x \in S_n} f(x)x$ of the group ring $\mathbb{C}S_n$. $R(S_n)$ has a scalar product defined by

$$\langle f, g \rangle = \frac{1}{|S_n|} \sum_{x \in S_n} f(x)\overline{g(x)}.$$

We put

$$R = \bigoplus_{n \geq 0} R(S_n)$$

and define a scalar product on R as

$$\langle f, g \rangle = \sum_{n \geq 0} n! \langle f_n, g_n \rangle \text{ for } f_n, g_n \in R(S_n).$$

R has a ring structure defined as follows. For $u \in R(S_n)$ and $v \in R(S_m)$, we define the multiplication of R by

$$fg = \mathrm{ind}_{S_n \times S_m}^{S_{n+m}} u \times v.$$

Classification number :33C45,05E35,05E05.

Let Λ be a ring of symmetric function. We define a $\mathbb{C}$-linear mapping

$$\mathrm{ch} : R \mapsto \Lambda$$

by

$$\mathrm{ch}(\sum_{x\in S_n} f(x)x) = \sum_{x\in S_n} f(x)p_{\sigma(x)},$$

where $f(x)\in\mathbb{C}$ and $\sigma(x)$ is a cycle type of x. This mapping is called the *characteristic map*. The characteristic map gives isometric isomorphism of R onto Λ. Let χ^λ_ρ be a irreducible character evaluated at a conjugacy class ρ. We obtain Schur functions as a image of a irreducible character of S_n:

$$\mathrm{ch}(\chi^\lambda) = S_\lambda,\ \lambda\vdash n.$$

In Macdonald's book(Chapter I, Appendix B) the theory above is extended to the character theory of the wreath products of any finite group with a symmetric group. We can also define the characteristic map as isometry isomorphism of the character ring of a wreath product onto the ring of symmetric functions. In this case we obtain c-times product of Schur functions as a image of irreducible characters. Here c is a number of irreducible characters of G.

We know a similar theory for the case of a Gelfand pair (S_{2n}, H_n)[6, VII7-2]. We consider zonal spherical functions of this pair. Although precise definition of zonal spherical functions appear in later (see Section 2). Here H_n is a subgroup of S_{2n} defined to be the centralizer of an element $(1,2)(3,4)\cdots(2n-1,2n)$. Littlewood's formula [5] says that

$$1^{S_{2n}}_{H_n} \sim \bigoplus_{\lambda\vdash n}\chi^{2\lambda}.$$

In fact, zonal spherical functions are unique H_n-invariant element of each irreducible component of $1^{S_{2n}}_{H_n}$ and constant on each double coset. It is known that double cosets of this pair are classified by the partition of n [6, VII-2(2.1)]. Let ω^λ_ρ be a zonal spherical function in V_i evaluated on a double coset indexed by ρ. We define *zonal polynomials* (cf. [2, 12, 13]) by

$$Z_\lambda = |H_n|\sum_{\rho\vdash n} z^{-1}_{2\rho}\omega^\lambda_\rho p_\rho,\ \lambda\vdash n.$$

Zonal polynomials are a special family of Jack symmetric function $J^\alpha_\lambda(x)$ [6, 11] with parameter $\alpha = 2$. In terms of Jack symmetric functions, zonal polynomials are formulated as follows: We consider a inner product on $\Lambda\otimes\mathbb{Q}(\alpha)$ defined by

$$\langle p_\rho, p_\sigma\rangle_\alpha = \delta_{\rho,\sigma} z_\rho \alpha^{\ell(\lambda)}.$$

Zonal polynomials are the unique homogenous basis of $\Lambda\otimes\mathbb{Q}(2)$ satisfying:

(1) $\langle Z_\lambda, Z_\mu\rangle_2 = h(2\lambda)\delta_{\lambda,\mu}$, where $h(\lambda)$ is the hook length product of λ
(2) We write $Z_\lambda = \sum_\mu v_{\lambda,\mu} m_\mu$, where m_μ is a monomial symmetric function. Then $v_{\lambda,\mu} = 0$ unless μ is less than λ as the dominance order(cf. [6] pp.7 Chapter 1-1).
(3) If $\lambda\vdash v$, then $v_{\lambda,1^n} = n!$

