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# Overpartitions, lattice paths and Rogers-Ramanujan identities

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Abstract. We define the notions of successive ranks and generalized Durfee squares for overpartitions. We show how these combinatorial statistics give extensions to overpartitions of combinatorial interpretations in terms of lattice paths of the generalizations of the Rogers-Ramanujan identities due to Burge, Andrews and Bressoud. All our proofs are combinatorial and use bijective techniques. Our result includes the Andrews-Gordon identities, the generalization of the Gordon-Göllnitz identities and Gordon's theorems for overpartitions.

Résumé. Nous définissons les notions de rangs successifs et de carré de Durfee généralisé pour les overpartitions. Nous montrons comment ces statistiques combinatoires permettent d'étendre aux overpartitions des interprétations combinatoires en termes de chemins des généralisations des identités de Rogers-Ramanujan dues à Burge, Andrews et Bressoud. Toutes nos preuves sont combinatoires et utilisent des techniques bijectives. Notre résultat englobe les identités d'Andrews-Gordon, les généralisations de l'identité de Gordon-Göllnitz et les theorèmes de Gordon pour les overpartitions.

## 1. Introduction

The starting point of this work is a result of Lovejoy of 2003 [**25**], called Gordon's theorem for overpartitions which states that

Theorem 1.1. [**25**] *Let $\overline{B}_k(n)$ denote the number of overpartitions of $n$ of the form $(\lambda_1, \lambda_2, \dots, \lambda_s)$, where $\lambda_j - \lambda_{j+k-1} \geq 1$ if $\lambda_{j+k-1}$ is overlined and $\lambda_j - \lambda_{j+k-1} \geq 2$ otherwise. Let $\overline{A}_k(n)$ denote the number of overpartitions of $n$ into parts not divisible by $k$. Then $\overline{A}_k(n) = \overline{B}_k(n)$.*

An overpartition here is a partition where the final occurrence of a part can be overlined [**16**]. For example there exist 8 overpartitions of 3

$$(3),\ (\overline{3}),\ (2,1),\ (\overline{2},1),\ (2,\overline{1}),\ (\overline{2},\overline{1}),\ (1,1,1),\ (1,1,\overline{1}).$$

Overpartitions have been recently heavily studied under different names and guises. They can be called joint partitions [**9**], or dotted partitions [**11**] and they are also closely related to 2-modular diagrams [**28**], jagged partitions [**21, 22**] and superpartitions [**20**]. Results on (for example) combinatorics of basic hypergeometric series identities [**17, 32**], $q$-series [**22, 25, 26**], congruences of the overpartition function [**21, 29**] and supersymmetric functions [**20**] have been discovered.

Gordon's theorem was proved in 1961 and is the following

Theorem 1.2. [**24**] *Let $B_{k,i}(n)$ denote the number of partitions of $n$ of the form $(\lambda_1, \lambda_2, \dots, \lambda_s)$, where $\lambda_j - \lambda_{j+k-1} \geq 2$ and at most $i-1$ of the parts are equal to 1. Let $A_{k,i}(n)$ denote the number of partitions of $n$ into parts not congruent to $0, \pm i$ modulo $2k+1$. Then $A_{k,i}(n) = B_{k,i}(n)$.*

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This theorem is an extension of the famous Rogers-Ramanujan identities proved by Rogers in 1894 [**31**] which correspond to the cases $k = i = 2$ and $k = 2$, $i = 1$. It is still a well known open problem to find a natural bijective proof of these identities, even though an impressive number of nearly combinatorial proofs have been published. A recent example was presented at FPSAC last year [**10**]. Lovejoy's result can be seen as an analog of Gordon's theorem, as the conditions on the $\overline{B}_k(n)$ reduce to the conditions on the $B_{k,k}(n)$ if the overpartition has no overlined parts and is indeed a partition.

Other combinatorial interpretations related to Gordon's theorem were given by Andrews and these became the Andrews-Gordon identities :

Theorem 1.3. [**4**] *Let $C_{k,i}(n)$ be the number of partitions of $n$ whose successive ranks lie in the interval $[-i+2, 2k-i-1]$ and let $D_{k,i}(n)$ be the number of partitions of $n$ with $i-1$ successive Durfee squares followed by $k-i$ successive Durfee rectangles. Then*

$$A_{k,i}(n) = B_{k,i}(n) = C_{k,i}(n) = D_{k,i}(n).$$

Details can be found in [**2**, Chapter 7]. It is well understood combinatorially that $B_{k,i}(n) = C_{k,i}(n) = D_{k,i}(n)$ and that result was established by some beautiful work of Burge [**14, 15**] using some recursive arguments. This work was reinterpreted by Andrews and Bressoud [**7**] who showed that Burge's argument could be rephrased in terms of binary words and that Gordon's theorem can be established thanks to these combinatorial arguments and the Jacobi Triple product identity [**23**]. Finally Bressoud [**12**] reinterpreted these in terms of ternary words and showed some direct bijections between the objects counted by $B_{k,i}(n)$, $C_{k,i}(n)$, $D_{k,i}(n)$ and the ternary words.

The purpose of this extended abstract is therefore to extend these works [**7, 12, 14, 15**] to overpartitions to try to generalize both Gordon's theorem for overpartitions and the Andrews-Gordon identities.

Our main result is the following and is proved totally combinatorially:

Theorem 1.4.

