Raising and Lowering Maps and Modules for the Quantum Affine Algebra $U_q(\widehat{\mathfrak{sl}}_2)$

Darren Neubauer

Abstract

Let $\mathbb{K}$ denote an algebraically closed field and let q denote a nonzero scalar in $\mathbb{K}$ that is not a root of unity. Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. Let $V_0, V_1, \ldots, V_d$ denote a sequence of nonzero subspaces whose direct sum is V. Suppose $R : V \to V$ and $L : V \to V$ are linear transformations such that

(i) $RV_i \subseteq V_{i+1} \qquad (0 \leq i \leq d-1), \qquad RV_d = 0,$

(ii) $LV_i \subseteq V_{i-1} \qquad (1 \leq i \leq d), \qquad LV_0 = 0,$

(iii) for $0 \leq i \leq d/2$ the restriction $R^{d-2i}|_{V_i} : V_i \to V_{d-i}$ is a bijection,

(iv) for $0 \leq i \leq d/2$ the restriction $L^{d-2i}|_{V_{d-i}} : V_{d-i} \to V_i$ is a bijection,

(v) $R^3L - [3]R^2LR + [3]RLR^2 - LR^3 = 0,$

(vi) $L^3R - [3]L^2RL + [3]LRL^2 - RL^3 = 0,$

where $[3] = (q^3 - q^{-3})/(q - q^{-1})$. Let $K : V \to V$ be the linear transformation such that, for $0 \leq i \leq d$, V_i is an eigenspace for K with eigenvalue q^{2i-d}. We show that there exists a unique $U_q(\widehat{\mathfrak{sl}}_2)$-module structure on V such that each of $R - e_1^-$, $L - e_0^-$, $K - K_0$, and $K^{-1} - K_1$ vanish on V, where e_1^-, e_0^-, K_0, K_1 are Chevalley generators for $U_q(\widehat{\mathfrak{sl}}_2)$. We determine which $U_q(\widehat{\mathfrak{sl}}_2)$-modules arise from our construction.

1 The quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$

Throughout this paper $\mathbb{K}$ will denote an algebraically closed field. We fix a nonzero scalar $q \in \mathbb{K}$ that is not a root of unity. We will use the following notation.

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \qquad n = 0, 1, \ldots$$

We now recall the definition of $U_q(\widehat{\mathfrak{sl}}_2)$.

Keywords: Quantum group, quantum affine algebra, affine Lie algebra $\widehat{\mathfrak{sl}}_2$, raising and lowering maps, tridiagonal pair.

2000 Mathematics Subject Classification. Primary: 17B37. Secondary: 16W35, 20G42, 81R50

Definition 1.1 [2, p. 262] The quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$ is the unital associative $\mathbb{K}$-alegbra with generators $e_i^{\pm}$, $K_i^{\pm 1}$, $i \in \{0, 1\}$ which satisfy the following relations:

$$\begin{aligned}
K_i K_i^{-1} &= K_i^{-1} K_i = 1, && (1)\\
K_0 K_1 &= K_1 K_0, && (2)\\
K_i e_i^{\pm} K_i^{-1} &= q^{\pm 2} e_i^{\pm}, && (3)\\
K_i e_j^{\pm} K_i^{-1} &= q^{\mp 2} e_j^{\pm}, \qquad i \neq j, && (4)\\
e_i^+ e_i^- - e_i^- e_i^+ &= \frac{K_i - K_i^{-1}}{q - q^{-1}}, && (5)\\
e_0^{\pm} e_1^{\mp} &= e_1^{\mp} e_0^{\pm}, && (6)
\end{aligned}$$

$$(e_i^{\pm})^3 e_j^{\pm} - [3](e_i^{\pm})^2 e_j^{\pm} e_i^{\pm} + [3] e_i^{\pm} e_j^{\pm} (e_i^{\pm})^2 - e_j^{\pm} (e_i^{\pm})^3 = 0 \qquad i \neq j. \tag{7}$$

We call $e_i^{\pm}$, $K_i^{\pm 1}$, $i \in \{0, 1\}$ the *Chevalley generators* for $U_q(\widehat{\mathfrak{sl}}_2)$ and refer to (7) as the *q-Serre relations.*

2 The Main Theorem

In this section we state our main result. We begin with two definitions.

Definition 2.1 Let V be a vector space over $\mathbb{K}$ with finite positive dimension. By a *decomposition* of V we mean a sequence $V_0, V_1, \ldots, V_d$ consisting of nonzero subspaces of V such that $V = \sum_{i=0}^{d} V_i$ (direct sum). For notational convenience we let $V_{-1} = 0, V_{d+1} = 0$.

