THE EXCEDANCE NUMBER OF SOME COLORED PERMUTATION GROUPS (EXTENDED ABSTRACT)

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ABSTRACT. We generalize the results of Ksavrelof and Zeng about the multidistribution of the excedance number of S_n with some natural parameters to the *colored permutation group* and to the Coxeter group of type D. We define two different orders on these groups which induce two different excedance numbers. Surprisingly, in the case of the *colored permutation group* we get the same generalized formulas for both orders.

1. INTRODUCTION

Let r and n be two positive integers. The *colored permutation group* $G_{r,n}$ consists of all permutations of the set

$$\Sigma = \{1, \dots, n, \bar{1}, \dots, \bar{n}, \dots, 1^{[r-1]}, \dots, n^{[r-1]}\}$$

satisfying $\pi(\bar{i}) = \overline{\pi(i)}$.

The symmetric group S_n is a special case of $G_{r,n}$ for $r = 1$ while for $r = 2$ we get the Weyl group of type B: B_n. In S_n one can define the following well-known parameters: Given $\sigma \in S_n$, $i \in [n]$ is *an excedance of* σ if and only if $\sigma(i) > i$. The number of excedances is denoted by $\mathrm{exc}(\sigma)$. Two other natural parameters on S_n are the number of fixed points and the number of cycles of σ, denoted by $\mathrm{fix}(\sigma)$ and $\mathrm{cyc}(\sigma)$ respectively.

Consider the following generating function over S_n:

$$P_n(q, t, s) = \sum_{\sigma \in S_n} q^{\mathrm{exc}(\sigma)} t^{\mathrm{fix}(\sigma)} s^{\mathrm{cyc}(\sigma)}.$$

$P_n(q, 1, 1)$ is the classical Eulerian polynomial, while $P_n(q, 0, 1)$ is the counter part for the derangements, i.e. the permutations without fixed points, see [**?**].

In the case $s = -1$, the two polynomials $P_n(q, 1, -1)$ and $P_n(q, 0, -1)$ have simple closed formulas:

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$$P_n(q,1,-1) = -(q-1)^{n-1} \tag{1}$$

$$P_n(q,0,-1) = -q[n-1]_q \tag{2}$$

Recently, Ksavrelof and Zeng [?] proved some new recursive formulas which induce Equations (??) and (??). A natural problem is to generalize the results of [?] to other groups. The main challenge here is to choose a suitable order on the alphabet Σ of $G_{r,n}$ and define the parameters properly.

In this paper we cope with this challenge. We define two different orders on Σ, one of them, the *absolute order* 'forgets' the colors, while the other is much more natural, since it takes into account the colorful structure of $G_{r,n}$. This order is called the *color order*. The parameter exc will be defined according to both orders in two different ways. The interesting point is that for the group $G_{r,n}$ we get the same recursive formulas for both cases.

Define

$$P^{\mathbf{ord}}_{G_{r,n}}(q,t,s) = \sum_{\pi \in G_{r,n}} q^{\mathrm{exc}^{\mathbf{ord}}(\pi)} t^{\mathrm{fix}(\pi)} s^{\mathrm{cyc}(\pi)}$$

where **ord** can be either the absolute order or the color order.

For $G_{r,n}$, we prove the following two main results:

Theorem 1.1.

$$P^{\mathrm{Abs}}_{G_{r,n}}(q,1,-1) = P^{\mathrm{Clr}}_{G_{r,n}}(q,1,-1) = (q^r-1)P_{G_{r,n-1}}(q,1,-1).$$

Hence,

$$P^{\mathrm{Abs}}_{G_{r,n}}(q,1,-1) = P^{\mathrm{Clr}}_{G_{r,n}}(q,1,-1) = -\frac{(q^r-1)^n}{q-1}.$$

Theorem 1.2.

$$P^{\mathrm{Abs}}_{G_{r,n}}(q,0,-1) = P^{\mathrm{Clr}}_{G_{r,n}}(q,0,-1) = [r]_q(P_{G_{r,n-1}}(q,0,-1) - q^{n-1}[r]_q^{n-1}).$$

Hence,

$$P^{\mathrm{Abs}}_{G_{r,n}}(q,0,-1) = P^{\mathrm{Clr}}_{G_{r,n}}(q,0,-1) = -q[r]_q^n[n-1]_q,$$

where $[r]_q = 1 + \cdots + q^{r-1}$.