- *Let $\overline{B}_{k,i}(n,j)$ be the number of overpartitions of $n$ of the form $(\lambda_1, \lambda_2, \ldots, \lambda_s)$ with $j$ overlined parts and where $\lambda_\ell - \lambda_{\ell+k-1} \geq 1$ if $\lambda_{\ell+k-1}$ is overlined and $\lambda_\ell - \lambda_{\ell+k-1} \geq 2$ otherwise and at most $i-1$ parts are equal to 1.*
- *Let $\overline{C}_{k,i}(n,j)$ be the number of overpartitions of $n$ with $j$ non-overlined parts in the bottom row of their Frobenius representation and whose successive ranks lie in $[-i+2, 2k-i-1]$.*
- *Let $\overline{D}_{k,i}(n,j)$ be the number of overpartitions of $n$ with $j$ overlined parts and $i-1$ successive Durfee squares followed by $k-i$ successive Durfee rectangles, the first one being a generalized Durfee square/rectangle.*
- *Let $\overline{E}_{k,i}(n,j)$ be the number of paths that use four kinds of unitary steps with special $(k,i)$-conditions, major index $n$, and $j$ South steps.*

*Then $\overline{B}_{k,i}(n,j) = \overline{C}_{k,i}(n,j) = \overline{D}_{k,i}(n,j) = \overline{E}_{k,i}(n,j)$.*

We use the classical $q$-series notations : $(a)_\infty = (a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i)$, $(a)_n = (a)_\infty/(aq^n)_\infty$ and $(a_1,\ldots,a_k;q)_\infty = (a_1;q)_\infty \ldots (a_k;q)_\infty$. The generating function $\overline{\mathcal{E}}_{k,i}(a,q) = \sum_{n,j} \overline{E}_{k,i}(n,j) q^n a^j$ is :

Theorem 1.5.

$$\overline{\mathcal{E}}_{k,i}(a,q) = \frac{(-aq)_\infty}{(q)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n a^n q^{kn^2+(k-i+1)n} \frac{(-1/a)_n}{(-aq)_n}. \tag{1.1}$$

In some cases, we can use the Jacobi Triple Product identity [**23**]:

$$(-1/z, -zq, q; q)_\infty = \sum_{n=-\infty}^{\infty} z^n q^{\binom{n+1}{2}}$$

and show that this generating function has a very nice form. For example,

COROLLARY 1.1.

$$\overline{\mathcal{E}}_{k,i}(0,q) = \frac{(q^i, q^{2k+1-i}, q^{2k+1}; q^{2k+1})_\infty}{(q)_\infty} \tag{1.2}$$

$$\overline{\mathcal{E}}_{k,i}(1/q,q^2) = \frac{(q^2;q^4)_\infty (q^{2i-1}, q^{4k+1-2i}, q^{4k}; q^{4k})_\infty}{(q)_\infty} \tag{1.3}$$

$$\overline{\mathcal{E}}_{k,i}(1,q) = \frac{(-q)_\infty}{(q)_\infty} \sum_{j=0}^{2(k-i)} (-1)^j (q^{i+j}, q^{2k-i-j}, q^{2k}; q^{2k})_\infty \tag{1.4}$$

$$\overline{\mathcal{E}}_{k,i}(1/q,q) = \frac{(-q)_\infty}{(q)_\infty} \left( (q^i, q^{2k-i}, q^{2k}; q^{2k})_\infty + (q^{i-1}, q^{2k+1-i}, q^{2k}; q^{2k})_\infty \right) \tag{1.5}$$

Hence our result gives a general view of different problems on partitions and overpartitions and shows how they are related.

- The case $a \to 0$ corresponds to the Andrews-Gordon identities [**4**].
- The case $q \to q^2$ and $a \to 1/q$ corresponds to Andrew's generalization of the Gordon-Göllnitz identities [**5, 7**].
- The cases $a \to 1$ and $i = k$ and $a \to 1/q$ and $i = 1$ correspond to the two Gordon's theorems for overpartitions of Lovejoy [**25**].

Therefore our extension of the work on the Andrews-Gordon identities [**7, 12, 14, 15**] to the case of overpartitions includes these identities, but it also includes Andrew's generalization of the Gordon-Göllnitz identities and Gordon's theorems for overpartitions.

We start by some definitions in Section 2. In Section 3 we present the paths counted by $\overline{E}_{k,i}(n,j)$ and compute the generating function. In Section 4 we present a direct bijection between the paths counted by $\overline{E}_{k,i}(n,j)$ and the overpartitions counted by $\overline{C}_{k,i}(n,j)$. In Section 5 we present a recursive bijection between the paths counted by $\overline{E}_{k,i}(n,j)$ and the overpartitions counted by $\overline{B}_{k,i}(n,j)$. We also give a generating function proof. In Section 6, we present a combinatorial argument that shows that the paths counted by $\overline{E}_{k,i}(n,j)$ and the overpartitions counted by $\overline{D}_{k,i}(n,j)$ are in bijection. All these bijections are refinements of Theorem 1.4. The number of the peaks of the paths will correspond respectively to the number of columns of the Frobenius representations, the number of weighted pairs and the size of the generalized Durfee square. We conclude in Section 7 with open further questions.

Due to the length of this extended abstract, we will most of the time present the sketch of the proofs. More details can be found in [**19, 30**].

## 2. Definitions on overpartitions

We will define all the notions in terms of overpartitions. We refer to [**2**] for definitions for partitions. In all of the cases the definitions coincide when the overpartition has no overlined parts.

An overpartition of $n$ is a non-increasing sequence of natural numbers whose sum is $n$ in which the final occurrence (equivalently, the first occurrence) of a number may be overlined. Alternatively $n$ can be called the weight of the overpartition. Since the overlined parts form a partition into distinct parts and the non-overlined parts form an ordinary partition, the generating function of overpartitions is $\frac{(-q)_\infty}{(q)_\infty}$.

The *multiplicity* of the part $j$ of an overpartition, denoted by $f_j$, is the number of occurrences of this part. We overline the multiplicity if the part appears overlined. For example, the multiplicity of the part 4 in the overpartition $(6, 6, 5, 4, 4, \overline{4}, 3, \overline{1})$ is $f_4 = \overline{3}$. The multiplicity sequence is the sequence $(f_1, f_2, \ldots)$. For example the previous overpartition has multiplicity sequence $(\overline{1}, 0, 1, \overline{3}, 1, 2)$.