Definition 2.2 Let V be a vector space over $\mathbb{K}$ with finite positive dimension. Let $V_0, V_1, \ldots, V_d$ be a decomposition of V. Let $K : V \to V$ denote the linear transformation such that, for $0 \leq i \leq d$, V_i is an eigenspace for K with eigenvalue q^{2i-d}. We refer to K as *the linear transformation corresponding to the decomposition* $V_0, V_1, \ldots, V_d$.

Note 2.3 With reference to Definition 2.2, we note that K is invertible. Moreover, for $0 \leq i \leq d$, V_i is the eigenspace for K^{-1} with eigenvalue q^{d-2i}. We observe that K^{-1} is the linear transformation corresponding to the decomposition $V_d, V_{d-1}, \ldots, V_0$.

We will be concerned with the following situation.

Assumption 2.4 Let V be a vector space over $\mathbb{K}$ with finite positive dimension. Let $V_0, V_1, \ldots V_d$ be a decomposition of V. Let K denote the linear transformation corresponding to $V_0, V_1, \ldots V_d$ as in Definition 2.2. Let $R : V \to V$ and $L : V \to V$ be linear transformations such that

(i) $RV_i \subseteq V_{i+1} \qquad (0 \leq i \leq d)$,

(ii) $LV_i \subseteq V_{i-1} \qquad (0 \leq i \leq d)$,

(iii) for $0 \leq i \leq d/2$ the restriction $R^{d-2i}|_{V_i} : V_i \to V_{d-i}$ is a bijection,

(iv) for $0 \leq i \leq d/2$ the restriction $L^{d-2i}|_{V_{d-i}} : V_{d-i} \to V_i$ is a bijection,

(v) $R^3L - [3]R^2LR + [3]RLR^2 - LR^3 = 0$,

(vi) $L^3R - [3]L^2RL + [3]LRL^2 - RL^3 = 0$.

We now state our main result.

Theorem 2.5 *Adopt Assumption 2.4. Then there exists a unique $U_q(\widehat{\mathfrak{sl}}_2)$-module structure on V such that $(R - e_1^-)V = 0$, $(L - e_0^-)V = 0$, $(K - K_0)V = 0$, $(K^{-1} - K_1)V = 0$, where e_1^-, e_0^-, K_0, K_1 are Chevelley generators for $U_q(\widehat{\mathfrak{sl}}_2)$.*

Theorem 2.5 is related to a result of G. Benkart and P. Terwilliger [1]. In [1] the authors adopt Assumption 2.4(i),(ii),(v),(vi). They replace Assumption 2.4(iii),(iv) with the assumption that V is irreducible as a (K, R, L)-module. From this assumption they obtain a $U_q(\widehat{\mathfrak{sl}}_2)$-module structure on V as in Theorem 2.5. The $U_q(\widehat{\mathfrak{sl}}_2)$-module structure that they obtain is irreducible while the $U_q(\widehat{\mathfrak{sl}}_2)$-module structure given by Theorem 2.5 is not necessarily irreducible. As far as we know Theorem 2.5 does not imply the result in [1] nor does the result in [1] imply Theorem 2.5. Both this paper and [1] use an adaptation of a construction which T. Ito and P. Terwilliger used to get $U_q(\widehat{\mathfrak{sl}}_2)$-modules from a certain type of tridiagonal pair [4]. In fact, the motivation for our work on $U_q(\widehat{\mathfrak{sl}}_2)$-modules came from the study of tridiagonal pairs [3].

The plan for the paper is as follows. In section 3 we present an overview of the argument used to prove Theorem 2.5. In sections 4 through 9 we summarize the proof of Theorem 2.5. In sections 10 and 11 we determine which $U_q(\widehat{\mathfrak{sl}}_2)$-modules arise from the construction in Theorem 2.5. Since the rest of the paper is meant to provide a summary, many of the proofs are omitted.

