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Schur positivity and Cell Transfer

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Abstract. We give combinatorial proofs that certain families of differences of products of Schur functions are monomial-positive. We show in addition that such monomial-positivity is to be expected of a large class of generating functions with combinatorial definitions similar to Schur functions. These generating functions are defined on posets with labelled Hasse diagrams and include for example generating functions of Stanley's (P, ω)-partitions. Then we prove Okounkov's conjecture, a conjecture of Fomin-Fulton-Li-Poon, and a special case of Lascoux-Leclerc-Thibon's conjecture on Schur positivity and give several more general statements using a recent result of Rhoades and Skandera. An alternative proof of this result is provided. We also give an intriguing log-concavity property of Schur functions. This text contains the material from [**LP, LPP**].

Résumé. Nous prouvons combinatoirement que certaines familles de différences de produits de fonctions de Schur sont monomiales-positives. Nous montrons de plus que l'on peut attendre une telle propriété pour une importante classe de fonctions génératrices définies combinatoirement d'une façon similaire aux fonctions de Schur. Ces fonctions génératrices sont définies en termes d'ensembles partiellement ordonnés dont le diagramme de Hasse est étiqueté et comprennent par exemple la fonction génératrice des (P, ω)-partitions de Stanley. Nous prouvons aussi la conjecture d'Okounkov, une conjecture de Fomin-Fulton-Li-Poon, et un cas particulier de la conjecture de Lascoux-Leclerc-Thibon sur la positivité de Schur, et nous donnons plusieurs énoncés plus généraux en utilisant un résultat récent de Rhoades et Skandera. Nous donnons aussi une nouvelle preuve de ce résultat et une propriété surprenante de log-concavité des fonctions de Schur.

1. Schur positivity conjectures

The Schur functions s_λ form an orthonormal basis of the ring of symmetric functions Λ. They have a remarkable number of combinatorial and algebraic properties, and are simultaneously the irreducible characters of $GL(N)$ and representatives of Schubert classes in the cohomology $H^*(Gr_{kn})$ of the Grassmannian; see [**Mac, Sta**]. In recent years, a lot of work has gone into studying whether certain expressions of the form

$$s_\lambda s_\mu - s_\nu s_\rho \tag{1.1}$$

The first aim of this article is to provide a large class of expressions of the form (1.1) which are *monomial-positive*, that is, expressible as a non-negative linear combination of monomials. In particular, we show that (1.1) is monomial-positive when $\lambda = \nu \vee \rho$ and $\mu = \nu \wedge \rho$ are the union and intersections of the Young diagrams of ν and ρ. However, we show in addition that such monomial-positivity is to be expected of many families of generating functions with combinatorial definitions similar to Schur functions, which are generating functions for semistandard Young tableaux.

We define a new combinatorial object called a $\mathbb{T}$-*labelled poset* and given a $\mathbb{T}$-labelled poset (P, O) we define another combinatorial object which we call (P, O)-*tableaux*. These (P, O)-tableaux include as special cases standard Young tableaux, semistandard Young tableaux, cylindric tableaux, plane partitions, and Stanley's (P, ω)-partitions. Our main theorem is the *cell transfer theorem*. It says that for a fixed $\mathbb{T}$-labelled poset (P, O), one obtains many expressions of the form (1.1) which are monomial-positive, where the Schur functions in (1.1) are replaced by generating functions for (P, O)-tableaux.

Key words and phrases. Schur functions, Schur positivity, Schur log-concavity, immanants, Kazhdan-Lusztig polynomials, Temperley-Lieb algebra, minors.

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A symmetric function is called *Schur nonnegative* if it is a linear combination with nonnegative coefficients of the Schur functions, or, equivalently, if it is the character of a certain representation of GL_n. In particular, *skew Schur functions* $s_{\lambda/\mu}$ are Schur nonnegative. We prove that our cell-transfer results for Schur functions hold not just for monomial-positivity but also for Schur-positivity. In particular, we prove the following theorem.

For two partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ and $\mu = (\mu_1, \mu_2, \dots)$, let us define partitions

$$\lambda \vee \mu := (\max(\lambda_1, \mu_1), \max(\lambda_2, \mu_2), \dots)$$

and

$$\lambda \wedge \mu := (\min(\lambda_1, \mu_1), \min(\lambda_2, \mu_2), \dots).$$

The Young diagram of $\lambda \vee \mu$ is the set-theoretical union of the Young diagrams of λ and μ. Similarly, the Young diagram of $\lambda \wedge \mu$ is the set-theoretical intersection of the Young diagrams of λ and μ. For two skew shapes, define $(\lambda/\mu) \vee (\nu/\rho) := \lambda \vee \nu / \mu \vee \rho$ and $(\lambda/\mu) \wedge (\nu/\rho) := \lambda \wedge \nu / \mu \wedge \rho$.

Theorem 1.1. *Let λ/μ and ν/ρ be any two skew shapes. Then we have*

$$s_{(\lambda/\mu)\vee(\nu/\rho)}\, s_{(\lambda/\mu)\wedge(\nu/\rho)} \geq_s s_{\lambda/\mu}\, s_{\nu/\rho}.$$

Using this theorem, we prove the following several Schur positivity conjectures due to Okounkov, Fomin-Fulton-Li-Poon, and Lascoux-Leclerc-Thibon.

Okounkov [**Oko**] studied branching rules for classical Lie groups and proved that the multiplicities were "monomial log-concave" in some sense. An essential combinatorial ingredient in his construction was the theorem about monomial nonnegativity of some symmetric functions. He conjectured that these functions are Schur nonnegative, as well. For a partition λ with all even parts, let $\frac{\lambda}{2}$ denote the partition $(\frac{\lambda_1}{2}, \frac{\lambda_2}{2}, \dots)$. For two symmetric functions f and g, the notation $f \geq_s g$ means that $f - g$ is Schur nonnegative.

Conjecture 1.2. Okounkov [**Oko**] *For two skew shapes λ/μ and ν/ρ such that $\lambda + \nu$ and $\mu + \rho$ both have all even parts, we have $(s_{\frac{(\lambda+\nu)}{2}/\frac{(\mu+\rho)}{2}})^2 \geq_s s_{\lambda/\mu}\, s_{\nu/\rho}$.*

Fomin, Fulton, Li, and Poon [**FFLP**] studied the eigenvalues and singular values of sums of Hermitian and of complex matrices. Their study led to two combinatorial conjectures concerning differences of products of Schur functions. Let us formulate one of these conjectures, which was also studied recently by Bergeron and McNamara [**BM**]. For two partitions λ and μ, let $\lambda \cup \mu = (\nu_1, \nu_2, \nu_3, \dots)$ be the partition obtained by rearranging all parts of λ and μ in the weakly decreasing order. Let $\mathrm{sort}_1(\lambda, \mu) := (\nu_1, \nu_3, \nu_5, \dots)$ and $\mathrm{sort}_2(\lambda, \mu) := (\nu_2, \nu_4, \nu_6, \dots)$.

Conjecture 1.3. Fomin-Fulton-Li-Poon [**FFLP**, Conjecture 2.7] *For two partitions λ and μ, we have $s_{\mathrm{sort}_1(\lambda,\mu)} s_{\mathrm{sort}_2(\lambda,\mu)} \geq_s s_\lambda s_\mu$.*

Lascoux, Leclerc, and Thibon [**LLT**] studied a family of symmetric functions $\mathcal{G}^{(n)}_\lambda(q, x)$ arising combinatorially from ribbon tableaux and algebraically from the Fock space representations of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_n)$. They conjectured that $\mathcal{G}^{(n)}_{n\lambda}(q, x) \geq_s \mathcal{G}^{(m)}_{m\lambda}(q, x)$ for $m \leq n$. For the case $q = 1$, their conjecture can be reformulated, as follows. For a partition λ and $1 \leq i \leq n$, let $\lambda^{[i,n]} := (\lambda_i, \lambda_{i+n}, \lambda_{i+2n}, \dots)$. In particular, $\mathrm{sort}_i(\lambda, \mu) = (\lambda \cup \mu)^{[i,2]}$, for $i = 1, 2$.