One can easily check that the formulas appeared in Theorem **??** and Theorem **??** indeed generalize the formulas of Ksavrelof and Zeng (for $r=1$).

As mentioned above, when $r=2$ we get the group B_n. This group has a well known normal subgroup called D_n consisting of the even signed permutations, i.e., permutations with an even number of minus signs. This group is also known as the Coxeter group of type D. With respect to D_n, we prove:

Theorem 1.3.
$$P_{D_n}^{\mathrm{Clr}}(q,1,-1) = (q^2-1)P_{D_{n-1}}(q,1,-1).$$
Hence,
$$P_{D_n}^{\mathrm{Clr}}(q,1,-1) = -(q^2-1)^{n-1}.$$

Theorem 1.4.
$$P_{D_n}^{\mathrm{Abs}}(q,1,-1) = -\frac{1}{2}(q-1)^{n-1}((1+q)^n + (1-q)^n).$$

2. Preliminaries

2.1. **Notations.** For $n \in \mathbb{N}$, let $[n] := \{1,2,\dots,n\}$ (where $[0] := \emptyset$).

Also, let:
$$[n]_q := \frac{1-q^n}{1-q} = 1 + q + \cdots + q^{n-1},$$
(so $[0]_q = 0$).

2.2. **The group of colored permutations.**

Definition 2.1. Let r and n be two positive integers. *The group of colored permutations of n digits with r colors* is the wreath product $G_{r,n} = \mathbb{Z}_r \wr S_n = \mathbb{Z}_r^n \rtimes S_n$, consisting of all the pairs (z,τ) where z is an n-tuple of integers between 0 and $r-1$ and $\tau \in S_n$. The multiplication is defined by the following rule: For $z = (z_1, ..., z_n)$ and $z' = (z'_1, ..., z'_n)$
$$(z,\tau)\cdot(z',\tau') = ((z_1 + z'_{\tau(1)}, ..., z_n + z'_{\tau(n)}), \tau \circ \tau')$$
(here $+$ is taken modulo r).

In particular, $G_{1,n} = C_1 \wr S_n$ is the symmetric group S_n while $G_{2,n} = C_2 \wr S_n$ is the group of signed permutations B_n, also known as the hyperoctahedral group, or the classical Weyl group of type B.

A natural way to present $G_{r,n}$, which justifies its name, is the following: Consider the alphabet $\Sigma = \{1,\dots,n,\bar{1},\dots,\bar{n},\dots,1^{[r-1]},\dots,n^{[r-1]}\}$ as the set $[n]$ colored by the colors $0,\dots,r-1$. Then, an element of $G_{r,n}$ is a *colored permutation*, i.e. a bijection $\pi : \Sigma \to \Sigma$ such that $\pi(\bar{i}) = \overline{\pi(i)}$.

Here are some conventions we use along this paper: For an element $\pi = (z,\tau) \in G_{r,n}$ with $z = (z_1, ..., z_n)$ we write $z_i(\pi) = z_i$. For $\pi = (z,\tau)$, we denote $|\pi| = (0,\tau), (0 \in \mathbb{Z}_r^n)$. The element $(z,\tau) = ((1,0,3,2),(2,1,4,3)) \in G_{3,4}$ will be written as $(\bar{2}1\bar{\bar{\bar{4}}}\bar{\bar{3}})$.

2.3. **The Coxeter group of type D.** We define here the following normal subgroup of B_n of index 2, called the *Coxeter group of type D*:

$$D_n = \{\pi \in B_n \mid \sum_{i=1}^{n} z_i(\pi) \equiv 0 \pmod 2\}.$$

3. Statistics on $G_{r,n}$

Given any ordered alphabet Σ', we recall the definition of the *excedance set* of a permutation π on Σ' by :

$$\mathrm{Exc}(\pi) = \{i \in \Sigma' \mid \pi(i) > i\}$$

and the *excedance number* to be $\mathrm{exc}(\pi) = |\mathrm{Exc}(\pi)|$.