The characteristic map make us understood equivalency of two definitions above. We define a graded ring of Hecke algebra

$$\mathcal{H} = \bigoplus_{n\geq 0} e_{H_n}\mathbb{C}S_n e_{H_n},$$

where $e_{H_n} = \frac{1}{|H_n|}\sum_{h\in H_n} h$. The multiplication of $\mathcal{H}$ is defined by

$$uv = e_{n+m}(u\times v)e_{n+m},\ u\in\mathcal{H}_n, v\in\mathcal{H}_m \text{ and } u\times v\in\mathcal{H}_n\times\mathcal{H}_m.$$

We can define the isometry isomorphism ch of $\mathcal{H}$ onto Λ and obtain

$$|H_n|\mathrm{ch}(\omega^\lambda) = Z_\lambda.$$

In this abstract our purpose is to generalize third case in the meaning of wreath products. Then we expect to obtain products of zonal polynomials as images of zonal spherical functions under proper isomorphism like the second case. We will consider $G\wr S_{2n}$ instead of S_{2n}. But what kind of subgroup should be chosen, we argue for this problem in Section 3. In Section 4, we classify double coset of the pair defined at Section 3 by using G-colored graphs. We recall the representation theory of wreath products in Section 5. Section 6 is devoted to the irreducible decomposition of the permutation representations. In Section 7-9 we see that our expectation is true. In fact, we obtain products of zonal polynomials and Schur functions as images of our zonal spherical function under a *'characteristic map'* (cf. Theorem 9.3).

2. Gelfand Pair of Finite Groups and Its Zonal Spherical Functions

We recall the theory of Gelfand pair of finite groups. Through this section we denote G by a finite group and H by its subgroup. Put

$$e_H = \frac{1}{|H|}\sum_{h\in H} h.$$

Let $\mathbb{C}G$ be the group ring of G and $e_H\mathbb{C}Ge_H$ a Hecke algebra. We regard

$$f = \sum_{x\in G} f(x)x \in \mathbb{C}G,\ f(x)\in\mathbb{C}$$

as a function $x\mapsto f(x)$ on G. Under this identity, the multiplication on $\mathbb{C}G$ is

$$(f*g)(x) = \sum_{yz=x} f(y)g(z).$$

We assume the induced representation $\mathbb{C}Ge$ is multiplicity free, i.e. (G,H) is a Gelfand pair. Under our assumption $\mathbb{C}Ge$ is direct sum of non-isomorphic irreducible G-module;

$$\mathbb{C}Ge = \bigoplus_{i=1}^{s} V_i,$$

where V_i's are irreducible representations. Let χ_i be a character of V_i. We define

$$e_i = \frac{\dim V_i}{|G|}\sum_{x\in G}\overline{\chi_i(x)}$$

to be a primitive idempotent affording to V_i-isotypic component of $\mathbb{C}Ge$. Then we have next proposition [1].

Proposition 2.1. *In the notation introduced above*

$$V_i = \mathbb{C}Ge_ie_H.$$

The Scalar product on $\mathbb{C}G$ is

$$\langle f, g\rangle_G = \frac{1}{|G|}\sum_{x\in G} f(x)\overline{g(x)}.$$

We can easy to see that this scalar product is G-invariant Hermitian scalar product. The Frobenius reciprocity gives us that $\frac{\dim V_i}{|G|}e_ie_H$ is a unique H-invariant element of V_i which equals to 1 at unit element of G. We call the functions,

$$\omega_i(x) = \frac{\dim V_i}{|G|}e_ie_H = \langle e_ie_H, xe_ie_H\rangle_G/\langle e_ie_H, e_ie_H\rangle_G,\ (1 \leq i \leq s,\ x \in G),$$

zonal spherical functions of a Gelfand pair (G, H). Zonal spherical functions are constant on each double coset $H\backslash G/H$ and have a orthogonality relation

$$\langle \omega_i, \omega_j\rangle_G = \delta_{ij}\frac{1}{\dim V_i}.$$

Proposition 2.2. [6, VIIpp. 389(1.3)] *Let F be a non-zero H-invariant element of $W \cong V_i$ and $\langle,\rangle$ be a G-invariant Hermitian scalar product on W. Then the zonal spherical function is written as*

$$\omega_i(x) = \langle F, xF\rangle/\langle F, F\rangle.$$

Some zonal spherical functions of Gelfand pair of wreath products are calculated in [7, 8, 9].