The Frobenius representation of an overpartition [**16, 27**] of $n$ is a two-rowed array

$$\begin{pmatrix} a_1 & a_2 & \ldots & a_N \\ b_1 & b_2 & \ldots & b_N \end{pmatrix}$$

where $(a_1, \ldots, a_N)$ is a partition into distinct nonnegative parts and $(b_1, \ldots, b_N)$ is an overpartition into nonnegative parts where the first occurrence of a part can be overlined and $N + \sum(a_i + b_i) = n$.

We now define the successive ranks.

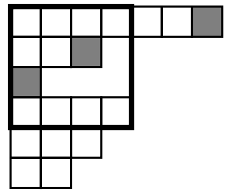

Figure 1. The generalized Durfee square of $\lambda = (\overline{7}, 4, 3, \overline{3}, 2, \overline{1})$ has side 4.

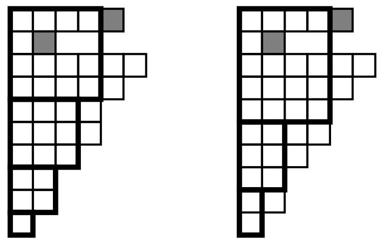

Figure 2. Successive Durfee squares and successive Durfee rectangles of $(6, 5, \overline{5}, 4, 4, 3, 2, 2, \overline{2}, 1)$.

Definition 2.1. The *successive ranks* of an overpartition can be defined from its Frobenius representation. If an overpartition has Frobenius representation $\begin{pmatrix} a_1 & a_2 & \cdots & a_N \\ b_1 & b_2 & \cdots & b_N \end{pmatrix}$ then its $i$th successive rank $r_i$ is $a_i - b_i$ minus the number of non-overlined parts in $\{b_{i+1}, \dots, b_N\}$.

This definition in an extension of Lovejoy's definition of the rank [**27**]. For example, the successive ranks of $\begin{pmatrix} 7 & 4 & 2 & 0 \\ \overline{3} & 3 & 1 & \overline{0} \end{pmatrix}$ are $(2, 0, 1, 0)$.

We say that the *generalized Durfee square* of an overpartition $\lambda$ has side $N$ if $N$ is the largest integer such that the number of overlined parts plus the number of non-overlined parts greater or equal to $N$ is greater than or equal to $N$ (see Figure 1). Thanks to the Algorithm Z [**8**], we can easily show that there exists a bijection between overpartitions whose Frobenius representation has $N$ columns and whose bottom line has $j$ overlined parts and overpartitions with generalized Durfee square of size $N$ and $N - j$ overlined parts. See [**19**] for details. The generating function of overpartitions with generalized Durfee square of size $N$ where the exponent of $q$ counts the weight and the exponent of $a$ the number of overlined parts is

$$\frac{a^N q^{\binom{N+1}{2}} (-1/a)_N}{(q)_N (q)_N}.$$

Definition 2.2. The *successive Durfee squares* of an overpartition are its generalized Durfee square and the successive Durfee squares of the partition below the generalized Durfee square, if we represent the partition as in Figure 1, with the overlined parts above the non-overlined ones. We can also define similarly the *successive Durfee rectangles* by dissecting the overpartition with $d \times (d+1)$-rectangles instead of squares.

These definitions imply that

$$\sum_{n_1 \geq \ldots \geq n_{k-1} \geq 0} \frac{q^{\binom{n_1+1}{2} + n_i + \ldots + n_{k-1}} (-1/a)_{n_1} a^{n_1}}{(q)_{n_1}} \left( q^{n_2^2} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}_q \right) \left( q^{n_3^2} \begin{bmatrix} n_2 \\ n_3 \end{bmatrix}_q \right) \cdots \left( q^{n_{k-1}^2} \begin{bmatrix} n_{k-2} \\ n_{k-1} \end{bmatrix}_q \right) \tag{2.1}$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q)_n}{(q)_k (q)_{n-k}}$$

is the generating function of overpartitions with $i - 1$ successive Durfee squares followed by $k - i$ successive Durfee rectangles, the first one being a generalized Durfee square/rectangle.

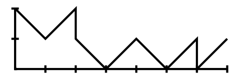

FIGURE 3. This path has four peaks : two NES peaks (located at $(2,2)$ and $(6,1)$) and two NESE peaks (located at $(4,1)$ and $(7,1)$). Its major index is $2+4+6+7=19$.

## 3. Paths and generating function

This part is an extension of papers of Andrews and Bressoud [**7, 12**] based on ideas of Burge [**12**]. We study paths in the first quadrant that use four kinds of unitary steps :

- North-East $NE$ : $(x,y)\to(x+1,y+1)$,
- South-East $SE$ : $(x,y)\to(x+1,y-1)$,
- South $S$ : $(x,y)\to(x,y-1)$,
- East $E$ : $(x,0)\to(x+1,0)$.

The *height* corresponds to the $y$-coordinate. A South step can only appear after a North-East step and an East step can only appear at height 0. The paths must end with a North-East or South step. A *peak* is a vertex preceeded by a North-East step and followed by a South step (in which case it will be called a *NES peak*) or by a South-East step (in which case it will be called a *NESE peak*). If the path ends with a North-East step, its last vertex is also a NESE peak. The *major index* of a path is the sum of the $x$-coordinates of its peaks (see Figure 3 for an example). When the paths have no South steps, this is the definition of the paths in [**12**].