3 An outline of the proof of Theorem 2.5

We begin by adopting Assumption 2.4. To start the construction of the $U_q(\widehat{\mathfrak{sl}}_2)$-action on V we require that the linear transformations $R - e_1^-$, $L - e_0^-$, $K^{\pm 1} - K_0^{\pm 1}$, $K^{\pm 1} - K_1^{\mp 1}$ vanish on V. This gives the actions of the elements $e_1^-, e_0^-, K_0^{\pm 1}, K_1^{\pm 1}$ on V. We define the actions of e_0^+, e_1^+ on V as follows. First we prove that $K + R$ and $K^{-1} + L$ are diagonalizable on V. Then we show that the set of distinct eigenvalues of both $K + R$ and $K^{-1} + L$ on V is $\{ q^{2i-d} \,|\, 0 \leq i \leq d \}$. For $0 \leq i \leq d$, we let W_i (resp. W_i^*) denote the eigenspace of $K + R$ (resp. $K^{-1} + L$) on V associated with the eigenvalue q^{2i-d}. Then $W_0, \ldots, W_d$

(resp. $W_0^*, \ldots, W_d^*$) is a decomposition of V. We show that the decomposition $V_0, V_1, \ldots, V_d$ satisfies

$$\begin{aligned} V_0 + \cdots + V_i &= W_{d-i}^* + \cdots + W_d^* && (0 \leq i \leq d) \\ V_i + \cdots + V_d &= W_i + \cdots + W_d && (0 \leq i \leq d). \end{aligned}$$

Then for $0 \leq i \leq d$ we define subspaces $Z_i = (W_0 + \cdots + W_{d-i}) \cap (W_{d-i}^* + \cdots + W_d^*)$. We argue that $Z_0, Z_1, \ldots, Z_d$ is a decomposition of V and that

$$\begin{aligned} Z_0 + \cdots + Z_i &= W_{d-i}^* + \cdots + W_d^* && (0 \leq i \leq d) \\ Z_i + \cdots + Z_d &= W_0 + \cdots + W_{d-i} && (0 \leq i \leq d). \end{aligned}$$

Next for $0 \leq i \leq d$ we define subspaces $Z_i^* = (W_{d-i} + \cdots + W_d) \cap (W_0^* + \cdots + W_{d-i}^*)$. We argue that $Z_0^*, Z_1^*, \ldots, Z_d^*$ is a decomposition of V and that

$$\begin{aligned} Z_0^* + \cdots + Z_i^* &= W_{d-i} + \cdots + W_d && (0 \leq i \leq d) \\ Z_i^* + \cdots + Z_d^* &= W_0^* + \cdots + W_{d-i}^* && (0 \leq i \leq d). \end{aligned}$$

We then define the linear transformation $B : V \to V$ (resp. $B^* : V \to V$) such that for $0 \leq i \leq d$, Z_i (resp. Z_i^*) is an eigenspace for B (resp. B^*) with eigenvalue q^{2i-d}. We let e_1^+ act on V as $I - K^{-1}B$ times $q^{-1}(q - q^{-1})^{-2}$. We let e_0^+ act on V as $I - KB^*$ times $q^{-1}(q - q^{-1})^{-2}$. Finally, we display some relations that are satisfied by $B, B^*, L, R, K^{\pm 1}$. Using these relations, we argue that the above actions of $e_0^{\pm}$, $e_1^{\pm}$, $K_0^{\pm 1}$, $K_1^{\pm 1}$ satisfy the defining relations for $U_q(\widehat{\mathfrak{sl}}_2)$. In this way, we obtain the required action of $U_q(\widehat{\mathfrak{sl}}_2)$ on V.

4 The linear transformations A and A^*

In this section we define and discuss two linear transformations that will be useful.

Definition 4.1 In reference to Assumption 2.4 let $A : V \to V$ and $A^* : V \to V$ denote the following linear transformations.

$$A = K + R, \qquad A^* = K^{-1} + L. \tag{8}$$

Lemma 4.2 *With reference to Definition 4.1 and Assumption 2.4 the following (i),(ii) hold.*

(i) $(A - q^{2i-d}I)V_i \subseteq V_{i+1}, \qquad 0 \leq i \leq d,$

(ii) $(A^* - q^{d-2i}I)V_i \subseteq V_{i-1}, \qquad 0 \leq i \leq d.$

Lemma 4.3 *With reference to Definition 4.1 and Assumption 2.4, the following (i),(ii) hold.*

(i) A is diagonalizable with eigenvalues $q^{-d}, q^{2-d}, \ldots, q^d$. Moreover, for $0 \leq i \leq d$, the dimension of the eigenspace for A associated with q^{2i-d} is equal to the dimension of V_i.