Example 3.1. *Given the order:* $\bar{\bar{1}} < \bar{\bar{2}} < \bar{\bar{3}} < \bar{1} < \bar{2} < \bar{3} < 1 < 2 < 3$, *we write* $\sigma = (3\bar{1}\bar{\bar{2}}) \in G_{3,3}$ *in an extended form:*

$$\begin{pmatrix} \bar{\bar{1}} & \bar{\bar{2}} & \bar{\bar{3}} & \bar{1} & \bar{2} & \bar{3} & 1 & 2 & 3 \\ \bar{\bar{3}} & 1 & \bar{2} & \bar{3} & \bar{\bar{1}} & 2 & 3 & \bar{1} & \bar{\bar{2}} \end{pmatrix}$$

and calculate: $\mathrm{Exc}(\sigma) = \{\bar{\bar{1}}, \bar{\bar{2}}, \bar{\bar{3}}, \bar{1}, \bar{3}, 1\}$ *and* $\mathrm{exc}(\sigma) = 6$.

We start by defining two orders on the set $\Sigma = \{1, \dots, n, \bar{1}, \dots, \bar{n}, \dots, 1^{[r-1]}, \dots, n^{[r-1]}\}$.

Definition 3.2. Define the *absolute order* on Σ to be:

$1^{[r-1]} < \cdots < \bar{1} < 1 < 2^{[r-1]} < \cdots < \bar{2} < 2 < \cdots < n^{[r-1]} < \cdots < \bar{n} < n$

and the *color order* on Σ by:

$$1^{[r-1]} < \cdots < n^{[r-1]} < \cdots < 1^{[1]} < \cdots < n^{[1]} < 1 < \cdots < n$$

Before defining the excedance number with respect to both orders, we have to introduce some notations.

Let $\sigma \in G_{r,n}$. We define:

$$\mathrm{csum}(\sigma) = \sum_{i=1}^{n} z_i(\sigma)$$

$$\mathrm{Exc}_A(\sigma) = \{i \in [n-1] \mid \sigma(i) > i\},$$

where the comparison is with respect to the color order.

$$\mathrm{exc}_A(\sigma) = |\mathrm{Exc}_A(\sigma)|$$

Let $\sigma \in G_{r,n}$. Recall that for $\sigma = (z, \tau) \in G_{r,n}$, $|\sigma|$ is the permutation of $[n]$ satisfying $|\sigma|(i) = \tau(i)$. For example, if $\sigma = (\bar{2}\bar{\bar{3}}1\bar{4})$ then $|\sigma| = (2314)$.

Now we can define the excedance numbers.

Definition 3.3. Define:

$$\mathrm{exc}^{\mathrm{Abs}}(\sigma) = \mathrm{exc}(|\sigma|) + \mathrm{csum}(\sigma)$$

$$\mathrm{exc}^{\mathrm{Clr}}(\sigma) = r \cdot \mathrm{exc}_A(\sigma) + \mathrm{csum}(\sigma)$$

Example 3.4. *Take* $\sigma = (\bar{1}\bar{\bar{3}}4\bar{2}) \in G_{3,4}$. *Then* $\mathrm{csum}(\sigma) = 4$, $\mathrm{Exc}_{\mathrm{A}}(\sigma) = \{3\}$, $\mathrm{Exc}(|\sigma|) = \{2,3\}$ *and thus* $\mathrm{exc}^{\mathrm{Abs}}(\sigma) = 6$ *and* $\mathrm{exc}^{\mathrm{Clr}}(\sigma) = 7$.

Recall that any permutation of S_n can be decomposed into a product of disjoint cycles. This notion can be easily generalized to the group $G_{r,n}$ as follows. Given any $\pi \in G_{r,n}$ we define the cycle number of $\pi = (z, \tau)$ to be the number of cycles in τ.

We say that $i \in [n]$ is an *absolute fixed point* of $\sigma \in G_{r,n}$ if $|\sigma(i)| = i$.