Let $\overline{E}_{k,i}(n,j,N)$ be the number of such paths of major index $n$ with $N$ peaks, $j$ South steps that start at height $k-i$ and whose height is less than $k$. Let $\overline{\mathcal{E}}_{k,i}(N)$ be the generating function of those paths, that is $\overline{\mathcal{E}}_{k,i}(N)=\overline{\mathcal{E}}_{k,i}(N,a,q)=\sum_{n,j}\overline{E}_{k,i}(n,j,N)a^jq^n$.

Then

PROPOSITION 3.1.

$$\begin{aligned}
\overline{\mathcal{E}}_{k,i}(N) &= q^N\overline{\mathcal{E}}_{k,i+1}(N)+q^N\overline{\Gamma}_{k,i-1}(N); \qquad i<k\\
\overline{\Gamma}_{k,i}(N) &= q^N\overline{\Gamma}_{k,i-1}(N)+(a+q^{N-1})\overline{\mathcal{E}}_{k,i+1}(N-1); \quad 0<i<k\\
\overline{\mathcal{E}}_{k,k}(N) &= \frac{q^N}{1-q^N}\overline{\Gamma}_{k,k-1}(N)\\
\overline{\mathcal{E}}_{k,i}(0)=1 &\qquad \overline{\Gamma}_{k,0}(N)=0
\end{aligned}$$

PROOF. We prove that by induction on the length of the path. If the path is empty, then its major index is 0 and $N=0$. Moreover if $N=0$ the only path counted in $\overline{\mathcal{E}}_{k,i}(0)$ is the empty path. If the path is not empty, then we take off its first step. If $i<k$, then a path counted in $\overline{\mathcal{E}}_{k,i}(N)$ starts with a North-East (defined by $q^N\overline{\Gamma}_{k,i-1}(N)$) or a South-East step ($q^N\overline{\mathcal{E}}_{k,i+1}(N)$). If $i>0$, $\overline{\Gamma}_{k,i}(N)$ is the generating function of paths counted in $\overline{\mathcal{E}}_{k,i+1}(N)$ where the first North-East step was deleted. These paths can start with a North-East step ($q^N\overline{\Gamma}_{k,i-1}(N)$), a South step ($a\overline{\mathcal{E}}_{k,i+1}(N-1)$) or a South-East step ($q^{N-1}\overline{\mathcal{E}}_{k,i+1}(N-1)$). If $i=k$ then a path counted in $\overline{\mathcal{E}}_{k,k}(N)$ starts with a North-East ($q^N\overline{\Gamma}_{k,k-1}(N)$) or an East step ($q^N\overline{\mathcal{E}}_{k,k}(N)$). The height of the paths is less than $k$, therefore no path which starts at height $k-1$ can start with a North-East step and $\overline{\Gamma}_{k,0}(N)=0$. □

These recurrences uniquely define the series $\overline{\mathcal{E}}_{k,i}(N)$ and $\overline{\Gamma}_{k,i}(N)$. We get that :

THEOREM 3.1.

$$\begin{aligned}
\overline{\mathcal{E}}_{k,i}(N) &= a^Nq^{\binom{N+1}{2}}(-1/a)_N\sum_{n=-N}^{N}(-1)^n\frac{q^{kn^2+n(k-i)-\binom{n}{2}}}{(q)_{N-n}(q)_{N+n}}\\
\overline{\Gamma}_{k,i}(N) &= a^Nq^{\binom{N}{2}}(-1/a)_N\sum_{n=-N}^{N-1}(-1)^n\frac{q^{kn^2+n(k-i)-\binom{n+1}{2}}}{(q)_{N-n-1}(q)_{N+n}}
\end{aligned}$$

The proof is omitted. It uses simple algebraic manipulation to prove that these generating functions satisfy the recurrence relations of Proposition 3.1.

We just need a proposition (which is in fact a aprticular case of the $q$-Gauss identity [**23**]) that can be proved combinatorially and analytically [**19**] to prove Theorem 1.5.

Proposition 3.2. *For any $n \in \mathbb{Z}$*

$$\sum_{N \geq |n|} \frac{(-azq)_n(-q^n/a)_{N-n} q^{\binom{N+1}{2}-\binom{n+1}{2}} z^{N-n} a^{N-n}}{(zq)_{N+n}(q)_{N-n}} = \frac{(-azq)_\infty}{(zq)_\infty}.$$

Summing on $N$ using the previous proposition we get

$$\sum_{N \geq 0} \overline{\mathcal{E}}_{k,i}(N) = \frac{(-aq)_\infty}{(q)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n a^n q^{kn^2+(k-i+1)n} \frac{(-1/a)_n}{(-aq)_n}.$$

This is equation (1.1).

## 4. Paths and successive ranks

This section is a generalization of Bressoud's correspondence for partitions presented in [**12**]. The aim of this section is the following:

Proposition 4.1. *There exists a one-to-one correspondence between the paths of major index $n$ with $j$ south steps counted by $\overline{E}_{k,i}(n,j)$ and the overpartitions of $n$ zith $j$ non-overlined parts in the bottom line of their Frobenius representation and whose successive ranks lie in $[-i+2, 2k-i-1]$ counted by $\overline{C}_{k,i}(n,j)$. This correspondence is such that the paths have $N$ peaks if and only if the Frobenius representation of the overpartition has $N$ columns.*

Given a lattice path which starts at $(0,a)$ and a peak $(x,y)$ with $u$ South steps to its left, we map this peak to the pair $(s,t)$ where

$$s = (x+a-y+u)/2$$
$$t = (x-a+y-2-u)/2$$

if there are an even number of East steps to the left of the peak, and

$$s = (x+a+y-1+u)/2$$
$$t = (x-a-y-1-u)/2$$

if there are an odd number of East steps to the left of the peak. Moreover, we overline $t$ if the peak is a NESE peak. In both cases, $s$ and $t$ are integers and we have $s+t+1 = x$. In the case of partitions treated in [**12**], $u$ is always 0.