(ii) A^* *is diagonalizable with eigenvalues* $q^{-d}, q^{2-d}, \ldots, q^d$*. Moreover, for* $0 \leq i \leq d$*, the dimension of the eigenspace for* A^* *associated with* q^{2i-d} *is equal to the dimension of* V_{d-i}*.*

Proof: (i) We start by displaying the eigenvalues for A. Notice that the scalars q^{2i-d} $(0 \leq i \leq d)$ are distinct since q is not a root of unity. Using Lemma 4.2(i) we see that, with respect to an appropriate basis for V, A can be represented as a lower triangular matrix that has diagonal entries $q^{-d}, q^{2-d}, \ldots, q^d$, with q^{2i-d} appearing $\dim(V_i)$ times for $0 \leq i \leq d$. Hence for $0 \leq i \leq d$, q^{2i-d} is a root of the characteristic polynomial of A with multiplicity $\dim(V_i)$. It remains to show that A is diagonalizable. To do this we show that the minimal polynomial of A has distinct roots. Recall that $V_0, V_1, \ldots, V_d$ is a decomposition of V. Using Lemma 4.2(i) we see that $\prod_{i=0}^{d}(A - q^{2i-d}I)V = 0$. By this and since q^{2i-d} $(0 \leq i \leq d)$ are distinct we see that the minimal polynomial of A has distinct roots. We conclude that A is diagonalizable and the result follows.
(ii) Similar to (i). □

Definition 4.4 With reference to Definition 4.1 and Lemma 4.3, for $0 \leq i \leq d$, we let W_i (resp. W_i^*) be the eigenspace for A (resp. A^*) with eigenvalue q^{2i-d}. Using Lemma 4.3 we observe that $W_0, W_1, \ldots, W_d$ (resp. $W_0^*, W_1^*, \ldots, W_d^*$) is a decomposition of V.

Lemma 4.5 *With reference to Assumption 2.4 and Definition 4.4, the following (i)–(iii) hold.*

(i) $V_0 + \cdots + V_i = W_{d-i}^* + \cdots + W_d^*, \qquad 0 \leq i \leq d,$

(ii) $V_i + \cdots + V_d = W_i + \cdots + W_d, \qquad 0 \leq i \leq d,$

(iii) $V_i = (W_i + \cdots + W_d) \cap (W_{d-i}^* + \cdots + W_d^*), \qquad 0 \leq i \leq d.$

Proof: (i) Let i be given. Define $T = V_0 + \cdots + V_i$ and $S = W_{d-i}^* + \cdots + W_d^*$. We show that $T = S$. First we show that $S \subseteq T$. Let $X = \prod_{h=0}^{d-i-1}(A^* - q^{2h-d}I)$. Recall that $W_0^*, W_1^*, \ldots, W_d^*$ is a decomposition of V and so we have $XV = S$. By Lemma 4.2(ii) we have $XV_j \subseteq T$ for $0 \leq j \leq d$. By this and since $V_0, V_1, \ldots V_d$ is a decomposition of V we find that $XV \subseteq T$. By these comments $S \subseteq T$. By Lemma 4.3(ii) and Definition 4.4 we have $\dim(W_{d-i}^*)$=$\dim(V_i)$. Thus $\dim(S)$=$\dim(T)$. We conclude $T = S$ and the result follows.
(ii) Similar to (i).
(iii) Immediate from (i),(ii) and the fact that $V_0, V_1, \ldots, V_d$ is a decomposition of V. □

5 The Subspaces Z_i, Z_i^*

Definition 5.1 With reference to Definition 4.4, for $0 \leq i \leq d$, we let Z_i denote the following subspace of V.

$$Z_i = (W_0 + \cdots + W_{d-i}) \cap (W_{d-i}^* + \cdots + W_d^*).$$

Using Assumption 2.4(i),(iii) and Lemma 4.5 the following theorem can be proven.

Theorem 5.2 *With reference to Definition 5.1, the following (i)–(iii) hold.*

(i) $Z_0, Z_1, \cdots, Z_d$ *is a decomposition of* V*,*

(ii) $Z_0 + \cdots + Z_i = W^*_{d-i} + \cdots + W^*_d, \qquad 0 \leq i \leq d,$

(iii) $Z_i + \cdots + Z_d = W_0 + \cdots + W_{d-i}, \qquad 0 \leq i \leq d.$

Definition 5.3 With reference to Definition 4.4, for $0 \leq i \leq d$, we let Z^*_i denote the following subspace of V.