4. Proof of Theorem ?? for the color order

In this section we prove Theorem **??** for the color order. The idea of proving such identities is constructing a subset S of $G_{r,n}$ whose contribution to the generating function is exactly the right side of the identity and a killing involution on $G_{r,n} - S$, i.e., an involution on $G_{r,n} - S$ which preserves the number of excedances but changes the sign of each element of $G_{r,n} - S$ and hence shows that $G_{r,n} - S$ contributes nothing to the generating function.

Therefore, we divide $G_{r,n}$ into $2r + 1$ disjoint subsets as follows:

$$K_{r,n} = \{\sigma \in G_{r,n} \mid |\sigma(n)| \neq n, |\sigma(n-1)| \neq n\}$$

$$T^i_{r,n} = \{\sigma \in G_{r,n} \mid \sigma(n) = n^{[i]}\}, \qquad (0 \leq i \leq r-1)$$

$$R^i_{r,n} = \{\sigma \in G_{r,n} \mid \sigma(n-1) = n^{[i]}\}, \qquad (0 \leq i \leq r-1)$$

We first construct a killing involution on the set $K_{r,n}$. Define $\varphi : K_{r,n} \to K_{r,n}$ by

$$\sigma' = \varphi(\sigma) = (\sigma(n-1), \sigma(n))\sigma, \qquad \sigma \in K_{r,n}$$

Note that φ exchanges $\sigma(n-1)$ with $\sigma(n)$. It is obvious that φ is indeed an involution.

We will show that $\mathrm{exc}^{\mathrm{Clr}}(\sigma) = \mathrm{exc}^{\mathrm{Clr}}(\sigma')$. First, for $i < n-1$, it is clear that $i \in \mathrm{Exc}_{\mathrm{A}}(\sigma)$ if and only if $i \in \mathrm{Exc}_{\mathrm{A}}(\sigma')$. Now, as $\sigma(n-1) \neq n$, $n-1 \notin \mathrm{Exc}_{\mathrm{A}}(\sigma)$ and thus $n \notin \mathrm{Exc}_{\mathrm{A}}(\sigma')$. Finally, $|\sigma(n)| \neq n$ implies that $n-1 \notin \mathrm{Exc}_{\mathrm{A}}(\sigma')$. Now since it is obvious that $\mathrm{csum}(\sigma) = \mathrm{csum}(\sigma')$, we have that $\mathrm{exc}^{\mathrm{Clr}}(\sigma) = \mathrm{exc}^{\mathrm{Clr}}(\sigma')$.

On the other hand, $\mathrm{cyc}(\sigma)$ and $\mathrm{cyc}(\sigma')$ have different parities due to a multiplication by a transposition. Hence, φ is indeed a killing involution on $K_{r,n}$.

We turn now to the sets $T^i_{r,n}$. Note that there is a natural bijection between $T^i_{r,n}$ and $G_{r,n-1}$ defined by ignoring the last digit. Denote the image of $\sigma \in T^i_{r,n}$ under this bijection by σ'. Since $n \notin \mathrm{Exc}_{\mathrm{A}}(\sigma)$, we have $\mathrm{exc}_{\mathrm{A}}(\sigma) = \mathrm{exc}_{\mathrm{A}}(\sigma')$. Now, since $z_n(\sigma) = i$ we have $\mathrm{csum}(\sigma') = \mathrm{csum}(\sigma) - \mathrm{i}$ and we get:

$$\mathrm{exc}^{\mathrm{Clr}}(\sigma) = \mathrm{exc}^{\mathrm{Clr}}(\sigma') + \mathrm{i}.$$

Now, since n is an absolute fixed point of σ, $\mathrm{cyc}(\sigma') = \mathrm{cyc}(\sigma) - 1$.

To summarize, we get that the total contribution of the elements in $T^i_{r,n}$ is:

$$P^{Clr}_{T^i_{r,n}} = -q^i P^{\mathrm{Clr}}_{G_{r,n-1}}(q, 1, -1)$$

for $0 \le i \le r-1$.