Conjecture 1.4. Lascoux-Leclerc-Thibon [**LLT**, Conjecture 6.4] *For integers $1 \leq m \leq n$ and a partition λ, we have $\prod_{i=1}^n s_{\lambda^{[i,n]}} \geq_s \prod_{i=1}^m s_{\lambda^{[i,m]}}$.*

Theorem 1.5. *Conjectures 1.2, 1.3 and 1.4 are true.*

In Section 6, we present and prove more general versions of these conjectures.

2. Posets and Tableaux

Let $(P, \leq)$ be a possibly infinite poset. Let $s, t \in P$. We say that s covers t and write $s \gtrdot t$ if for any $r \in P$ such that $s \geq r \geq t$ we have $r = s$ or $r = t$. The *Hasse diagram* of a poset P is the graph with vertex set equal to the elements of P and edge set equal to the set of covering relations in P. If $Q \subset P$ is a subset of the elements of P then Q has a natural induced subposet structure. If $s, t \in Q$ then $s \leq t$ in Q if and only if $s \leq t$ in P. Call a subset $Q \subset P$ *connected* if the elements in Q induce a connected subgraph in the Hasse diagram of P.

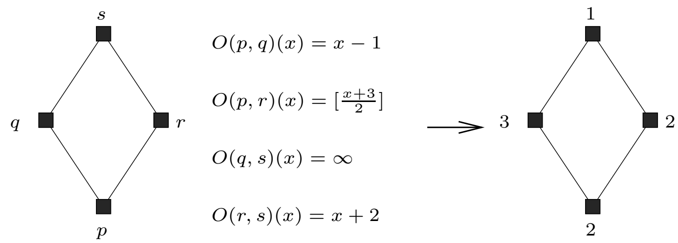


FIGURE 1. An example of a $\mathbb{T}$-labelled poset (P, O) and a (P, O)-tableaux.

An *order ideal* I of P is an induced subposet of P such that if $s \in I$ and $s \geq t \in P$ then $t \in I$. A subposet $Q \subset P$ is called *convex* if for any $s, t \in Q$ and $r \in P$ satisfying $s \leq r \leq t$ we have $r \in Q$. Alternatively, a convex subposet is one which is closed under taking intervals. A convex subset Q is determined by specifying two order ideals J and I so that $J \subset I$ and $Q = \{s \in I \mid s \notin J\}$. We write $Q = I/J$. If $s \notin Q$ then we write $s < Q$ if $s < t$ for some $t \in Q$ and similarly for $s > Q$. If $s \in Q$ or s is incomparable with all elements in Q we write $s \sim Q$. Thus for any $s \in P$, exactly one of $s < Q$, $s > Q$ and $s \sim Q$ is true.

Let $\mathbb{P}$ denote the set of positive integers and $\mathbb{Z}$ denote the set of integers. Let $\mathbb{T}$ denote the set of all weakly increasing functions $f : \mathbb{P} \to \mathbb{Z} \cup \{\infty\}$.

DEFINITION 2.1. A $\mathbb{T}$-*labelling* O of a poset P is a map $O : \{(s, t) \in P^2 \mid s \gtrdot t\} \to \mathbb{T}$ labelling each edge (s, t) of the Hasse diagram by a weakly increasing function $O(s, t) : \mathbb{P} \to \mathbb{Z} \cup \{\infty\}$. A $\mathbb{T}$-*labelled poset* is an an ordered pair (P, O) where P is a poset, and O is a $\mathbb{T}$-labelling of P.

We shall refer to a $\mathbb{T}$-labelled poset (P, O) as P when no ambiguity arises. If $Q \subset P$ is a convex subposet of P then the covering relations of Q are also covering relations in P. Thus a $\mathbb{T}$-labelling O of P naturally induces a $\mathbb{T}$-labelling $O|_Q$ of Q. We denote the resulting $\mathbb{T}$-labelled poset by $(Q, O) := (Q, O|_Q)$.

DEFINITION 2.2. A (P, O)-tableau is a map $\sigma : P \to \mathbb{P}$ such that for each covering relation $s \lessdot t$ in P we have
$$\sigma(s) \leq O(s, t)(\sigma(t)).$$
If $\sigma : P \to \mathbb{P}$ is any map, then we say that σ *respects* O if σ is a (P, O)-tableau.

Figure 1 contains an example of a $\mathbb{T}$-labelled poset (P, O) and a corresponding (P, O)-tableau.

Denote by $\mathcal{A}(P, O)$ the set of all (P, O)-tableaux. If P is finite then one can define the formal power series $K_{P,O}(x_1, x_2, \ldots) \in \mathbb{Q}[[x_1, x_2, \ldots]]$ by
$$K_{P,O}(x_1, x_2, \ldots) = \sum_{\sigma \in \mathcal{A}(P,O)} x_1^{\#\sigma^{-1}(1)} x_2^{\#\sigma^{-1}(2)} \cdots .$$
The composition $\mathrm{wt}(\sigma) = (\#\sigma^{-1}(1), \#\sigma^{-1}(2), \ldots)$ is called the weight of σ.

EXAMPLE 2.3. Any Young diagram $P = \lambda$ can be considered as a $\mathbb{T}$-labelled poset. Indeed, consider its cells to be elements of the poset, and let O be the labelling of the horizontal edges with the function $f^{\mathrm{weak}}(x) = x$ and label the vertical edges with the function $f^{\mathrm{strict}}(x) = x - 1$. A (λ, O)-tableau is just a semistandard Young tableaux and $K_{\lambda,O}(x_1, x_2, \cdots)$ is the Schur function $s_\lambda(x_1, x_2, \cdots)$.

EXAMPLE 2.4. Another interesting example are cylindric tableaux and cylindric Schur functions. Let $1 \leq k < n$ be two positive integers. Let $\mathbb{C}_{k,n}$ be the quotient of $\mathbb{Z}^2$ given by
$$\mathbb{C}_{k,n} = \mathbb{Z}^2/(k - n, k)/Z.$$
In other words, the integer points (a, b) and $(a + k - n, b + k)$ are identified in $\mathbb{C}_{k,n}$. We can give $\mathbb{C}_{k,n}$ the structure of a poset by the generating relations $(i, j) \lessdot (i+1, j)$ and $(i, j) \lessdot (i, j+1)$. We give $\mathbb{C}_{k,n}$ a $\mathbb{T}$-labelling O by labelling the edges $(i, j) \lessdot (i + 1, j)$ with the function $f^{\mathrm{weak}}(x) = x$ and the edges $(i, j) \lessdot (i, j + 1)$ with the function $f^{\mathrm{strict}}(x) = x - 1$. A finite convex subposet P of $\mathbb{C}_{k,n}$ is known as a *cylindric skew shape*; see [**GK, Pos, McN**]. The (P, O)-tableau are known as *semistandard cylindric tableaux* of shape P and the generating function $K_{P,O}(x_1, x_2, \cdots)$ is the cylindric Schur function defined in [**BS, Pos**].