3. A pair (SG_{2n}, HG_n)

Through this section, S_{2n} is a permutation group on $[2n] = \{1, 2, \cdots, 2n\}$ and its subgroup H_n is the centralizer of an element $(1, 2)(3, 4)\cdots(2n-1, 2n) \in S_{2n}$. We remark that H_n can be considered the permutation groups on $\{\{2i-1, 2i\}; 1 \leq i \leq n\}$ and

$$H_n \cong W(B_n),$$

where $W(B_n)$ is the Weyl group of type B. Let G be a finite group. We denote by G_* the set of conjugacy class of G. We consider a wreath product

$$SG_{2n} = G \wr S_{2n}.$$

Let ΔG be a diagonal subgroup of $G \times G$ defined by

$$\Delta G = \{(g, g) \mid g \in G\}.$$

We restrict the action of S_{2n} on G^{2n} to H_n and define a subgroup of SG_{2n} by

$$HG_n = (\Delta G)^n \rtimes_{\tilde{\theta}} H_n.$$

From the definition of H_n it is clear that HG_n is well defined.

4. Description of Double Cosets

Through a combinatorial argument, we can describe a complete representatives of each double coset of (SG_{2n}, HG_n) [10]. In this section, without proofs, we introduce a method of identification of each double coset.

For an element $x = (g_1, g_2, \cdots, g_{2n}; \sigma)$ of SG_{2n}, the *G-colored graph* $\Gamma_G(x) = \{V_G(x), E_G(x)\}$ is a graph with vertices

$$V_G(x) = \{g_1, g_2, \cdots, g_{2n}\}$$

and edges

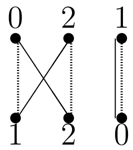
$$E_G(x) = \left\{\{g_{2i-1}, g_{2i}\}, \left\{g_{\sigma(2j-1)}, g_{\sigma(2j)}\right\}; 1 \leq i, j \leq n\right\}.$$

Here we call the edge $\{g_{2i-1}, g_{2i}\}$ *"broken"* and $\left\{g_{\sigma(2i-1)}, g_{\sigma(2i)}\right\}$ *"staright"*.

Example 4.1. $G = \mathbb{Z}/3\mathbb{Z} = \{0, 1, 2\}$. We consider SG_6 and take an element

$$x = (0, 1, 2, 2, 1, 0; (123)(56)).$$

Then the graph of x is

$\Gamma_G(x) =$  .

This graph gives a two-sided HG_n-invariant: Fix an element $x = (g_1, g_2, \cdots, g_{2n}; \sigma)$ of SG_{2n}. Let L be a cycle of $\Gamma_G(x)$. We assume that L has vertices $\left\{g_{i_j}; 1 \leq j \leq 2k\right\}$. Let

$$\left\{\left\{g_{i_{2j-1}}, g_{i_{2j}}\right\}; (1 \leq j \leq k)\right\}$$

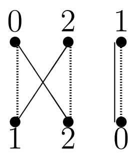
be broken edges of L and

$$\left\{\left\{g_{i_{2j}}, g_{i_{2j+1}}\right\}, \left\{g_{i_{2k}}, g_{i_1}\right\}; (1 \leq j \leq k-1)\right\}$$

be staright edges of L. We define a *circuit product* of L by

$$p(L) = \prod_{j=1}^{k} g_{i_{2j-1}}^{-1} g_{i_{2j}}.$$

Example 4.2. In the case of example 4.1, from a graph

$\Gamma_G(x) =$  ,

we compute circuit products

$$-0 + 1 - 2 + 2 = 1, -1 + 0 = 2.$$

If L has $2k$ edges then we call $p(L)$ a *circuit product of length k*.