Now, we treat the sets $R^i_{r,n}$. There is a bijection between $R^i_{r,n}$ and $T^i_{r,n}$ using the same function φ we used above. Define $\varphi : R^i_{r,n} \to T^i_{r,n}$ by

$$\sigma' = \varphi(\sigma) = (\sigma(n-1), \sigma(n))\sigma.$$

When we compute the change in the excedance, we split our treatment into two cases: $i = 0$ and $i > 0$.

We start with the case $i = 0$. Note that $n - 1 \in \mathrm{Exc}_{\mathrm{A}}(\sigma)$. On the other hand, in σ', $n-1, n \notin \mathrm{Exc}_{\mathrm{A}}(\sigma')$. Hence, $\mathrm{exc}_{\mathrm{A}}(\sigma) - 1 = \mathrm{exc}_{\mathrm{A}}(\sigma')$.

Now, for the case $i > 0 : n-1, n \notin \mathrm{Exc}_{\mathrm{A}}(\sigma)$ (since $\sigma(n-1) = n^{[i]}$ is not an excedance with respect to the color order). We also have: $n-1, n \notin \mathrm{Exc}_{\mathrm{A}}(\sigma')$ and thus $\mathrm{Exc}_{\mathrm{A}}(\sigma) = \mathrm{Exc}_{\mathrm{A}}(\sigma')$ for $\sigma \in R^i_{r,n}$ where $i > 0$.

In both cases, we have that $\mathrm{csum}(\sigma) = \mathrm{csum}(\sigma')$. Hence, $\mathrm{exc}^{\mathrm{Clr}}(\sigma) - \mathrm{r} = \mathrm{exc}^{\mathrm{Clr}}(\sigma')$ for $i = 0$ and $\mathrm{exc}^{\mathrm{Clr}}(\sigma) = \mathrm{exc}(\sigma')$ for $i > 0$.

As before, the number of cycles changes its parity due to the multiplication by a transposition, and hence: $(-1)^{\mathrm{cyc}(\sigma)} = -(-1)^{\mathrm{cyc}(\sigma')}$.

Hence, the total contribution of elements in $R^i_{r,n}$ is

$$q^r P^{\mathrm{Clr}}_{G_{r,n-1}}(q, 1, -1)$$

for $i = 0$, and

$$q^i P^{\mathrm{Clr}}_{G_{r,n-1}}(q, 1, -1)$$

for $i > 0$.

Now, if we sum up all the parts, we get:

$$P^{\mathrm{Clr}}_{G_{r,n}}(q,1,-1) = \sum_{i=0}^{r-1}(-q^i P^{\mathrm{Clr}}_{G_{r,n-1}}(q,1,-1)) + q^r P^{\mathrm{Clr}}_{G_{r,n-1}}(q,1,-1) +$$

$$\sum_{i=1}^{r-1} q^i P^{\mathrm{Clr}}_{G_{r,n-1}}(q,1,-1) = (q^r-1)P^{\mathrm{Clr}}_{G_{r,n-1}}(q,1,-1)$$

as needed.

Now, for $n=1$, $G_{r,1}$ is the cyclic group of order r and thus

$$P^{\mathrm{Clr}}_{G_{r,1}}(q,1,-1) = -(1+q+\cdots+q^{r-1}) = -\frac{q^r-1}{q-1}$$

so we have

$$P^{\mathrm{Clr}}_{G_{r,n}}(q,1,-1) = -\frac{(q^r-1)^n}{q-1}$$

5. Proof of Theorem ?? for the color order

We recall that D_n is the subgroup of B_n consisting of the even signed permutations, i.e., permutations with an even number of minus signs. We divide D_n into 5 subsets:

$$K_n = \{\sigma \in D_n \mid |\sigma(n)| \neq n, |\sigma(n-1)| \neq n\}.$$

$$T^0_n = \{\sigma \in D_n \mid \sigma(n) = n\}.$$
$$T^1_n = \{\sigma \in D_n \mid \sigma(n) = \bar{n}\}.$$
$$R^0_n = \{\sigma \in D_n \mid \sigma(n-1) = n\}.$$
$$R^1_n = \{\sigma \in D_n \mid \sigma(n-1) = \bar{n}\}.$$

Now we denote:

$$a_n = P^{\mathrm{Clr}}_{D_n}(q,1,-1),$$
$$b_n = P^{\mathrm{Clr}}_{D^c_n}(q,1,-1),$$

where D^c_n is the complement of D_n in B_n.