$$Z^*_i = (W_{d-i} + \cdots + W_d) \cap (W^*_0 + \cdots + W^*_{d-i}).$$

Theorem 5.4 *With reference to Definition 5.3, the following (i)–(iii) hold:*

(i) $Z^*_0, Z^*_1, \cdots, Z^*_d$ *is a decomposition of* V*,*

(ii) $Z^*_0 + \cdots + Z^*_i = W_{d-i} + \cdots + W_d, \qquad 0 \leq i \leq d,$

(iii) $Z^*_i + \cdots + Z^*_d = W^*_0 + \cdots + W^*_{d-i}, \qquad 0 \leq i \leq d.$

6 The linear transformations B and B^*

Definition 6.1 With reference to Definition 5.1 and Definition 5.3, we define the following linear transformations.
(i) Let $B : V \to V$ be the unique linear transformation such that for $0 \leq i \leq d$, Z_i is an eigenspace for B with eigenvalue q^{2i-d}.
(ii) Let $B^* : V \to V$ be the unique linear transformation such that for $0 \leq i \leq d$, Z^*_i is an eigenspace for B^* with eigenvalue q^{2i-d}.

7 Some relations involving A,A^*,B,B^*,$K^{\pm 1}$

Lemma 7.1 *In reference to Definition 4.1 and Definition 6.1, the following hold.*

$$\frac{qAB - q^{-1}BA}{q - q^{-1}} = I, \tag{9}$$

$$\frac{qA^*B^* - q^{-1}B^*A^*}{q - q^{-1}} = I, \tag{10}$$

$$\frac{qBA^* - q^{-1}A^*B}{q - q^{-1}} = I, \tag{11}$$

$$\frac{qB^*A - q^{-1}AB^*}{q - q^{-1}} = I. \tag{12}$$

Lemma 7.2 *With reference to Assumption 2.4 and Definition 6.1, the following hold.*

$$\frac{qBK^{-1} - q^{-1}K^{-1}B}{q - q^{-1}} = I, \tag{13}$$

$$\frac{qB^*K - q^{-1}KB^*}{q - q^{-1}} = I. \tag{14}$$

Lemma 7.3 *With reference to Defintion 6.1, the following (i),(ii) hold.*

(i) $B^3B^* - [3]B^2B^*B + [3]BB^*B^2 - B^*B^3 = 0,$

(ii) $B^{*3}B - [3]B^{*2}BB^* + [3]B^*BB^{*2} - BB^{*3} = 0.$

8 The proof of Theorem 2.5 (existence)

This section is devoted to proving the existence part of Theorem 2.5.

Definition 8.1 With reference to Assumption 2.4 and Definition 6.1, let $r : V \to V$, $l : V \to V$ be the following linear transformations.

$$r = \frac{I - KB^*}{q(q - q^{-1})^2}, \qquad l = \frac{I - K^{-1}B}{q(q - q^{-1})^2}.$$

Lemma 8.2 *With reference to Definition 8.1, the following (i),(ii) hold.*

(i) $B = K - q(q - q^{-1})^2Kl,$

(ii) $B^* = K^{-1} - q(q - q^{-1})^2K^{-1}r.$

Proof: Immediate from Definition 8.1. □

Theorem 8.3 *With reference to Assumption 2.4 and Definition 8.1, the following (i)–(ix) hold.*

(i) $KK^{-1} = K^{-1}K = I,$

(ii) $KR = q^2RK, \qquad KL = q^{-2}LK,$

(iii) $Kr = q^2rK, \quad Kl = q^{-2}lK,$

(iv) $rR = Rr, \qquad lL = Ll,$

(v) $lR - Rl = \frac{K^{-1}-K}{q-q^{-1}}$, $rL - Lr = \frac{K-K^{-1}}{q-q^{-1}},$

(vi) $R^3L - [3]R^2LR + [3]RLR^2 - LR^3 = 0,$

(vii) $L^3R - [3]L^2RL + [3]LRL^2 - RL^3 = 0,$

(viii) $r^3l - [3]r^2lr + [3]rlr^2 - lr^3 = 0,$

(ix) $l^3r - [3]l^2rl + [3]lrl^2 - rl^3 = 0.$

Proof: (i) Immediate from Note 2.3.
(ii) Since $V_0, V_1, \ldots, V_d$ is a decompositon of V to prove the first equation it suffices to show that $KR - q^2RK$ vanishes of V_i for $0 \leq i \leq d$. Let i be given, and let $v \in V_i$. Recall that v is an eigenvector for K with eigenvalue q^{2i-d}. By Assumption 2.4(i), Rv is an eigenvector for K with eigenvalue q^{2i+2-d}. From these comments we see that $(KR - q^2RK)v = 0$. The second equation follows in a similar fashion.
(iii) Evaluate the equations in Lemma 7.2 using Lemma 8.2.
(iv),(v) Evaluate (9)–(12) of Lemma 7.1 using Definition 4.1, Lemma 8.2, and Theorem 8.3(ii),(iii).
(vi),(vii) These relations hold by Assumption 2.4(v),(vi).
(viii), (ix) Substiute the expressions in Lemma 8.2 into Lemma 7.3(i),(ii), and simply using Theorem 8.3(iii). $\square$