Definition 4.3. Put

$$G_{**} = \left\{ R = C \cup C^{-1}; C \in G_* \right\},$$

where $C^{-1} = \{g^{-1}; g \in C\}$. We call a conjugacy class *real*(resp. *complex*) when $C = C^{-1}$(resp.$C \neq C^{-1}$). Put

$$m_k(R) = \sharp\{L; L \text{ is a 2k-cycle of } \Gamma(x) \text{ and } p(L) \in G_{**}. \}$$

We define a tuple of partitions

$$\underline{\rho}(x) = (\rho(R); R \in G_{**}),$$

where $\rho(R) = (1^{m_1(R)}, 2^{m_2(R)}, \cdots, n^{m_n(R)})$. This tuple of partitions $\underline{\rho}(x)$ is called *circuit type of* x.

Example 4.4. If $G = \mathbb{Z}/3\mathbb{Z}$ we have $G_{**} = \{\{0\}, \{1, 2\}\}$. In the case of example 4.1, we obtain

$$\underline{\rho}(x) = ((\emptyset), (2, 1)).$$

Definition 4.3 gives us next theorem.

Theorem 4.5. (1) $x \in HG_n y HG_n \Leftrightarrow \underline{\rho}(x) = \underline{\rho}(y)$.

(2) $\underline{\rho}(x) = \underline{\rho}(x^{-1})$.

Furthermore we can see the cardinality of each double coset.

Proposition 4.6. *Let x be an element such that whose circuit type is $\underline{\rho}(x) = (\rho(R); R \in G_{**})$ where $\rho = (1^{m_1(R)}, 2^{m_2(R)}, \cdots, n^{m_n(R)})$ and $\zeta_C = \frac{|G|}{|C|}$ for $C \in G_*$. Then we have*

$$\begin{aligned} {\mathcal{Z}_{\underline{\rho}(x)}}^{-1} = |HG_n x HG_n| &= \frac{|H_n|^2 |G|^{2n}}{\prod_{R \in G_{**}} z_{2\rho(R)}} \times \frac{\prod_{R \in G_{**}} |R|^{\ell(\rho(R))}}{|G|^{\ell(\rho)}} \\ &= |H_n|^2 |G|^{2n} \prod_{\substack{R = C \in G_{**} \\ C = C^{-1}}} \frac{1}{z_{2\rho(R)} \zeta_C^{\ell(\rho(R))}} \times \prod_{\substack{R = C \cup C^{-1} \in G_{**} \\ C \neq C^{-1}}} \frac{1}{z_{\rho(R)} \zeta_C^{\ell(\rho(R))}}. \end{aligned}$$

This result is important to determine the weight of inner product on the ring of symmetric functions see Section 7 .

5. Representation Theory of Wreath Products

In this section we recall the representation theory of wreath products (cf. [4]). Let G be a finite group. We write

$$SG_n = G \wr S_n.$$

Let G^* be a set of irreducible characters of G and c its cardinality. We introduce a construction method of the irreducible representations.

Let

$$\mathcal{C}_n = \left\{ \underline{n} = (n_\chi; \chi \in G^*); \sum_{\chi \in G^*} n_\chi = n,\ n_i \geq 0 \right\}$$

be a set of c-composition of n. We take an element $\underline{n} \in \mathcal{C}_n$ and define a set of c-tuple of partitions;

$$\mathcal{P}(\underline{n}) = \{(\lambda^\chi | \chi \in G^*); \lambda^\chi \vdash n_\chi\}$$

We define a subgroup of SG_n by

$$SG(\underline{n}) = \prod_{\chi \in G^*} SG_{n_\chi}.$$

Taking $\underline{n} \in \mathcal{C}_n$ and $\underline{\lambda} = (\lambda(\chi) | \chi \in G^*) \in \mathcal{P}(\underline{n})$, we define two representations $\mathcal{R}(\underline{n})$ and $\mathcal{S}(\underline{\lambda})$ of $SG(\underline{n})$ as follows:

$$\mathcal{R}(\underline{n}) \cong \bigotimes_{\chi \in G^*} {V_\chi}^{\otimes n_\chi},$$

$$\mathcal{S}(\underline{\lambda}) \cong \bigotimes_{\chi \in G^*} S^{\lambda(\chi)},$$

where S^λ is the Specht module indexed by a partition λ. The action of $SG(\underline{n})$ is defined by

$$(g_1, \cdots, g_n; \sigma) v_1 \otimes \cdots \otimes v_n = g_1 v_{\sigma^{-1}(1)} \otimes \cdots \otimes g_n v_{\sigma^{-1}(n)} \text{ on } \mathcal{R}(\underline{n}),$$

and

$$(g_1, \cdots, g_n; \sigma) v = \sigma v \text{ on } \mathcal{S}(\underline{\lambda}).$$