Define $\varphi : K_n \to K_n$ by

$$\sigma' = \varphi(\sigma) = (\sigma(n-1), \sigma(n))\sigma.$$

Note that φ exchanges $\sigma(n-1)$ with $\sigma(n)$. It is easy to see that φ is a killing involution on K_n.

We turn now to the set T^0_n. Note that there is a natural bijection between T^0_n and D_{n-1} defined by ignoring the last digit. Let $\sigma \in T^0_n$. Denote the image of $\sigma \in T^0_n$ under this bijection by σ'. Note that $\mathrm{csum}(\sigma') = \mathrm{csum}(\sigma)$, $\mathrm{Exc_A}(\sigma') = \mathrm{Exc_A}(\sigma)$ and $\mathrm{Exc^{Clr}}(\sigma') = \mathrm{Exc^{Clr}}(\sigma)$.

On the other hand, $\mathrm{cyc}(\sigma') = \mathrm{cyc}(\sigma) - 1$ and thus the restriction of a_n to T_n^0 is just $-a_{n-1}$.

For the contribution of the set T_n^1 note that the function φ defined above gives us a bijection between T_n^1 and D_{n-1}^c. In this case, $\mathrm{csum}(\sigma') = \mathrm{csum}(\sigma) - 1$, $\mathrm{exc}_{\mathrm{A}}(\sigma') = \mathrm{exc}_{\mathrm{A}}(\sigma)$ and $\mathrm{exc}(\sigma')^{\mathrm{Clr}} = \mathrm{exc}^{\mathrm{Clr}}(\sigma)$. On the other hand, $\mathrm{cyc}(\sigma') = \mathrm{cyc}(\sigma) - 1$ as before. Hence, the restriction of a_n to T_n^1 is $-qb_{n-1}$.

Now, for the set R_n^0, we have the following bijection between R_n^0 and D_{n-1}: for $\sigma \in R_n^0$, exchange the last two digits, and then ignore the last digit. If we denote the image of σ by σ', we have: $\mathrm{csum}(\sigma') = \mathrm{csum}(\sigma)$, $\mathrm{exc}_{\mathrm{A}}(\sigma') = \mathrm{exc}_{\mathrm{A}}(\sigma) - 1$, $\mathrm{exc}^{\mathrm{Clr}}(\sigma') = \mathrm{exc}^{\mathrm{Clr}}(\sigma) - 2$ and $\mathrm{cyc}(\sigma') \equiv \mathrm{cyc}(\sigma) \pmod 2$. Hence, the restriction of a_n to R_n^0 is $q^2 a_{n-1}$.

Similarly, for the set R_n^1, we have a bijection between R_n^1 and D_{n-1}^c: for $\sigma \in R_n^1$, exchange the last two digits, and then ignore the last digit. Denoting the image of σ by σ', we have $\mathrm{csum}(\sigma') = \mathrm{csum}(\sigma) - 1$, $\mathrm{exc}_{\mathrm{A}}(\sigma') = \mathrm{exc}_{\mathrm{A}}(\sigma)$, and hence $\mathrm{exc}^{\mathrm{Clr}}(\sigma') = \mathrm{exc}^{\mathrm{Clr}}(\sigma) - 1$. Also, we have $\mathrm{cyc}(\sigma') \equiv \mathrm{cyc}(\sigma) \pmod 2$. Hence, the restriction of a_n to R_n^1 is qb_{n-1}.

We summarize all the contributions over all the four subsets, and we have:

$$a_n = -a_{n-1} - qb_{n-1} + q^2 a_{n-1} + qb_{n-1} = (q^2 - 1)a_{n-1}.$$

For computing a_1, note that $D_1 = \{1\}$ and thus $a_1 = -1$.

Therefore, we have:

$$P_{D_n}^{\mathrm{Clr}}(q, 1, -1) = a_n = -(q^2 - 1)^{n-1},$$

and we are done. □

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