Theorem 8.4 *With reference to Assumption 2.4 and Definition 8.1, V supports a $U_q(\widehat{\mathfrak{sl}}_2)$-module structure for which the Chevalley generators act as follows.*

generator	e_1^-	e_0^-	e_0^+	e_1^+	K_0	K_1	K_0^{-1}	K_1^{-1}
action on V	R	L	r	l	K	K^{-1}	K^{-1}	K

Proof: To see that the above action on V determines a $U_q(\widehat{\mathfrak{sl}}_2)$-module compare the equations in Theorem 8.3 with the defining relations for $U_q(\widehat{\mathfrak{sl}}_2)$ in Definition 1.1. $\square$

Proof of Theorem 2.5 (existence): The existence part of Theorem 2.5 is immediate from Theorem 8.4. $\square$

9 The proof of Theorem 2.5 (uniqueness)

This section is devoted to proving the uniqueness part of Theorem 2.5.

In proving uniqueness we will make use of the quantum algebra $U_q(sl_2)$ and its representations. We now recall the definition of $U_q(sl_2)$.

Definition 9.1 [5, p. 9] The quantum algebra $U_q(sl_2)$ is the unital associative $\mathbb{K}$-algebra generated by k, k^{-1}, e, f subject to the following relations:

$$
\begin{aligned}
kk^{-1} = k^{-1}k &= 1,\\
ke &= q^2ek,\\
kf &= q^{-2}fk,\\
ef - fe &= \frac{k-k^{-1}}{q-q^{-1}}.
\end{aligned}
$$

We now recall the irreducible finte-dimensional modules for $U_q(sl_2)$.

Lemma 9.2 *[5, p. 20] With reference to Defintion 9.1, there exist a family*

$$V_{\epsilon,d} \qquad \epsilon \in \{-1,1\}, \qquad d = 0,1,2,\ldots$$

of irreducible finite-dimensional $U_q(sl_2)$-modules with the following properties. The module $V_{\epsilon,d}$ has a basis $u_0, u_1, \ldots, u_d$ satisfying:

$$ku_i = \epsilon q^{d-2i} u_i, \qquad 0 \le i \le d, \tag{15}$$

$$fu_i = [i+1]u_{i+1}, \qquad 0 \le i \le d-1, \qquad fu_d = 0, \tag{16}$$

$$eu_i = \epsilon[d-i+1]u_{i-1}, \qquad 1 \le i \le d, \qquad eu_0 = 0. \tag{17}$$

Moreover, every irreducible finite-dimensional $U_q(sl_2)$-module is isomorphic to exactly one of the modules $V_{\epsilon,d}$.

We now show how $U_q(sl_2)$-modules and $U_q(\widehat{\mathfrak{sl}}_2)$-modules are related.

Lemma 9.3 *Let V be a finite-dimensional $U_q(\widehat{\mathfrak{sl}}_2)$-module. For $i \in \{0,1\}$, V supports a $U_q(sl_2)$-module structure such that each of $K_i - k$, $e_i^+ - e$, $e_i^- - f$ vanish on V, where k, e, f are the generators from Definition 9.1.*

Proof: Immediate from Definition 1.1 and Definition 9.1. □

Lemma 9.4 *Let k, e, f be the generators for $U_q(sl_2)$ as in Definition 9.1. Let V be a finite-dimensional $U_q(sl_2)$-module. Assume the action of k on V is diagonalizable. Suppose $e' : V \to V$ is a linear transfomation such that*

$$ke' = q^2 e' k, \tag{18}$$

$$e'f - fe' = \frac{k - k^{-1}}{q - q^{-1}}, \tag{19}$$

hold on V. Then $(e - e')V = 0$.