Let $N$ be the number of peaks in the path and $j$ the number of South steps of the paths. If the $i$th peak from the right has coordinates $(x_i, y_i)$ and the corresponding pair is $(s_i, t_i)$, then we show in [**19**] that the sequence $(s_1, s_2, \ldots, s_N)$ is a partition into distinct nonnegative parts and the sequence $(t_1, t_2, \ldots, t_N)$ is an overpartition into nonnegative parts with $j$ non-overlined parts. Therefore $\begin{pmatrix} s_1 & s_2 & \cdots & s_N \\ t_1 & t_2 & \cdots & t_N \end{pmatrix}$ is the Frobenius representation of an overpartition whose weight is

$$\sum_{i=1}^{N} (s_i + t_i + 1) = \sum_{i=1}^{N} x_i$$

i.e. the major index of the corresponding path.

As an example, the path in Figure 4 corresponds to the partition $\begin{pmatrix} 14 & 11 & 6 & 4 & 2 \\ 7 & \overline{6} & \overline{5} & 4 & \overline{3} \end{pmatrix}$.

The peaks all have height at least one, thus for a peak $(x,y)$ which is preceeded by an even number of East steps, we have :

$$\begin{aligned} & 1 \leq y = a+1+t-s+u \\ \Leftrightarrow\ & s-t-u \leq a \\ \Leftrightarrow\ & \text{the corresponding successive rank is } \leq a \end{aligned}$$

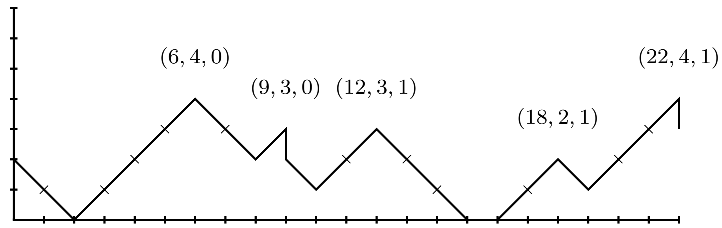


FIGURE 4. Illustration of the correspondence between paths and successive ranks. The values of $x$, $y$ and $u$ are given for each peak.

and if the peak is preceeded by and odd number of East steps, we have :

$$\begin{aligned} & 1 \leq y = s - t - u - a \\ \Leftrightarrow\ & s - t - u \geq a + 1 \\ \Leftrightarrow\ & \text{the corresponding successive rank is} \geq a + 1 \end{aligned}$$

Thus, given a Frobenius representation of an overpartition and a nonnegative integer $a$, there is a unique corresponding path which starts at $(0, a)$.

In our paths, all peaks have height at most $k-1$ and $a = k-i$, therefore in the first case the successive rank $r \in [-i+2, k-i]$ and in the second case $r \in [k-i+1, 2k-i-1]$.

The map is easily reversible. This proves Proposition 4.1.

## 5. Paths and multiplicities

Recall that $\overline{B}_{k,i}(n,j)$ is the number of overpartitions $\lambda$ of $n$ with $j$ overlined parts such that for all $\ell$,

$$\begin{cases} \lambda_\ell - \lambda_{\ell+k-1} \leq \begin{cases} 1 & \text{if } \lambda_{\ell+k-1} \text{ is overlined} \\ 2 & \text{otherwise} \end{cases} \\ f_1 < i \end{cases}$$

or equivalently

$$\begin{cases} \forall \ell,\ f_\ell + f_{\ell+1} \leq \begin{cases} k+1 & \text{if a part } \ell \text{ is overlined} \\ k & \text{otherwise} \end{cases} \\ f_1 < i \end{cases}$$

The aim of this section is the following:

PROPOSITION 5.1. *There exists a one-to-one correspondence between the paths counted by $\overline{E}_{k,i}(n,j)$ and the overpartitions counted by $\overline{B}_{k,i}(n,j)$. This correspondence is such that the paths have $N$ peaks if and only if the overpartition has $N$ weighted pairs.*

We will first give a generating function proof of that proposition (without the refinement). Then we will give a combinatorial proof which is a generalization of Burge's correspondence for partitions presented in [**14**].

### 5.1. A generating function proof.

Let $\overline{\mathcal{B}}_{k,i}(a,q) = \sum_{n \geq 0} \overline{B}_{k,i}(n,j) a^j q^n$. We prove that

PROPOSITION 5.2.

$$\overline{\mathcal{B}}_{k,i}(a,q) = \overline{\mathcal{E}}_{k,i}(a,q)$$

PROOF. We generalize Lovejoy's proof of Theorem 1.1 of [**25**]. Let

$$\begin{aligned} J_{k,i}(a,x,q) &= H_{k,i}(a,xq,q) - axqH_{k,i-1}(a,xq,q) \\ H_{k,i}(a,x,q) &= \sum_{n=0}^{\infty} \frac{x^{kn} q^{kn^2+n-in} a^n (1 - x^i q^{2ni})(axq^{n+1})_\infty (1/a)_n}{(q)_n (xq^n)_\infty}. \end{aligned}$$

Andrews showed in [**2**, p. 106-107] that for $2 \leq i \leq k$,

$$\begin{aligned} J_{k,i}(a,x,q) - J_{k,i-1}(a,x,q) &= (xq)^{i-1} J_{k,k-i+1}(a,xq,q) - a(xq)^{i-1} J_{k,k-i+2}(a,xq,q) \\ J_{k,1}(a,x,q) &= J_{k,k}(a,x,q). \end{aligned}$$