Example 2.5. Let N be the number of elements in a poset P, and let $\omega : P \longrightarrow [N]$ be a bijective labelling of elements of P with numbers from 1 to N. Recall that a (P,ω)-partition (see [**Sta**]) is a map $\sigma : P \longrightarrow \mathbb{P}$ such that $s \le t$ in P implies $\sigma(s) \le \sigma(t)$, while if in addition $\omega(s) > \omega(t)$ then $\sigma(s) < \sigma(t)$. Label now each edge (s,t) of the Hasse diagram of P with f^{weak} or f^{strict}, depending on whether $\omega(s) \le \omega(t)$ or $\omega(s) > \omega(t)$ correspondingly. It is not hard to see that for this labelling O the (P,O)-tableaux are exactly the (P,ω)-partitions. Similarly, if we allow any labelling of the edges of P with f^{weak} and f^{strict}, we get the *oriented posets* of McNamara; see [**McN**].

3. The Cell Transfer Theorem

A generating function $f \in \mathbb{Q}[[x_1, x_2, \ldots]]$ is *monomial-positive* if all coefficients in its expansion into monomials are non-negative. If f is actually a symmetric function then this is equivalent to f being a non-negative linear combination of monomial symmetric functions.

Let (P,O) be a $\mathbb{T}$-labelled poset. Let Q and R be two finite convex subposets of P. The subset $Q \cap R$ is also a convex subposet. Define two convex subposets $Q \wedge R$ and $Q \vee R$ by

$$Q \wedge R = \{s \in R \mid s < Q\} \cup \{s \in Q \mid s \sim R \text{ or } s < R\} \tag{3.1}$$

and

$$Q \vee R = \{s \in Q \mid s > R\} \cup \{s \in R \mid s \sim Q \text{ or } s > Q\} \tag{3.2}$$

Recall that if A and B are sets then $A \backslash B = \{a \in A \mid a \notin B\}$ denotes the *set difference*.

Lemma 3.1. *The subposets $Q \wedge R$ and $Q \vee R$ are both convex subposets of P. We have $(Q \wedge R) \cup (Q \vee R) = Q \cup R$ and $(Q \wedge R) \cap (Q \vee R) = Q \cap R$.*

Proof. Suppose $s < t$ lie in $Q \wedge R$ and $s < r < t$ for some $r \in P$ but $r \notin Q \wedge R$. Then either $s \in R\backslash Q$ and $t \in Q\backslash R$ or $s \in Q\backslash R$ and $t \in R\backslash Q$. In the first case, since $t > s$ we must have $t > R$ which is impossible by definition. In the second case, we have $t > Q$ which is again impossible. The proof for $Q \vee R$ is analogous. The second statement of the lemma is straightforward. $\square$

Note that the operations $\wedge$ and $\vee$ are stable so that $(Q \wedge R) \wedge (Q \vee R) = Q \wedge R$ and $(Q \wedge R) \vee (Q \vee R) = Q \vee R$.

Theorem 3.2 (Cell Transfer Theorem). *The difference*

$$K_{Q \wedge R, O} K_{Q \vee R, O} - K_{Q,O} K_{R,O}$$

is monomial-positive.

Proof. We prove Theorem 3.2 by exhibiting an injection

$$\eta : \mathcal{A}(Q,O) \times \mathcal{A}(R,O) \longrightarrow \mathcal{A}(Q \wedge R, O) \times \mathcal{A}(Q \vee R, O)$$

which is weight preserving. We call this the *cell transfer procedure*. The name comes from our main examples where elements of a poset are cells of a Young diagram. For convenience, in this paper we call elements of any poset *cells*.

Let ω be a (Q,O)-tableau and σ be a (R,O)-tableau. We now describe how to construct a $(Q \wedge R, O)$-tableau $\omega \wedge \sigma$ and a $(Q \vee R, O)$-tableau $\omega \vee \sigma$. Define a subset of $Q \cap R$, depending on ω and σ, by

$$(Q \cap R)^+ = \{x \in Q \cap R \mid \omega(x) < \sigma(x)\}.$$

We give $(Q \cap R)^+$ the structure of a graph by inducing from the Hasse diagram of $Q \cap R$.

Let $\mathrm{bd}(R) = \{x \in Q \cap R \mid x \gtrdot y \text{ for some } y \in R\backslash Q\}$ be the "lower boundary" of $Q \cap R$ which touches elements in R. Let $\mathrm{bd}(R)^+ \subset (Q \cap R)^+$ be the union of the connected components of $(Q \cap R)^+$ which contain an element of $\mathrm{bd}(R)$. Similarly, let $\mathrm{bd}(Q) = \{x \in Q \cap R \mid x \lessdot y \text{ for some } y \in Q\backslash R\}$ be the "upper boundary" of $Q \cap R$ which touches elements in Q. Let $\mathrm{bd}(Q)^+ \subset (Q \cap R)^+$ be the union of the connected components of $(Q \cap R)^+$ which contain an element of $\mathrm{bd}(Q)$. The elements in $\mathrm{bd}(Q)^+ \cup \mathrm{bd}(R)^+$ are amongst the cells that we might "transfer".

Let $S \subset Q \cap R$. Define $(\omega \wedge \sigma)_S : Q \wedge R \to \mathbb{P}$ by

$$(\omega \wedge \sigma)_S(x) = \begin{cases} \sigma(x) & \text{if } x \in R\backslash Q \text{ or } x \in S, \\ \omega(x) & \text{otherwise.} \end{cases}$$

And define $(\omega \vee \sigma)_S : Q \vee R \to \mathbb{P}$ by

$$(\omega \vee \sigma)_S(x) = \begin{cases} \omega(x) & \text{if } x \in Q \backslash R \text{ or } x \in S, \\ \sigma(x) & \text{otherwise.} \end{cases}$$

One checks directly that $\mathrm{wt}(\sigma) + \mathrm{wt}(\omega) = \mathrm{wt}((\omega \wedge \sigma)_S) + \mathrm{wt}((\omega \vee \sigma)_S)$. We claim that when $S = S^* := \mathrm{bd}(Q)^+ \cup \mathrm{bd}(R)^+$, both $(\omega \wedge \sigma)^*_S$ and $(\omega \vee \sigma)_{S^*}$ respect O. We check this for $(\omega \wedge \sigma)_{S^*}$ and the claim for $(\omega \vee \sigma)_{S^*}$ follows from symmetry.

Let $s \lessdot t$ be a covering relation in $Q \wedge R$. Since σ and ω are assumed to respect O, we need only check the conditions when $(\omega \wedge \sigma)_{S^*}(s) = \omega(s)(\neq \sigma(s))$ and $(\omega \wedge \sigma)_{S^*}(t) = \sigma(t)(\neq \omega(t))$; or when $(\omega \wedge \sigma)_{S^*}(s) = \sigma(s)(\neq \omega(s))$ and $(\omega \wedge \sigma)_{S^*}(t) = \omega(t)(\neq \sigma(t))$.

In the first case, we must have $s \in Q$ and $t \in R$. If $t \in R$ but $t \notin Q$ then by the definition of $Q \wedge R$ we must have $t < Q$ and so $t < t'$ for some $t' \in Q$. This is impossible since Q is convex. Thus $t \in Q \cap R$ and so $t \in S^*$. We compute that $\omega(s) \leq O(s,t)(\omega(t)) \leq O(s,t)(\sigma(t))$ since $\omega(t) < \sigma(t)$ and $O(s,t)$ is weakly increasing.

In the second case, we must have $s \in R$ and $t \in Q$. By the definition of $Q \wedge R$ we must have $t \in R$ as well. So $t \in Q \cap R$ but $t \notin S^*$ which means that $\omega(t) > \sigma(t)$. Thus $\sigma(s) \leq O(s,t)(\sigma(t)) \leq O(s,t)(\omega(t))$ and $\omega \wedge \sigma$ respects O here.