We consider an irreducible representation of $S(\underline{n})$

$$S(\underline{\lambda}) = \mathcal{R}(\underline{n}) \otimes \mathcal{S}(\underline{\lambda})$$

We write

$$\mathfrak{S}(\underline{\lambda}) = S(\underline{\lambda}) \uparrow_{SG(\underline{n})}^{SG_n}.$$

Theorem 5.1. [4] *The complete system of irreducible representations of SG_n are given by*

$$\{\mathfrak{S}(\underline{\lambda}); \underline{n} \in \mathcal{C}_n,\ \underline{\lambda} \in \mathcal{P}(\underline{n})\}.$$

As can be seen from the theorem above, there is a one-to-one correspondence between SG_n^* and the set of c-tuple of partitions of n.

6. Gelfand pair (SG_{2n}, HG_n)

From the second claim of Theorem 4.5, we see that $x \in SG_{2n}$ and x^{-1} are in same double coset. Therefore we have the following proposition [6, VII(1.2)].

Proposition 6.1. *(SG_{2n}, HG_n) is a Gelfand pair.*

We consider irreducible decomposition of the permutation representation

$$\mathrm{ind}_{HG_n}^{SG_{2n}} 1 = 1_{HG_n}^{SG_{2n}}.$$

Let G^* be a set of irreducible characters of G. We call a character $\chi \in G^*$ *real* (resp. *complex*) if $\chi = \overline{\chi}$(resp. $\chi \neq \overline{\chi}$). Let G_R^* be a set of real characters and G_C^* a set of complex characters. We define a relation in G_C^* as

$$\chi \sim \chi' \Leftrightarrow \overline{\chi} = \chi'$$

and

$$G^{**} = G_R^* \cup G_C^*/\sim.$$

Taking proper representatives, we consider $G^{**}/\sim$ to be a subset of G^*. Next propositions are elementary in this section.

Proposition 6.2. (1) *(S_{2n}, H_n) is a Gelfand pair.*

(2) *$(G \times G, \Delta G)$ is a Gelfand pair.*

(3) *Especially, $(S_n \times S_n, \Delta S_n)$ is a Gelfand pair.*

Proposition 6.3. (1) $1_{H_n}^{S_{2n}} = \bigoplus_{\lambda \vdash n} S^{2\lambda}$.

(2) $1_{\Delta G}^{G \times G} = \bigoplus_{\chi \in G_R^*} \chi \otimes \chi \oplus \bigoplus_{\chi \in G_C^*} \chi \otimes \overline{\chi}$.

(3) *Especially,* $1_{\Delta S_n}^{S_n \times S_n} = \bigoplus_{\lambda \vdash n} S^\lambda \otimes S^\lambda$.

We use the notation appeared in Section 5. We put

$$\mathcal{C}_{2n} \supset \mathcal{C}_{2n}^{**} = \left\{(n_\chi; n_\chi \equiv 0 \pmod 2\ \chi \in G_R^*, n_\chi = n_{\overline{\chi}}\ \chi \in G_C^*)\right\}.$$

Example 6.4. The case of $G = \mathbb{Z}/2\mathbb{Z}$:

$$\mathcal{C}_{2n}^{**} = \left\{(2n - 2k, 2k); 0 \leq k \leq n\right\}.$$

The case of $G = \mathbb{Z}/3\mathbb{Z}$:

$$\mathcal{C}_{2n}^{**} = \left\{(2n - 2k, k, k); 0 \leq k \leq n\right\}.$$

For $\underline{n} \in \mathcal{C}_{2n}^{**}$ we define

$$\mathcal{P}(\underline{n}) \supset \mathcal{P}^{**}(\underline{n}) = \left\{(\lambda^\chi; \lambda^\chi = 2^\exists \mu^\chi,\ \chi \in G_R^* \text{ and } \ \lambda^\chi = \lambda^{\overline{\chi}},\ \chi \in G_C^*)\right\},$$

where 2λ means $(2\lambda_1, 2\lambda_2, \cdots)$ for $\lambda = (\lambda_1, \lambda_2, \cdots)$.