This functional equation of $J_{k,i}(a,x,q)$ implies that

$$\overline{\mathcal{B}}_{k,i}(a,q) = J_{k,i}(-a,1,q).$$

Hence

$$\begin{aligned}
\overline{\mathcal{B}}_{k,i}(a,q) &= \frac{(-aq)_\infty}{(q)_\infty}\sum_{n=0}^{\infty}(-1)^n a^n \frac{q^{kn^2+n(k-i+1)}(-1/a)_n(1-q^{(2n+1)i})}{(-aq)_{n+1}} \\
&\quad +aq\frac{(-aq)_\infty}{(q)_\infty}\sum_{n=0}^{\infty}(-1)^n a^n \frac{q^{kn^2+n(k-i+2)}(-1/a)_n(1-q^{(2n+1)(i-1)})}{(-aq)_{n+1}} \\
&= \frac{(-aq)_\infty}{(q)_\infty}\sum_{n=0}^{\infty}(-1)^n a^n \frac{q^{kn^2+n(k+1)}(-1/a)_n(q^{-in}+aq^{1-(i-1)n})}{(-aq)_{n+1}} \\
&\quad -\frac{(-aq)_\infty}{(q)_\infty}\sum_{n=0}^{\infty}(-1)^n a^n \frac{q^{kn^2+n(k+1)}(-1/a)_n(q^{(n+1)i}+aq^{(n+1)(i-1)+1})}{(-aq)_{n+1}} \\
&= \frac{(-aq)_\infty}{(q)_\infty}\left(\sum_{n=0}^{\infty}(-1)^n a^n \frac{q^{kn^2+n(k+1-i)}(-1/a)_n}{(-aq)_n} - \sum_{n=0}^{\infty}(-1)^n a^{n+1} \frac{q^{kn^2+n(k+i)+i}(-1/a)_{n+1}}{(-aq)_{n+1}}\right) \\
&= \frac{(-aq)_\infty}{(q)_\infty}\left(\sum_{n=0}^{\infty}(-1)^n a^n \frac{q^{kn^2+n(k+1-i)}(-1/a)_n}{(-aq)_n} + \sum_{n=-\infty}^{-1}(-1)^n a^{-n} \frac{q^{kn^2+n(k-i)}(-1/a)_{-n}}{(-aq)_{-n}}\right) \\
&= \frac{(-aq)_\infty}{(q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n a^n \frac{q^{kn^2+n(k+1-i)}(-1/a)_n}{(-aq)_n} = \overline{\mathcal{E}}_{k,i}(a,q)
\end{aligned}$$

□

**5.2. A combinatorial proof.** This part is a generalization of [**14**, Section 3]. Like Burge, we define operations on overpartitions represented by their multiplicity sequence.

The operation $\alpha$ is defined as follows. We divide the overpartition into $(\ell+1)$-tuples of the form $(f_m,\dots,f_{m+\ell})$ with $\ell \geq 1$ starting at the smallest part. When we find a multiplicity $f_m > 0$, we open a parenthesis to its left. If $f_m$ is not overlined then we close the parenthesis to the right of $f_{m+1}$. Otherwise, we look for the next non-overlined multiplicity, say $f_p$. If $f_p = 0$ then we close the parenthesis to its right, otherwise we close the parenthesis to the right of $f_{p+1}$. Then we look for the next positive multiplicity, and so on. Finally, for each $(\ell+1)$-tuple $(f_m,\dots,f_{m+\ell})$, we do :

- $f_m \leftarrow f_m - 1$
- $f_{m+\ell} \leftarrow f_{m+\ell} + 1$
- if $f_m$ is overlined, we remove its overlining and we overline the smallest non-overlined multiplicity in the $(\ell+1)$-tuple.

The operation $\beta$ (resp. $\delta$) consists in setting $f_0 = 1$ (resp. $f_0 = \overline{1}$) and applying $\alpha$.

The inverse operation $\alpha^{-1}$ is performed by first dividing the overpartition into $(\ell+1)$-tuples of the form $(f_m,\dots,f_{m+\ell})$, with $\ell \geq 1$ starting at the largest part, such that :

- $f_{m+\ell} > 0$
- $f_m$ is not overlined
- $f_{m+p}$ is overlined for $1 \leq p \leq \ell - 1$

(for an example, see the first line of Table 1, which corresponds to the overpartition $(5,5,\overline{5},\overline{4},3,\overline{2})$) and then doing for each $(\ell+1)$-tuple :

- if $f_{m+\ell} = \overline{1}$ :
  - remove the overlining of $f_{m+\ell}$
  - underline $f_m$
- else if $\ell > 1$ :
  - remove the overlining of $f_{m+\ell-1}$
  - underline $f_m$
- $f_{m+\ell} \leftarrow f_{m+\ell} - 1$
- $f_m \leftarrow f_m + 1$

| Operation | $N$ | $i$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|---|
| $\alpha^{-1}$ | 3 | 1 | 0 | (0 | $\overline{1}$) | (1 | $\overline{1}$ | $\overline{3}$) |
| $\delta^{-1}$ | 3 | 2 | (0 | $\overline{1}$) | (0 | $\overline{2}$) | (1 | $\overline{2}$) |
| $\alpha^{-1}$ | 2 | 2 | 0 | 0 | (1 | $\overline{1}$) | (2 | $\overline{1}$) |
| $\alpha^{-1}$ | 2 | 3 | 0 | (0 | $\overline{2}$) | (0 | $\overline{3}$) | |
| $\alpha^{-1}$ | 2 | 4 | 0 | (1 | $\overline{1}$) | (1 | $\overline{2}$) | |
| $\beta^{-1}$ | 2 | 4 | (0 | $\overline{2}$) | 0 | (2 | $\overline{1}$) | |
| $\delta^{-1}$ | 2 | 3 | (0 | $\overline{1}$) | (0 | $\overline{3}$) | | |
| $\alpha^{-1}$ | 1 | 3 | 0 | 0 | (1 | $\overline{2}$) | | |
| $\alpha^{-1}$ | 1 | 4 | 0 | 0 | (2 | $\overline{1}$) | | |
| $\alpha^{-1}$ | 1 | 4 | 0 | (0 | $\overline{3}$) | | | |
| $\alpha^{-1}$ | 1 | 4 | 0 | (1 | $\overline{2}$) | | | |
| $\alpha^{-1}$ | 1 | 4 | 0 | (2 | $\overline{1}$) | | | |
| $\beta^{-1}$ | 1 | 4 | (0 | $\overline{3}$) | | | | |
| $\beta^{-1}$ | 1 | 3 | (0 | $\overline{2}$) | | | | |
| $\delta^{-1}$ | 1 | 2 | (0 | $\overline{1}$) | | | | |
| | 0 | 2 | 0 | | | | | |