For each (ω, σ), say a subset $S \subseteq S^*$ is *transferrable* if both $(\omega \wedge \sigma)_S$ and $(\omega \vee \sigma)_S$ respect O. If S' and S'' are both transferrable then it is easy to check that so is $S' \cap S''$. Thus there exists a unique smallest transferrable subset $S^\diamond \subseteq S^*$. Now define $\eta : \mathcal{A}(Q,O) \times \mathcal{A}(R,O) \to \mathcal{A}(Q \wedge R, O) \times \mathcal{A}(Q \vee R, O)$ by

$$(\omega, \sigma) \longmapsto ((\omega \wedge \sigma)_{S^\diamond}, (\omega \vee \sigma)_{S^\diamond}).$$

Note that $S^\diamond$ depends on ω and σ, though we have suppressed the dependence from the notation.

We now show that this η is injective. Given $(\alpha, \beta) \in \eta(\mathcal{A}(Q,O) \times \mathcal{A}(R,O))$, we show how to recover ω and σ. As before, for a subset $S \subset Q \cap R$, define $\omega_S = \omega(\alpha,\beta)_S : Q \to \mathbb{P}$ by

$$\omega_S(x) = \begin{cases} \beta(x) & \text{if } x \in (Q \backslash R) \cap (Q \vee R) \text{ or } x \in S, \\ \alpha(x) & \text{otherwise.} \end{cases}$$

And define $\sigma_S = \sigma(\alpha,\beta)_S : R \to \mathbb{P}$ by

$$\sigma_S(x) = \begin{cases} \alpha(x) & \text{if } x \in (R \backslash Q) \cap (Q \wedge R) \text{ or } x \in S, \\ \beta(x) & \text{otherwise.} \end{cases}$$

Note that if $(\alpha, \beta) = ((\omega \wedge \sigma)_{S^\diamond}, (\omega \vee \sigma)_{S^\diamond}))$ then $\omega = \omega_{S^\diamond}$ and $\sigma = \sigma_{S^\diamond}$. Let $S^\square \subset Q \cap R$ be the unique smallest subset such that $\omega_{S^\square}$ and $\sigma_{S^\square}$ both respect O. Since we have assumed that $(\alpha, \beta) \in \eta(\mathcal{A}(Q,O) \times \mathcal{A}(R,O))$, such a $S^\square$ must exist. (As before the intersection of two transferrable subsets with respect to (α, β) is transferrable.)

We now show that if $(\alpha, \beta) = ((\omega \wedge \sigma)_{S^\diamond}, (\omega \vee \sigma)_{S^\diamond}))$ then $S^\square = S^\diamond$. By definition, $S^\square \subset S^\diamond$. Let $C \subset S^\diamond \backslash S^\square$ be a connected component of $S^\diamond \backslash S^\square$, viewed as an induced subgraph of the Hasse diagram of P. We claim that $S^\diamond \backslash C$ is a transferrable set for (ω, σ); this means that changing $\alpha|_C$ to $\omega|_C$ and $\beta|_C$ to $\sigma|_C$ gives a pair in $\mathcal{A}(Q \wedge R, O) \times \mathcal{A}(Q \vee R, O)$. Suppose first that $c \in C$ and $s \in S^\square$ is so that $c \lessdot s$. By the definition of $S^\square$, we must have $\alpha(c) \leq O(c,s)(\beta(s))$ and $\beta(c) \leq O(c,s)(\alpha(s))$. Now suppose that $c \in C$ and $s \in Q \backslash R$ such that $c \lessdot s$. Then we must have $O(c,s)(\omega(s)) = O(c,s)(\beta(s)) \geq \alpha(c) = \sigma(c)$. Similar conclusions hold for $c \gtrdot s$. Thus we have checked that $S^\diamond \backslash C$ is a transferrable set for (ω, σ).

This shows that the map $(\omega, \sigma) \mapsto ((\omega \wedge \sigma)_{S^\diamond}, (\omega \vee \sigma)_{S^\diamond}))$ is injective, completing the proof. $\square$

On Figure 2 we can see how shapes $P \vee Q$ and $P \wedge Q$ are formed in the case of SSYT. On Figure 3 an example of cell transfer for those shapes is given. Note that $S^\diamond$ does not contain one cell which is is in S^*.

Note that $(\omega, \sigma) \mapsto ((\omega \wedge \sigma)_{S^*}, (\omega \vee \sigma)_{S^*})$ also defines a weight-preserving map $\eta^* : \mathcal{A}(Q,O) \times \mathcal{A}(R,O) \to \mathcal{A}(Q \wedge R, O) \times \mathcal{A}(Q \vee R, O)$. Unfortunately, η^* is not always injective.

Suppose P is a locally-finite poset with a minimal element. Let $J(P)$ be the lattice of finite order ideals of P; see [**Sta**]. If $I, J \in J(P)$ then the sets $I \wedge J$ and $I \vee J$ defined in (3.1) and (3.2) are finite order ideals and agree with the the meet $\wedge_{J(P)}$ and join $\vee_{J(P)}$ of I and J respectively within $J(P)$. In fact, by defining $(Q \wedge R)' = \{s \in R \mid s < Q\} \cup \{s \in Q \mid s \in R \text{ or } s < R\}$ and $(Q \vee R)' = \{s \in Q \mid s \sim R\} \cup \{s \in R \mid s \sim$

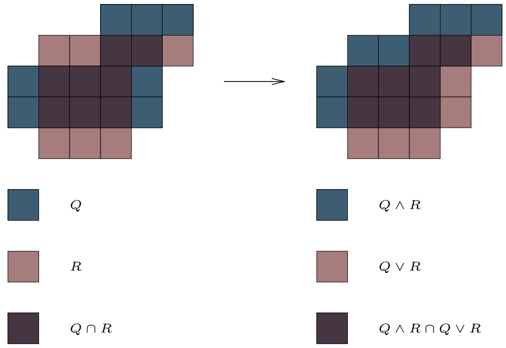


Figure 2. An example of $P \wedge Q$ and $P \vee Q$ for semistandard Young tableaux.

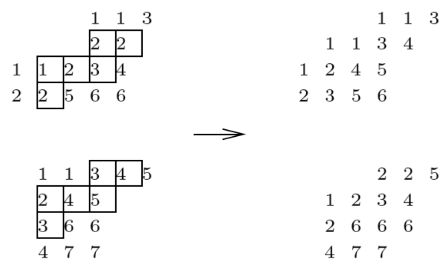


Figure 3. An example of cell transfer for semistandard Young tableaux, cells in $S^\diamond$ are marked.

Q or $s > Q\}$, the order ideals $(I \wedge J)' = I \wedge_{J(P)} J$ and $(I \vee J)' = I \vee_{J(P)} J$ agree with the meet and join in $J(P)$ even when P does not contain a minimal element.

Corollary 3.3. *Let P be a locally-finite poset and $I, J \in J(P)$. Then the generating function*

$$K_{I \wedge_{J(P)} J, O} K_{I \vee_{J(P)} J, O} - K_{I,O} K_{J,O}$$

is monomial-positive.

Proof. The elements altered going from $(Q \wedge R)$ to $(Q \wedge R)'$ do not involve the intersection $Q \cap R$, and in fact are incomparable to the elements of $Q \cap R$. The cells being transferred in the proof of Theorem 3.2 are not affected by changing $(Q \wedge R)$ to $(Q \wedge R)'$ and changing $(Q \vee R)$ to $(Q \vee R)'$. Thus the same proof works here. $\square$

4. Background for Schur positivity proof

In this section we give an overview of some results of Haiman [**Hai**] and Rhoades-Skandera [**RS2, RS1**]. We include an alternative proof Rhoades-Skandera's result.