Example 6.5. The case of $G = \mathbb{Z}/2\mathbb{Z}$:

$$\mathcal{P}^{**}(\underline{n}) = \left\{(2\lambda, 2\mu); |\lambda| + |\mu| = n\right\}.$$

The case of $G = \mathbb{Z}/3\mathbb{Z}$:

$$\mathcal{P}^{**}(\underline{n}) = \left\{(2\lambda, \mu, \mu); |\lambda| + |\mu| = n\right\}.$$

We consider a representation $\chi(\underline{n}) \otimes S(\underline{\lambda})$ for $\underline{n} \in \mathcal{C}_{2n}^{**}$ and $\underline{\lambda} \in \mathcal{P}^{**}(\underline{n})$. Propositions 6.2 and 6.3 give us the following fact.

Proposition 6.6. *$\chi(\underline{n}) \otimes S(\underline{\lambda})$ has $\prod_{\chi \in G_R^*} HG_{n_\chi} \times \prod_{\chi \in G_C^*/\sim} \Delta SG_{n_\chi}$-invariant element. Here we think the following embedding*

$$HG_{n_\chi} \subset SG_{2n_\chi},\ \Delta SG_{n_\chi} \subset SG_{n_\chi} \times SC_{n_{\overline{\chi}}}$$

and

$$\prod_{\chi \in G_R^*} HG_{n_\chi} \times \prod_{\chi \in G_C^*/\sim} \Delta SG_{n_\chi} \subset SG(\underline{\lambda}).$$

A construction method of the irreducible representation of wreath product, see Section 5, gives us the reverse of Proposition 6.6. Next proposition is a corollary of a lemma due to Brauer(cf. [3, Chapter6 (6.32)]).

Proposition 6.7.

$$|G^{**}| = |G_{**}|.$$

Therefore we have the following theorem from Proposition 6.6 and 6.7

Theorem 6.8.

$$1_{HG_n}^{SG_{2n}} = \bigoplus_{\underline{n}\in\mathcal{C}_{2n}^{**}} \bigoplus_{\underline{\lambda}\in\mathcal{P}^{**}(\underline{n})} \mathfrak{S}(\underline{\lambda})$$

The end of this section we see an example.

Example 6.9. The case of $G = \mathbb{Z}/2\mathbb{Z}$:

$$1_{HG_n}^{SG_{2n}} = \bigoplus_{|\lambda|+|\mu|=n} \mathfrak{S}(2\lambda, 2\mu).$$

The case of $G = \mathbb{Z}/3\mathbb{Z}$:

$$1_{HG_n}^{SG_{2n}} = \bigoplus_{|\lambda|+|\mu|=n} \mathfrak{S}(2\lambda, \mu, \mu).$$

7. The Ring $\tilde{\Lambda}(G)$

In this section we define a suitable ring of symmetric function for considering our zonal spherical functions. Let $p_r(R)(r \geq 1)$ be the power sum symmetric function with variables $x(R) = (x(R)_1, x(R)_2, \cdots)$ for $R \in G_{**}$ and $\tilde{\Lambda}(G)$ a ring generated by $p_r(R)(r \geq 1, R \in G_{**})$. Let $\underline{\rho} = (\rho(R); R \in G_{**})$ be a $|G_{**}|$-tuple of partitions. Put

$$P_{\underline{\rho}}(G_{**}) = \prod_{R\in G_{**}} p_{\rho(R)}(R)$$

for $\underline{\rho}$. We change variables $p_r(R)$'s to

$$p_r(\chi) = \sum_{\substack{R=C\cup C^{-1}\in G_{**}\\ C=C^{-1}}} \frac{\chi(C)}{\zeta_C} p_r(R) + \sum_{\substack{R=C\cup C^{-1}\in G_{**}\\ C\neq C^{-1}}} \frac{\chi(C)+\overline{\chi(C)}}{\zeta_C} p_r(R),$$

where $\chi \in G^{**}$ and $\chi(C)$ is a value of χ at conjugacy class C. We also put

$$P_{\underline{\lambda}}(G^{**}) = \prod_{\lambda^\chi\in G^{**}} p_{\lambda^\chi}(\chi)$$

for a tuple of partition $\underline{\lambda} = (\lambda^\chi; \chi \in G^{**})$. We define an inner product on $\tilde{\Lambda}(G)$ by

$$\langle P_{\underline{\rho}}(G_{**}), P_{\underline{\sigma}}(G_{**})\rangle_{\tilde{\Lambda}(G)} = \delta_{\underline{\rho}\underline{\sigma}} \mathcal{Z}_{\underline{\rho}}.$$

$\mathcal{Z}_{\underline{\rho}}$ is given in Proposition 4.6. Here we write a polynomial with variables $p_r(\chi)$ like as $S_\lambda(\chi)$.