TABLE 1. Reduction of the overpartition $(5,5,\overline{5},\overline{4},3,\overline{2})$.

If there is an $(\ell+1)$-tuple $(f_0,\dots,f_\ell)$, the operation $\alpha^{-1}$ will produce a zero part, which may be overlined or not. The operation $\beta^{-1}$ (resp. $\delta^{-1}$) consists in applying $\alpha^{-1}$ and removing the non-overlined (resp. overlined) zero part.

The inverse operations allow us to define a reduction process for overpartitions which is similar to Burge's reduction for partitions [**14**]. An example is shown on Table 1.

Let $\overline{B}_{k,i}(n,j,N)$ be the number of partitions counted by $\overline{B}_{k,i}(n,j)$ such that $N=\sum_{(\ell+1)-tuples}\ell$. We call $N$ the number of weighted pairs (for partitions, we always have $\ell=1$ and $N$ is the number of pairs [**14**]). Let $\overline{\mathcal{B}}_{k,i}(N)=\sum_{n,j}\overline{B}_{k,i}(n,j,N)q^na^j$. Starting with an overpartition counted in $\overline{\mathcal{B}}_{k,i}(N)$, when we apply the reduction the weight will decrease by $N$. We can only apply a $\beta^{-1}$ or $\delta^{-1}$ if $i>0$. We show in [**19**] that when we apply $\alpha^{-1}$ (resp. $\beta^{-1}$, (resp. $\delta^{-1}$)), $N$ stays the same (resp. stays the same or decreases by 1 [in which case the next reduction is an $\alpha^{-1}$], (resp. decreases by 1)) and $i$ increases by 1 (resp. decreases by 1, (resp. stays the same)). These observations imply that $\overline{\mathcal{B}}_{k,i}(N)$ satisfies exactly the same recurrences relations as $\overline{\mathcal{E}}_{k,i}(N)$ defined in Proposition 3.1. Therefore $\overline{\mathcal{B}}_{k,i}(N)=\overline{\mathcal{E}}_{k,i}(N)$. This proves Proposition 5.1.

## 6. Paths and successive Durfee squares

We will prove here that

PROPOSITION 6.1.

$$\frac{q^{\binom{n_1+1}{2}+n_2^2+\cdots+n_{k-1}^2+n_i+\cdots+n_{k-1}}(-1/a)_{n_1}a^{n_1}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-2}-n_{k-1}}(q)_{n_{k-1}}}$$

*is the generating function of the paths counted by major index and number of South steps starting at height $k-i$, whose height is less than $k$ and having $n_j$ peaks of relative height $\geq j$ for $1\leq j\leq k-1$.*

The relative height of a peak was defined by Bressoud in [**12**] when he proved that

LEMMA 6.1 (Bressoud).

$$\frac{q^{n_1^2+n_2^2+\cdots+n_{k-1}^2+n_i+\cdots+n_{k-1}}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-2}-n_{k-1}}(q)_{n_{k-1}}}$$

*is the generating function of the paths with no South steps starting at height $k-i$, whose height is less than $k$ and having $n_j$ peaks of relative height $\geq j$ for $1\leq j\leq k-1$.*

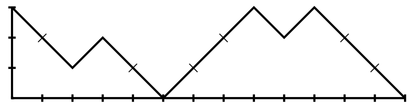

Figure 5. Example of a path.

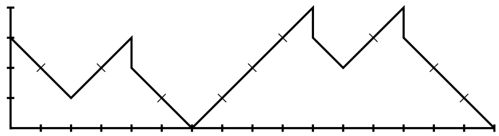

Figure 6. Effect of the "volcanic uplift".

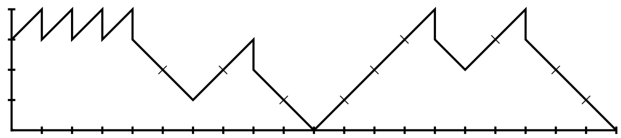

Figure 7. After adding the $n_1 - n_2 = 4$ NES peaks of relative height one.

An example of such a path, taken from [**12**], is shown on Figure 5.

A mountain in a path is a portion of the path that starts at the beginning of the path or at height 0 stays above the $x$-axis and ends at height 0. We recall Bressoud's definition of the relative height of a peak [**12**]. We first map each peak of the path to a pair $(y, y')$ where $y$ is the height of the peak and $y'$ is defined as follows. In each mountain, we choose the leftmost peak of maximal height relative to that mountain. For this peak, $y'$ is the minimal height over all vertices to its left. Then, if there are any unchosen peaks left, we cut all the mountains off at height one. This may divide some moutains into several mountains relative to height one. For each mountain relative to height one in which no peaks have been chosen, we choose the leftmost peak of maximal height relative to that mountain ; for this peak, $y'$ is the greater of one and the minimal height over all vertices to its left. We continue cutting the mountains off at height 2, 3, etc. until all peaks have been chosen.

Definition 6.2. [**12**] The *relative height of a peak* is then defined by $y - y'$.