Proof of Theorem 2.5 (uniqueness): By the existence part of Theorem 2.5 we know that there exists a $U_q(\widehat{\mathfrak{sl}}_2)$-module structure on V under the action of the Chevalley generators $e_i^{\pm}, K_i^{\pm 1}, i \in \{0,1\}$ such that each of $R - e_1^-$, $L - e_0^-$, $K - K_0$, and $K^{-1} - K_1$ vanish on V. Now suppose there exists another $U_q(\widehat{\mathfrak{sl}}_2)$-module structure on V under the action of the Chevalley generators $(e_i^{\pm})', (K_i^{\pm 1})', i \in \{0,1\}$ such that each of $R - (e_1^-)'$, $L - (e_0^-)'$, $K - K_0'$, and $K^{-1} - K_1'$ vanish on V. To prove uniqueness it suffices to show that for $i \in \{0,1\}$, $e_i^{\pm} - (e_i^{\pm})'$ and $K_i^{\pm 1} - (K_i^{\pm 1})'$ vanish on V. Since $R - e_1^-$ and $R - (e_1^-)'$ vanish on V then $(e_1^- - (e_1^-)')V = 0$. Similarly, we have that

$$e_0^- - (e_0^-)', \quad K_0^{\pm 1} - (K_0^{\pm 1})', \quad K_1^{\pm 1} - (K_1^{\pm 1})', \tag{20}$$

vanish on V. We now show that $(e_0^+ - (e_0^+)')V = 0$. By Lemma 9.3 we can view V as a $U_q(sl_2)$-module under the action of K_0, e_0^-, e_0^+. Using Definition 1.1 and (20) we see that $K_0(e_0^+)' = q^2(e_0^+)'K_0$ and $(e_0^+)'e_0^- - e_0^-(e_0^+)' = \frac{K_0-K_0^{-1}}{q-q^{-1}}$. Therefore, by Lemma 9.4, we have $(e_0^+ - (e_0^+)')V = 0$. The proof that $(e_1^+ - (e_1^+)')V = 0$ is similar. □

10 Which $U_q(\widehat{\mathfrak{sl}}_2)$-modules arise from Theorem 2.5?

Theorem 2.5 gives a way to constuct finite dimensional $U_q(\widehat{\mathfrak{sl}}_2)$-modules. Not all finite dimensional $U_q(\widehat{\mathfrak{sl}}_2)$-modules arise from this construction; in this section we determine which ones do.

Definition 10.1 Let V denote a nonzero finite dimensional $U_q(\widehat{\mathfrak{sl}}_2)$-module. Let d denote a nonnegative integer. We say V is *basic of diameter* d whenever there exists a decomposition $V_0, V_1, \ldots, V_d$ of V and linear transformations $R : V \to V$ and $L : V \to V$ satisfying Assumption 2.4(i)–(vi) such that the given $U_q(\widehat{\mathfrak{sl}}_2)$-module structure on V agrees with the $U_q(\widehat{\mathfrak{sl}}_2)$-module structure on V given by Theorem 2.5.

Our goal for the remainder of this section is to determine which $U_q(\widehat{\mathfrak{sl}}_2)$-modules are basic.

Lemma 10.2 *Let V be a finite dimensional $U_q(\widehat{\mathfrak{sl}}_2)$-module. If the characteristic of $\mathbb{K}$ is not equal to 2 then the actions of K_0 and K_1 on V are diagonalizable.*

Theorem 10.3 *Let d be a nonnegative integer and let V be a finite dimensional $U_q(\widehat{\mathfrak{sl}}_2)$-module. With reference to Definition 10.1, the following are equivalent.*

(i) V is basic of diameter d.

(ii) $(K_0K_1 - I)V = 0$, the action of K_0 on V is diagonalizable, and the set of distinct eigenvalues for K_0 on V is $\{q^{2i-d}, 0 \le i \le d\}$.

11 The relationship between general $U_q(\widehat{\mathfrak{sl}}_2)$-modules and basic $U_q(\widehat{\mathfrak{sl}}_2)$-modules

Throughout this section V will denote a finite dimensional $U_q(\widehat{\mathfrak{sl}}_2)$-module (not necessarily irreducible) on which the actions of K_0 and K_1 are diagonalizable (see Lemma 10.2).

In this section we will show, roughly speaking, that V is made up of basic $U_q(\widehat{\mathfrak{sl}}_2)$-modules. We will use the following definition.

Definition 11.1 Let $\epsilon_0, \epsilon_1 \in \{1, -1\}$. Define $V_{even}^{(\epsilon_0,\epsilon_1)}$ (resp. $V_{odd}^{(\epsilon_0,\epsilon_1)}$) to be the subspace of V spanned by all the vectors $v \in V$ such that $K_0v = \epsilon_0 q^i v$, $K_1 v = \epsilon_1 q^{-i} v$, $i \in \mathbb{Z}$, i even (resp. i odd).