4.1. Haiman's Schur positivity result. Let $H_n(q)$ be the *Hecke algebra* associated with the symmetric group S_n. The Hecke algebra has the standard basis $\{T_w \mid w \in S_n\}$ and the *Kazhdan-Lusztig basis* $\{C'_w(q) \mid w \in S_n\}$ related by

$$q^{l(v)/2} C'_v(q) = \sum_{w \leq v} P_{w,v}(q)\, T_w \quad \text{and} \quad T_w = \sum_{v \leq w} (-1)^{l(vw)} Q_{v,w}(q)\, q^{l(v)/2} C'_v(q),$$

where $P_{w,v}(q)$ are the *Kazhdan-Lusztig polynomials* and $Q_{v,w}(q) = P_{w_\circ w, w_\circ v}(q)$, for the longest permutation $w_\circ \in S_n$, see [**Hum**] for more details.

For $w \in S_n$ and a $n \times n$ matrix $X = (x_{ij})$, the *Kazhdan-Lusztig immanant* was defined in [**RS2**] as

$$\mathrm{Imm}_w(X) := \sum_{v \in S_n} (-1)^{l(vw)} Q_{w,v}(1)\, x_{1,v(1)} \cdots x_{n,v(n)},$$

Let $h_k = \sum_{i_1 \le \cdots \le i_k} x_{i_1} \cdots x_{i_k}$ be the k-th homogeneous symmetric function, where $h_0 = 1$ and $h_k = 0$ for $k < 0$. A *generalized Jacobi-Trudi matrix* is a $n \times n$ matrix of the form $\left(h_{\mu_i - \nu_j}\right)_{i,j=1}^{n}$, for partitions $\mu = (\mu_1 \ge \mu_2 \cdots \ge \mu_n \ge 0)$ and $\nu = (\nu_1 \ge \nu_2 \cdots \ge \nu_n \ge 0)$. Haiman's result can be reformulated as follows, see [**RS2**].

THEOREM 4.1. Haiman [**Hai**, Theorem 1.5] *The immanants* Imm_w *of a generalized Jacobi-Trudi matrix are Schur non-negative.*

Haiman's proof of this result is based on the Kazhdan-Lusztig conjecture proven by Beilinson-Bernstein and Brylinski-Kashiwara. This conjecture expresses the characters of Verma modules as sums of the characters of some irreducible highest weight representations of $\mathfrak{sl}_n$ with multiplicities equal to $P_{w,v}(1)$. One can derive from this conjecture that the coefficients of Schur functions in Imm_w are certain tensor product multiplicities of irreducible representations.

4.2. Temperley-Lieb algebra. The *Temperley-Lieb algebra* $TL_n(\xi)$ is the $\mathbb{C}[\xi]$-algebra generated by $t_1, \dots, t_{n-1}$ subject to the relations $t_i^2 = \xi\, t_i$, $t_i t_j t_i = t_i$ if $|i-j| = 1$, $t_i t_j = t_j t_i$ if $|i-j| \ge 2$. The dimension of $TL_n(\xi)$ equals the n-th Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. A *321-avoiding permutation* is a permutation $w \in S_n$ that has no reduced decomposition of the form $w = \cdots s_i s_j s_i \cdots$ with $|i-j| = 1$. (These permutations are also called *fully-commutative.*) A natural basis of the Temperley-Lieb algebra is $\{t_w \mid w \text{ is a 321-avoiding permutation in } S_n\}$, where $t_w := t_{i_1} \cdots t_{i_l}$, for a reduced decomposition $w = s_{i_1} \cdots s_{i_l}$.

The map $\theta : T_{s_i} \mapsto t_i - 1$ determines a homomorphism $\theta : H_n(1) = \mathbb{C}[S_n] \to TL_n(2)$. Indeed, the elements $t_i - 1$ in $TL_n(2)$ satisfy the Coxeter relations.

THEOREM 4.2. Fan-Green [**FG**] *The homomorphism* θ *acts on the Kazhdan-Lusztig basis* $\{C'_w(1)\}$ *of* $H_n(1)$ *as follows:*

$$\theta(C'_w(1)) = \begin{cases} t_w & \text{if } w \text{ is 321-avoiding,} \\ 0 & \text{otherwise.} \end{cases}$$

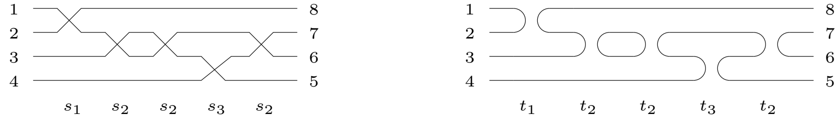
For any permutation $v \in S_n$ and a 321-avoiding permutation $w \in S_n$, let $f_w(v)$ be the coefficient of the basis element $t_w \in TL_n(2)$ in the basis expansion of $\theta(T_v) = (t_{i_1} - 1) \cdots (t_{i_l} - 1) \in TL_n(2)$, for a reduced decomposition $v = s_{i_1} \cdots s_{i_l}$. Rhoades and Skandera [**RS1**] defined the *Temperley-Lieb immanant* $\mathrm{Imm}_w^{\mathrm{TL}}(x)$ of an $n \times n$ matrix $X = (x_{ij})$ by

$$\mathrm{Imm}_w^{\mathrm{TL}}(X) := \sum_{v \in S_n} f_w(v)\, x_{1,v(1)} \cdots x_{n,v(n)}.$$

THEOREM 4.3. Rhoades-Skandera [**RS1**] *For a 321-avoiding permutation* $w \in S_n$*, we have* $\mathrm{Imm}_w^{\mathrm{TL}}(X) = \mathrm{Imm}_w(X)$.

PROOF. Applying the map θ to $T_v = \sum_{w \le v} (-1)^{l(vw)} Q_{w,v}(1)\, C'_w(1)$ and using Theorem 4.2 we obtain $\theta(T_v) = \sum (-1)^{l(vw)} Q_{w,v}(1)\, t_w$, where the sum is over 321-avoiding permutations w. Thus $f_w(v) = (-1)^{l(vw)} Q_{w,v}(1)$ and $\mathrm{Imm}_w^{\mathrm{TL}} = \mathrm{Imm}_w$. $\square$

A product of generators (decomposition) $t_{i_1} \cdots t_{i_l}$ in the Temperley-Lieb algebra TL_n can be graphically presented by a *Temperley-Lieb diagram* with n non-crossing strands connecting the vertices $1, \dots, 2n$ and, possibly, with some internal loops. This diagram is obtained from the wiring diagram of the decomposition $w = s_{i_1} \cdots s_{i_l} \in S_n$ by replacing each crossing "$\times$" with a *vertical uncrossing* ")(". For example, the following figure shows the wiring diagram for $s_1 s_2 s_2 s_3 s_2 \in S_4$ and the Temperley-Lieb diagram for $t_1 t_2 t_2 t_3 t_2 \in TL_4$.



Pairs of vertices connected by strands of a wiring diagram are $(2n+1-i, w(i))$, for $i = 1, \ldots, n$. Pairs of vertices connected by strands in a Temperley-Lieb diagram form a *non-crossing matching*, i.e., a graph on the vertices $1, \ldots, 2n$ with n disjoint edges that contains no pair of edges (a, c) and (b, d) with $a < b < c < d$. If two Temperley-Lieb diagrams give the same matching and have the same number of internal loops, then the corresponding products of generators of TL_n are equal to each other. If the diagram of a is obtained from the diagram of b by removing k internal loops, then $b = \xi^k a$ in TL_n.

The map that sends t_w to the non-crossing matching given by its Temperley-Lieb diagram is a bijection between basis elements t_w of TL_n, where w is 321-avoiding, and non-crossing matchings on the vertex set $[2n]$. For example, the basis element $t_1t_3t_2$ of TL_4 corresponds to the non-crosssing matching with the edges $(1, 2), (3, 4), (5, 8), (6, 7)$.

4.3. An identity for products of minors. For a subset $S \subset [2n]$, let us say that a Temperley-Lieb diagram (or the associated element in TL_n) is *S-compatible* if each strand of the diagram has one end-point in S and the other end-point in its complement $[2n] \setminus S$. Coloring vertices in S black and the remaining vertices white, a basis element t_w is S-compatible if and only if each edge in the associated matching has two vertices of different colors. Let $\Theta(S)$ denote the set of all 321-avoiding permutation $w \in S_n$ such that t_w is S-compatible.