8. The Ring $\mathcal{H}(G)$

We define a Hecke algebra by

$$\mathcal{H}(SG_{2n}, HG_n) = e_{HG_n}\mathbb{C}SG_{2n}e_{HG_n}.$$

It is true that zonal spherical functions of (SG_{2n}, HG_n) are orthonormal basis of $\mathcal{H}(SG_{2n}, HG_n)$. We define a graded vector space

$$\mathcal{H}(G) = \bigoplus_{n\geq 0}\mathcal{H}(SG_{2n}, HG_n).$$

The multiplication of $\mathcal{H}(G)$ is defined by

$$uv = e_{HG_{n+m}}(u\times v)e_{HG_{n+m}},$$

where we think that $u \times v$ is a function on a diagonal subalgebra $\mathbb{C}S_{2n} \times \mathbb{C}S_{2m}$ of $\mathbb{C}S_{2n+2m}$. Since $\mathcal{H}(G)$ has a structure of a graded algebra. We see that zonal spherical functions are basis of $\mathcal{H}(G)$.

9. Main result

By using group theoretical method as in the book [1] we can obtain zonal spherical functions as a product of some primitive idempotents. To describe details of this fact is a little complicated. So we omit to explain how to get our zonal spherical functions here. We only show the final final form of them.

Our zonal spherical functions can be described as follows. Let $S^\lambda(\chi)$ be an irreducible representation of SG_n isomorphic to $\chi(\underline{n})\otimes S^\lambda$ for $\underline{n} = (n_\eta; \eta\in G^*$ and $n_\eta = \delta_{\chi\eta}n)$. We define $e_\lambda(\chi), (\ \lambda\vdash n$ and $\chi\in G^*)$ to be a primitive idempotent of $\mathbb{C}SG_n$ which afford to $S^\lambda(\chi)$-isotypic component in $\mathbb{C}SG_n$. We avoid to write concrete equations in the below so we use the notation "$\propto$". Then we have

Proposition 9.1. *We put* $\underline{n}\in\mathcal{C}^{**}_{2n}$ *and* $\underline{\lambda} = (\lambda^\chi; \lambda^\chi = 2^\exists\mu^\chi,\ \chi\in G^*_R$ and $\lambda^\chi = \lambda^{\overline{\chi}}) \in \mathcal{P}^{**}(\underline{n})$. *Then we have zonal spherical functions in a irreducible component* $\chi(\underline{n})\otimes S(\lambda)\uparrow^{SG_{2n}}_{HG_n}$ *of* $1^{SG_{2n}}_{HG_n}$ *as*

$$\omega^{\underline{\lambda}} \propto e_{HG_n}\left(\prod_{\chi\in G^*_R} e_{2\mu^\chi}(\chi)\times\prod_{\chi\in G^*_C/\sim} e_{\lambda^\chi}(\chi)\times e_{\lambda^\chi}(\overline{\chi})\right)e_{HG_n}.$$

We define a characteristic map

$$ch_{\mathcal{H}} : \mathcal{H}(G)\mapsto\tilde{\Lambda}(G)$$

by

$$ch_{\mathcal{H}}(x) = P_{\underline{\rho}(x)}(G^{**}),\ (x\in SG_{2n}).$$

This characteristic map gives an isometric isomorphism of $\mathcal{H}(G)$ onto $\tilde{\Lambda}(G)$. We have the following proposition.