Lemma 11.2 *With reference to Definition 11.1 the following holds.*

$$V = \sum_{(\epsilon_0,\epsilon_1)} \sum_{\sigma} V_{\sigma}^{(\epsilon_0,\epsilon_1)} \qquad (\textit{direct sum of } U_q(\widehat{\mathfrak{sl}}_2) - \textit{modules}) \tag{21}$$

where the first sum is over all ordered pairs (ϵ_0, ϵ_1) with $\epsilon_0, \epsilon_1 \in \{1, -1\}$ and the second sum is over all $\sigma \in \{even, odd\}$.

Lemma 11.3 *With reference to Definition 10.1, Definition 11.1, and Lemma 11.2 the following are equivalent.*

(i) $V = V_{even}^{(1,1)}$.

(ii) V *is basic of even diameter.*

(iii) The spaces $V_{even}^{(-1,1)}$, $V_{even}^{(1,-1)}$, $V_{even}^{(-1,-1)}$, $V_{odd}^{(1,1)}$, $V_{odd}^{(-1,1)}$, $V_{odd}^{(1,-1)}$, $V_{odd}^{(-1,-1)}$ *are all zero.*

Lemma 11.4 *With reference to Definition 10.1, Definition 11.1, and Lemma 11.2 the following are equivalent.*

(i) $V = V_{odd}^{(1,1)}$.

(ii) V *is basic of odd diameter.*

(iii) The spaces $V_{odd}^{(-1,1)}$, $V_{odd}^{(1,-1)}$, $V_{odd}^{(-1,-1)}$, $V_{even}^{(1,1)}$, $V_{even}^{(-1,1)}$, $V_{even}^{(1,-1)}$, $V_{even}^{(-1,-1)}$ *are all zero.*

Refering to (21) even though the six terms $V_{even}^{(-1,1)}$, $V_{even}^{(1,-1)}$, $V_{even}^{(-1,-1)}$, $V_{odd}^{(-1,1)}$, $V_{odd}^{(1,-1)}$, $V_{odd}^{(-1,-1)}$ are not basic modules they can easily be modified to become basic modules. We now state a lemma that makes this precise.

Lemma 11.5 *[2, Prop. 3.2] For any choice of scalars $\epsilon_0, \epsilon_1 \epsilon \{1, -1\}$ there exists a $\mathbb{K}$-algebra automorphism of $U_q(\widehat{\mathfrak{sl}}_2)$ such that*

$$K_i \to \epsilon_i K_i, \qquad e_i^+ \to e_i^+, \qquad e_i^- \to \epsilon_i e_i^-.$$

for $i \in \{1, -1\}$.

Remark 11.6 With reference to Definition 10.1 and Definition 11.1 we can alter each of the modules $V_{even}^{(-1,1)}$, $V_{even}^{(1,-1)}$, $V_{even}^{(-1,-1)}$ to a basic $U_q(\widehat{\mathfrak{sl}}_2)$-module of even diameter by applying an automorphism as in Lemma 11.5. Furthermore, we can alter each of the modules $V_{odd}^{(-1,1)}$, $V_{odd}^{(1,-1)}$, $V_{odd}^{(-1,-1)}$ to a basic $U_q(\widehat{\mathfrak{sl}}_2)$-module of odd diameter by applying an automorphism as in Lemma 11.5.

References

[1] G. Benkart and P. Terwilliger. Irreducible modules for the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$ and its Borel subalgebra, Journal of Algebra 282 (2004) 172–194.

[2] V. Chari and A. Pressley, Quantum affine algebras, *Commun. Math. Phys.* **142** (1991), 261–283.

[3] T. Ito, K. Tanabe, and P. Terwilliger. Some algebra related to P- and Q-polynomial association schemes. In *Codes and association schemes (Piscataway NJ, 1999)*, 167–192, DIMACS Ser. Discrete Math. Theoret. Comput. Sci. **56**, Amer. Math. Soc., Providence RI, 2001.

[4] T. Ito and P. Terwilliger. Tridiagonal pairs and the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$. *Ramanujan J.* Submitted.

[5] J.C. Jantzen, *Lectures on Quantum Groups.* American Mathematical Society, Providence RI, 1996.

Darren Neubauer (email: `neubauer@math.wisc.edu`)

Department of Mathematics
University of Wisconsin
480 Lincoln Drive
Madison, WI 53706-1388 USA