For two subsets $I, J \subset [n]$ of the same cardinality let $\Delta_{I,J}(X)$ denote the *minor* of an $n \times n$ matrix X in the row set I and the column set J. Let $\bar{I} := [n] \setminus I$ and let $I^\wedge := \{2n + 1 - i \mid i \in I\}$.

THEOREM 4.4. Rhoades-Skandera [**RS1**, Proposition 4.3], cf. Skandera [**Ska**] *For two subsets $I, J \subset [n]$ of the same cardinality and $S = J \cup (\bar{I})^\wedge$, we have*

$$\Delta_{I,J}(X) \cdot \Delta_{\bar{I},\bar{J}}(X) = \sum_{w \in \Theta(S)} \mathrm{Imm}_w^{\mathrm{TL}}(X).$$

The proof given in [**RS1**] employs planar networks. We give a more direct proof that uses the involution principle.

PROOF. Let us fix a permutation $v \in S_n$ with a reduced decomposition $v = s_{i_1} \cdots s_{i_l}$. The coefficient of the monomial $x_{1,v(1)} \cdots x_{n,v(n)}$ in the expansion of the product of two minors $\Delta_{I,J}(X) \cdot \Delta_{\bar{I},\bar{J}}(X)$ equals

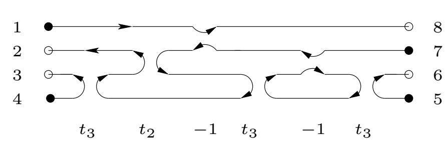
$$\begin{cases} (-1)^{\mathrm{inv}(I)+\mathrm{inv}(\bar{I})} & \text{if } v(I) = J, \\ 0 & \text{if } v(I) \neq J, \end{cases}$$

where $\mathrm{inv}(I)$ is the number of inversions $i < j$, $v(i) > v(j)$ such that $i, j \in I$.

On the other hand, by the definition of $\mathrm{Imm}_w^{\mathrm{TL}}$, the coefficient of $x_{1,v(1)} \cdots x_{n,v(n)}$ in the right-hand side of the identity equals the sum $\sum (-1)^r\, 2^s$ over all diagrams obtained from the wiring diagram of the reduced decomposition $s_{i_1} \cdots s_{i_l}$ by replacing each crossing “ $\times$ ” with either a *vertical uncrossing* “) (” or a *horizontal uncrossing* “ $\asymp$ ” so that the resulting diagram is S-compatible, where r is the number of horizontal uncrossings “ $\asymp$ ” and s is the number of internal loops in the resulting diagram. Indeed, the choice of “) (” corresponds to the choice of “t_{i_k}” and the choice of “ $\asymp$ ” corresponds to the choice of “-1” in the k-th term of the product $(t_{i_1} - 1) \cdots (t_{i_l} - 1) \in TL_n(2)$, for $k = 1, \ldots, l$.

Let us pick directions of all strands and loops in such diagrams so that the initial vertex in each strand belongs to S (and, thus, the end-point is not in S). There are 2^s ways to pick directions of s internal loops. Thus the above sum can be written as the sum $\sum (-1)^r$ over such *directed Temperley-Lieb* diagrams.

Here is an example of a directed diagram for $v = s_3s_2s_1s_3s_2s_3$ and $S = \{1, 4, 5, 7\}$ corresponding to the term $t_3t_2(-1)t_3(-1)t_3$ in the expansion of the product $(t_3 - 1)(t_2 - 1)(t_1 - 1)(t_3 - 1)(t_2 - 1)(t_3 - 1)$. This diagram comes with the sign $(-1)^2$.



Let us construct a sign reversing partial involution ι on the set of such directed Temperley-Lieb diagrams. If a diagram has a *misaligned uncrossing*, i.e., an uncrossing of the form “) (”, “) (”, “ $\asymp$ ”, or “ $\asymp$ ”, then ι switches the leftmost such uncrossing according to the rules $\iota :$) ($\leftrightarrow$ $\asymp$ and $\iota :$) ($\leftrightarrow$ $\asymp$. Otherwise, when the diagram involves only *aligned uncrossings* “ ”, “ ”, “ ”, “ ”, the involution ι is not defined.

For example, in the above diagram, the involution ι switches the second uncrossing, which has the form “ ”, to “ ”. The resulting diagram corresponds to the term $t_3(-1)(-1)t_3(-1)t_3$.

Since the involution ι reverses signs, this shows that the total contribution of all diagrams with at least one misaligned uncrossing is zero. Let us show that there is at most one S-compatible directed Temperley-Lieb diagram with all aligned uncrossings. If we have a such diagram, then we can direct the strands of the wiring diagram for $v = s_{i_1} \dots s_{i_l}$ so that each segment of the wiring diagram has the same direction as in the Temperley-Lieb diagram. In particular, the end-points of strands in the wiring diagram should have different colors. Thus each strand starting at an element of J should finish at an element of $I^\wedge$, or, equivalently, $v(I) = J$. The directed Temperley-Lieb diagram can be uniquely recovered from this directed wiring diagram by replacing the crossings with uncrossings, as follows: $\to$, $\to$, $\to$, $\to$. Thus the coefficient of $x_{1,v(1)} \cdots x_{n,v(n)}$ in the right-hand side of the needed identity is zero, if $v(I) \neq J$, and is $(-1)^r$, if $v(I) = J$, where r is the number of crossings of the form “ ” or “ ” in the wiring diagram. In other words, r equals the number of crossings such that the right end-points of the pair of crossing strands have the same color. This is exactly the same as the expression for the coefficient in the left-hand side of the needed identity. $\square$

5. Proof of Theorem 1.1

For two subsets $I, J \subseteq [n]$ of the same cardinality, let $\Delta_{I,J}(H)$ denote the minor of the Jacobi-Trudi matrix $H = (h_{j-i})_{1 \leq i,j \leq n}$ with row set I and column set J, where h_i is the i-th homogeneous symmetric function, as before. According to the Jacobi-Trudi formula, see [**Mac**], the minors $\Delta_{I,J}(H)$ are precisely the skew Schur functions

$$\Delta_{I,J}(H) = s_{\lambda/\mu},$$

where $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k \geq 0)$, $\mu = (\mu_1 \geq \cdots \geq \mu_k \geq 0)$ and the associated subsets are $I = \{\mu_k + 1 < \mu_{k-1} + 2 < \cdots < \mu_1 + k\}$, $J = \{\lambda_k + 1 < \lambda_{k-1} + 2 < \cdots < \lambda_1 + k\}$.

For two sets $I = \{i_1 < \cdots < i_k\}$ and $J = \{j_1 < \cdots < j_k\}$, let us define $I \vee J := \{\max(i_1, j_1) < \cdots < \max(i_k, j_k)\}$ and $I \wedge J := \{\min(i_1, j_1) < \cdots < \min(i_k, j_k)\}$.

Theorem 1.1 can be reformulated in terms of minors, as follows. Without loss of generality we can assume that all partitions λ, μ, ν, ρ in Theorem 1.1 have the same number k of parts, some of which might be zero. Note that generalized Jacobi-Trudi matrices are obtained from H by skipping or duplicating rows and columns.

Theorem 5.1. *Let I, J, I', J' be k element subsets in $[n]$. Then we have*

$$\Delta_{I \vee I',\, J \vee J'}(X) \cdot \Delta_{I \wedge I',\, J \wedge J'}(X) \geq_s \Delta_{I,J}(X) \cdot \Delta_{I',J'}(X),$$

for a generalized Jacobi-Trudi matrix X.