This definition extends naturally to overpartitions. We can now move on to the proof of Proposition 6.1.

Proof. We prove the proposition using Bressoud's result. We consider a path counted by

$$\frac{q^{n_2^2+\cdots+n_{k-1}^2+n_i+\cdots+n_{k-1}}}{(q)_{n_2-n_3}\cdots(q)_{n_{k-2}-n_{k-1}}(q)_{n_{k-1}}}$$

where $2 \le i \le k$. Thanks to Lemma 6.1, we know that this path starts at height $k-i$, its height is less than $k-1$ and having $n_j$ peaks of relative height $\ge j-1$ for $2 \le j \le k-1$. We first insert a NES peak at each peak (see Figure 6). This "volcanic uplift" operation increases the weight of the path by

$$1 + 2 + \cdots + n_2 = \binom{n_2+1}{2}$$

and the relative height of each peak by one.

We then insert $n_1 - n_2$ NES peaks at the beginning of the path (see Figure 7). These new peaks have total weight $\binom{n_1-n_2+1}{2}$ and they increase the weight of each of the old peaks by $n_1 - n_2$. Altogether, the two operations introduce a factor

$$q^{\binom{n_2+1}{2}+\binom{n_1-n_2+1}{2}+n_2(n_1-n_2)} = q^{\binom{n_1+1}{2}}.$$

If $i = 2$, so that the path starts at $(0, k-2)$, we have the option to introduce an extra step at the beginning of the path, from $(0, k-1)$ to $(1, k-2)$. This introduces the factor $q^{n_1}$.

The factor $(-1)_{n_1}$ corresponds to a partition into distinct parts which lie in $[0, n_1 - 1]$. If this partition contains a part $j-1$ $(1 \le j \le n_1)$, we transform the $j$th NES peak from the right into a NESE peak (see Figure 8). This operation increases the weight of the path by $j-1$.

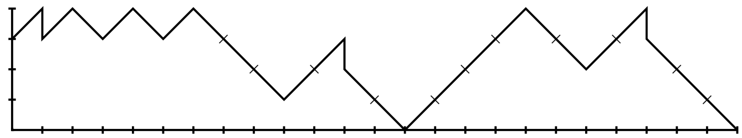

FIGURE 8. Effect of transforming some NES peaks into NESE peaks. The partition into distinct parts is $(5, 4, 3, 1)$.

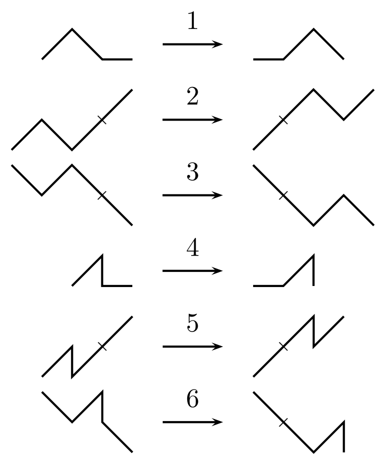


FIGURE 9. The rules for moving peaks.

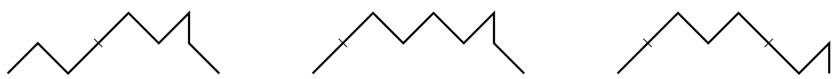

FIGURE 10. We want to move the leftmost peak to the right twice, but after the first move, we come up against a sequence of adjacent peaks. We then move the rightmost peak in this sequence.

The factor $\frac{1}{(q)_{n_1-n_2}}$ corresponds to a partition $(b_1, b_2, \ldots, b_{n_1-n_2})$ where $b_1 \geq b_2 \geq \ldots \geq b_{n_1-n_2} \geq 0$. For $1 \leq j \leq n_1 - n_2$, we move the $j$th peak of relative height one from the right $b_j$ times according to the rules illustrated in Figure 9. See [**19**] for details.

When we move a peak, it can meet the next peak to the right. We say that a peak $(x, y)$ meets a peak $(x', y')$ if

$$x' - x = \begin{cases} 2 \text{ if } (x, y) \text{ is a NESE peak} \\ 1 \text{ if } (x, y) \text{ is a NES peak} \end{cases} .$$

If this happens, we abandon the peak we have been moving and move the next one. If we come up against a sequence of adjacent peaks, we move the rightmost peak in the sequence (see Figure 10).

It can be shown that the distribution of relative heights is not modified by the operations of Figure 9 and that the construction procedure is uniquely reversible. □

The multiple series

$$\sum_{n_1 \geq \ldots \geq n_{k-1} \geq 0} \frac{q^{\binom{n_1+1}{2}+n_2^2+\cdots+n_{k-1}^2+n_i+\cdots+n_{k-1}}(-1/a)_{n_1} a^{n_1}}{(q)_{n_1-n_2} \cdots (q)_{n_{k-2}-n_{k-1}} (q)_{n_{k-1}}}$$

can be re-expressed as (2.1), which is the generating function of overpartitions with $i-1$ successive Durfee squares followed by $k-i$ successive Durfee rectangles, the first one being a generalized Durfee square/rectangle.

## 7. Conclusion

We showed in this work how the combinatorial interpretation of the Andrews-Gordon identities can be generalized to the case of overpartitions, when the combinatorial statistics (successive ranks, generalized Durfee square, weighted pairs) are defined properly. There exist other generalizations of the Rogers-Ramanujan identities, see for example [**13**]. It was shown that the combinatorial interpretation in terms of lattice paths can also be done for these identities [**1, 12, 14, 15**]. Our work can also be extended in that direction and the results are presented in [**18**]. Finally there exists an extension of the concept of successive ranks for partitions due to Andrews, Baxter, Bressoud, Burge, Forrester and Viennot [**6**] and our goal now is to extend that notion to overpartitions.

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