Proposition 9.2. *Let* λ *be a partition of* n.

$$ch_{\mathcal{H}}(e_{2\lambda}(\chi)e_{HG_n}) \propto Z_\lambda(\chi),\ \chi\in G^*_R,$$
$$ch_{\mathcal{H}}(e_\lambda(\chi)\times e_\lambda(\overline{\chi})e_{\Delta SG_n}) \propto h(\lambda)S_\lambda(\chi),\ \chi\in G^*_C/\sim .$$

Bh combining two propositions above, we obtain the main theorem of this abstract.

Theorem 9.3. *We put* $\underline{n} \in \mathcal{C}_{2n}^{**}$ *and* $\underline{\lambda} = (\lambda^\chi; \lambda^\chi = 2\mu^\chi,\ \chi \in G_R^*$ and $\lambda^\chi = \lambda^{\overline{\chi}}) \in \mathcal{P}^{**}(\underline{n})$. *Let* $\omega^{\underline{\lambda}}$ *be a zonal spherical function in a irreducible component* $\mathfrak{S}(\lambda)$ *of* $1_{HG_n}^{SG_{2n}}$. *Then we have*

$$\begin{aligned} ch_{\mathcal{H}}(\omega^{\underline{\lambda}}) &\propto \prod_{\chi \in G_R^*} Z_{\mu^\chi}(\chi) \times \prod_{\chi \in G_C^*/\sim} h(\lambda^\chi) S_{\lambda^\chi}(\chi) \\ &= \prod_{\chi \in G_R^*} J_{\mu^\chi}^{(2)}(\chi) \times \prod_{\chi \in G_C^*/\sim} J_{\lambda^\chi}^{(1)}(\chi). \end{aligned}$$

Example 9.4. In the case of $G = \mathbb{Z}/2\mathbb{Z} = \{-1, 1\}$: Put $\chi_i(j) = j^i (i = 0, 1, j \in G)$ We obtain

$$\{Z_\mu(\chi_0) Z_\lambda(\chi_1)\}$$

as images of zonal spherical functions.
In the case of $G = \mathbb{Z}/3\mathbb{Z} = \{1, \xi, \xi^2\}$: Put $\chi_i(j) = j^i (i = 0, 1, 2, j \in G)$ We obtain

$$\{Z_\mu(\chi_0) S_\lambda(\chi_1)\}$$

as images of zonal spherical functions.

References

[1] C. W. Curtis and I. Reiner, *Methods of Representations Theory, Vol. 1* A Wiley-Interscience, 1981.

[2] N. Bergeron and A. M. Garsia, Zonal polynomials and domino tableaux, Discrete Mathematics 99, 1992 pp. 3-15.

[3] I. M. Issacs, *Character Theory of Finite Groups,* Dover Publications Inc. 1994.

[4] G. James and A. Kerber, *The Representation Theory of the Symmetric Group,* Encyclopedia of Mathematics and its Applications, 16, 1981.

[5] D. E. Littlewood, *The Theory of Group Characters, 2nd. ed. ,* Oxford, 1950.

[6] I. G. Macdonald, *Symmetric Functions and Hall Polynomials, 2nd. ed. ,* Oxford, Clarendon Press, 1995.

[7] H. Mizukawa, Zonal spherical functions on the complex reflection groups and $(m+1, n+1)$-hypergeometric functions Advances in Math. 184, 2004 pp. 1-17.

[8] Hiroki Akazawa and Hiroshi Mizukawa, Orthogonal polynomials arising from the wreath products of dihedral group J. Combin. Theory Ser. A 104 2003 pp. 371-380.

[9] Hiroshi Mizukawa and Hajime Tanaka , $(n+1, m+1)$-hypergeometric functions associated to character algebras, Proc. AMS. 132, 2004, no. 9, pp. 2613-2618.

[10] H. Mizukawa, Zonal polynomials for wreath products, preprint 2004.

[11] R. Stanley, Some combinatorial properties of Jack symmetric functions, Advances in Math. vol. 77,1989, pp.76-115

[12] J. R. Stembridge, On Schur's Q-functions and the primitive idempotents of a commutative Hecke algebra", J. Alg. Comb. 1 1992, pp.71-95.

[13] A. Takemura *Zonal Polynomials,* Inst. of Math. Statistics Lecture Notes,-Monograph Series vol.4 .Hayward, California,1984

Department of Mathematics, National Defense Academy in Japan, Yokosuka 239-8686, Japan.
E-mail address: mzh@nda.ac.jp