Proof. Let us denote $\bar{I} := [n] \setminus I$ and $\check{S} := [2n] \setminus S$. By skipping or duplicating rows and columns of the matrix X, we may assume that $I' = \bar{I}$ and $J' = \bar{J}$. Then $I \vee I' = \overline{I \wedge I'}$ and $J \vee J' = \overline{J \wedge J'}$. Let $S := J \cup (\bar{I})^\wedge$ and $T := (J \vee J') \cup (\overline{I \vee I'})^\wedge$. Then we have $T = S \vee \check{S}$ and $\check{T} = S \wedge \check{S}$.

Let us show that $\Theta(S) \subseteq \Theta(T)$, i.e., every S-compatible non-crossing matching on $[2n]$ is also T-compatible. Let $S = \{s_1 < \cdots < s_n\}$ and $\check{S} = \{\check{s}_1 < \cdots < \check{s}_n\}$. Then $T = \{\max(s_1, \check{s}_1), \dots, \max(s_n, \check{s}_n)\}$ and $\check{T} = \{\min(s_1, \check{s}_1), \dots, \min(s_n, \check{s}_n)\}$. Let M be an S-compatible non-crossing matching on $[2n]$ and let $(a < b)$ be an edge of M. Without loss of generality we may assume that $a = s_i \in S$ and $b = \check{s}_j \in \check{S}$. We must show that either ($a \in T$ and $b \in \check{T}$) or ($a \in \check{T}$ and $b \in T$). Since no edge of M can cross (a, b), the elements of S in the interval $[a + 1, b - 1]$ are matched with the elements of $\check{S}$ in this interval. Let $k = \#(S \cap [a+1, b-1]) = \#(\check{S} \cap [a+1, b-1])$. Suppose that $a, b \in T$, or, equivalently, $\check{s}_i < s_i$ and $s_j < \check{s}_j$. Since there are at least k elements of $\check{S}$ in the interval $[\check{s}_i + 1, \check{s}_j - 1]$, we have $i + k + 1 \leq j$. On the other hand, since there are at most $k - 1$ elements of S in the interval $[s_i + 1, s_j - 1]$, we have $i + k \geq j$. We obtain a contradiction. The case $a, b \in \check{T}$ is analogous.

Now Theorem 4.4 implies that the difference $\Delta_{I \vee I',\, J \vee J'} \cdot \Delta_{I \wedge I',\, J \wedge J'} - \Delta_{I,J} \cdot \Delta_{I',J'}$ is a nonnegative combination of Temperley-Lieb immanants. Theorems 4.1 and 4.3 imply its Schur nonnegativity. $\square$

6. Proof of conjectures and generalizations

In this section we prove generalized versions of Conjectures 1.2-1.4, which were conjectured by Kirillov [**Kir**, Section 6.8]. Corollary 6.2 was also conjectured by Bergeron-McNamara [**BM**, Conjecture 5.2] who showed that it implies Theorem 6.3.

Let $\lfloor x \rfloor$ be the maximal integer $\leq x$ and $\lceil x \rceil$ be the minimal integer $\geq x$. For vectors v and w and a positive integer n, we assume that the operations $v+w$, $\frac{v}{n}$, $\lfloor v \rfloor$, $\lceil v \rceil$ are performed coordinate-wise. In particular, we have well-defined operations $\lfloor \frac{\lambda+\nu}{2} \rfloor$ and $\lceil \frac{\lambda+\nu}{2} \rceil$ on pairs of partitions.

The next claim extends Okounkov's conjecture (Conjecture 1.2).

Theorem 6.1. *Let λ/μ and ν/ρ be any two skew shapes. Then we have*

$$s_{\lfloor \frac{\lambda+\nu}{2} \rfloor / \lfloor \frac{\mu+\rho}{2} \rfloor}\, s_{\lceil \frac{\lambda+\nu}{2} \rceil / \lceil \frac{\mu+\rho}{2} \rceil} \geq_s s_{\lambda/\mu}\, s_{\nu/\rho}.$$

Proof. We will assume that all partitions have the same fixed number k of parts, some of which might be zero. For a skew shape $\lambda/\mu = (\lambda_1, \ldots, \lambda_k)/(\mu_1, \ldots, \mu_k)$, define

$$\overrightarrow{\lambda/\mu} := (\lambda_1+1, \ldots, \lambda_k+1)/(\mu_1+1, \ldots, \mu_k+1),$$

that is, $\overrightarrow{\lambda/\mu}$ is the skew shape obtained by shifting the shape λ/μ one step to the right. Similarly, define the left shift of λ/μ by

$$\overleftarrow{\lambda/\mu} := (\lambda_1-1, \ldots, \lambda_k-1)/(\mu_1-1, \ldots, \mu_k-1),$$

assuming that the result is a legitimate skew shape. Note that $s_{\lambda/\mu} = s_{\overleftarrow{\lambda/\mu}} = s_{\overrightarrow{\lambda/\mu}}$.

Let θ be the operation on pairs of skew shapes given by

$$\theta : (\lambda/\mu, \nu/\rho) \longmapsto ((\lambda/\mu) \vee (\nu/\rho), (\lambda/\mu) \wedge (\nu/\rho)).$$

According to Theorem 1.1, the product of the two skew Schur functions corresponding to the shapes in $\theta(\lambda/\mu, \nu/\rho)$ is $\geq_s s_{\lambda/\mu}\, s_{\nu/\rho}$. Let us show that we can repeatedly apply the operation θ together with the left and right shifts of shapes and the flips $(\lambda/\mu, \nu/\rho) \mapsto (\nu/\rho, \lambda/\mu)$ in order to obtain the pair of skew shapes $(\lfloor \frac{\lambda+\nu}{2} \rfloor / \lfloor \frac{\mu+\rho}{2} \rfloor, \lceil \frac{\lambda+\nu}{2} \rceil / \lceil \frac{\mu+\rho}{2} \rceil)$ from $(\lambda/\mu, \nu/\rho)$.

Let us define two operations ϕ and ψ on ordered pairs of skew shapes by conjugating θ with the right and left shifts and the flips, as follows:

$$\phi : (\lambda/\mu, \nu/\rho) \longmapsto ((\lambda/\mu) \wedge (\overrightarrow{\nu/\rho}), \overleftarrow{(\lambda/\mu) \vee (\overrightarrow{\nu/\rho})}),$$

$$\psi : (\lambda/\mu, \nu/\rho) \longmapsto (\overleftarrow{(\overrightarrow{\lambda/\mu}) \vee (\nu/\rho)}, (\overrightarrow{\lambda/\mu}) \wedge (\nu/\rho)).$$

In this definition the application of the left shift "$\leftarrow$" always makes sense. Indeed, in both cases, before the application of "$\leftarrow$", we apply "$\rightarrow$" and then "$\vee$". As we noted above, both products of skew Schur functions for shapes in $\phi(\lambda/\mu, \nu/\rho)$ and in $\psi(\lambda/\mu, \nu/\rho)$ are $\geq_s s_{\lambda/\mu}\, s_{\nu/\rho}$.

It is convenient to write the operations ϕ and ψ in the coordinates $\lambda_i, \mu_i, \nu_i, \rho_i$, for $i = 1, \ldots, k$. These operations independently act on the pairs (λ_i, ν_i) by

$$\begin{aligned} \phi &: (\lambda_i, \nu_i) \mapsto (\min(\lambda_i, \nu_i+1), \max(\lambda_i, \nu_i+1)-1), \\ \psi &: (\lambda_i, \nu_i) \mapsto (\max(\lambda_i+1, \nu_i)-1, \min(\lambda_i+1, \nu_i)), \end{aligned}$$

and independently act on the pairs (μ_i, ρ_i) by exactly the same formulas. Note that both operations ϕ and ψ preserve the sums $\lambda_i + \nu_i$ and $\mu_i + \rho_i$.

The operations ϕ and ψ transform the differences $\lambda_i - \nu_i$ and $\mu_i - \rho_i$ according to the following piecewise-linear maps:

$$\bar{\phi}(x) = \begin{cases} x & \text{if } x \leq 1, \\ 2-x & \text{if } x \geq 1, \end{cases} \qquad \text{and} \qquad \bar{\psi}(x) = \begin{cases} x & \text{if } x \geq -1, \\ -2-x & \text{if } x \leq -1. \end{cases}$$

Whenever we apply the composition $\phi \circ \psi$ of these operations, all absolute values $|\lambda_i - \nu_i|$ and $|\mu_i - \rho_i|$ strictly decrease, if these absolute values are ≥ 2. It follows that, for a sufficiently large integer N, we have $(\phi \circ \psi)^N(\lambda/\mu, \nu/\rho) = (\tilde{\lambda}/\tilde{\mu}, \tilde{\nu}/\tilde{\rho})$ with $\tilde{\lambda}_i + \tilde{\nu}_i = \lambda_i + \nu_i$, $\tilde{\mu}_i + \tilde{\rho}_i = \mu_i + \rho_i$, and $|\tilde{\lambda}_i - \tilde{\nu}_i| \leq 1$, $|\tilde{\mu}_i - \tilde{\rho}_i| \leq 1$, for all i. Finally, applying the operation θ, we obtain $\theta(\tilde{\lambda}/\tilde{\mu}, \tilde{\nu}/\tilde{\rho}) = (\lceil \frac{\lambda+\nu}{2} \rceil / \lceil \frac{\mu+\rho}{2} \rceil, \lfloor \frac{\lambda+\nu}{2} \rfloor / \lfloor \frac{\mu+\rho}{2} \rfloor)$, as needed. $\square$

The following conjugate version of Theorem 6.1 extends Fomin-Fulton-Li-Poon's conjecture (Conjecture 1.3) to skew shapes.

Corollary 6.2. *Let λ/μ and ν/ρ be two skew shapes. Then we have*

$$s_{\mathrm{sort}_1(\lambda,\nu)/\mathrm{sort}_1(\mu,\rho)}\, s_{\mathrm{sort}_2(\lambda,\nu)/\mathrm{sort}_2(\mu,\rho)} \geq_s s_{\lambda/\mu}\, s_{\nu/\rho}.$$

Proof. This statement is obtained from Theorem 6.1 by conjugating the shapes. Indeed, $\lceil\frac{\lambda+\mu}{2}\rceil' = \mathrm{sort}_1(\lambda',\mu')$ and $\lfloor\frac{\lambda+\mu}{2}\rfloor' = \mathrm{sort}_2(\lambda',\mu')$. Here λ' denote the partition conjugate to λ. □

Theorem 6.3. *Let $\lambda^{(1)}/\mu^{(1)},\dots,\lambda^{(n)}/\mu^{(n)}$ be n skew shapes, let $\lambda = \bigcup\lambda^{(i)}$ be the partition obtained by the decreasing rearrangement of the parts in all $\lambda^{(i)}$, and, similarly, let $\mu = \bigcup\mu^{(i)}$. Then we have $\prod_{i=1}^n s_{\lambda^{[i,n]}/\mu^{[i,n]}} \geq_s \prod_{i=1}^n s_{\lambda^{(i)}/\mu^{(i)}}$.*

This theorem extends Corollary 6.2 and Conjecture 1.3. Also note that Lascoux-Leclerc-Thibon's conjecture (Conjecture 1.4) is a special case of Theorem 6.3 for the n-tuple of partitions $(\lambda^{[1,m]},\dots,\lambda^{[m,m]},\emptyset,\dots,\emptyset)$.

Proof. Let us derive the statement by applying Corollary 6.2 repeatedly. For a sequence $v = (v_1, v_2, \dots, v_l)$ of integers, the *anti-inversion number* is $\mathrm{ainv}(v) := \#\{(i,j) \mid i<j,\ v_i<v_j\}$. Let $L = (\lambda^{(1)}/\mu^{(1)},\dots,\lambda^{(n)}/\mu^{(n)})$ be a sequence of skew shapes. Define its anti-inversion number as

$$\begin{aligned}\mathrm{ainv}(L) &= \mathrm{ainv}(\lambda_1^{(1)},\lambda_1^{(2)},\dots,\lambda_1^{(n)},\lambda_2^{(1)},\dots,\lambda_2^{(n)},\lambda_3^{(1)},\dots,\lambda_3^{(n)},\dots)\\ &\quad+\mathrm{ainv}(\mu_1^{(1)},\mu_1^{(2)},\dots,\mu_1^{(n)},\mu_2^{(1)},\dots,\mu_2^{(n)},\mu_3^{(1)},\dots,\mu_3^{(n)},\dots).\end{aligned}$$

If $\mathrm{ainv}(L)\neq 0$ then there is a pair $k<l$ such that $\mathrm{ainv}(\lambda^{(k)}/\mu^{(k)},\lambda^{(l)}/\mu^{(l)})\neq 0$. Let $\tilde{L}$ be the sequence of skew shapes obtained from L by replacing the two terms $\lambda^{(k)}/\mu^{(k)}$ and $\lambda^{(l)}/\mu^{(l)}$ with the terms

$$\mathrm{sort}_1(\lambda^{(k)},\lambda^{(l)})/\mathrm{sort}_1(\mu^{(k)},\mu^{(l)}) \quad\text{and}\quad \mathrm{sort}_2(\lambda^{(k)},\lambda^{(l)})/\mathrm{sort}_2(\mu^{(k)},\mu^{(l)}),$$

correspondingly. Then $\mathrm{ainv}(\tilde{L}) < \mathrm{ainv}(L)$. Indeed, if we rearrange a subsequence in a sequence in the decreasing order, the total number of anti-inversions decreases. According to Corollary 6.2, we have $s_{\tilde{L}} \geq_s s_L$, where $s_L := \prod_{i=1}^n s_{\lambda^{(i)}/\mu^{(i)}}$. Note that the operation $L\mapsto\tilde{L}$ does not change the unions of partitions $\bigcup\lambda^{(i)}$ and $\bigcup\mu^{(i)}$. Let us apply the operations $L\mapsto\tilde{L}$ for various pairs (k,l) until we obtain a sequence of skew shapes $\hat{L} = (\hat{\lambda}^{(1)}/\hat{\mu}^{(1)},\dots,\hat{\lambda}^{(n)}/\hat{\mu}^{(n)})$ with $\mathrm{ainv}(\hat{L}) = 0$, i.e., the parts of all partitions must be sorted as $\hat{\lambda}_1^{(1)}\geq\cdots\geq\hat{\lambda}_1^{(n)}\geq\hat{\lambda}_2^{(1)}\geq\cdots\geq\hat{\lambda}_2^{(n)}\geq\hat{\lambda}_3^{(1)}\geq\cdots\geq\hat{\lambda}_3^{(n)}\geq\cdots$, and the same inequalities hold for the $\hat{\mu}_j^{(i)}$. This means that $\hat{\lambda}^{(i)}/\hat{\mu}^{(i)} = \lambda^{[i,n]}/\mu^{[i,n]}$, for $i=1,\dots,n$. Thus $s_{\hat{L}} = \prod s_{\lambda^{[i,n]}/\mu^{[i,n]}} \geq_s s_L$, as needed. □

Let us define $\lambda^{\{i,n\}} := ((\lambda')^{[i,n]})'$, for $i=1,\dots,n$. Here λ' again denotes the partition conjugate to λ. The partitions $\lambda^{\{i,n\}}$ are uniquely defined by the conditions $\lceil\frac{\lambda}{n}\rceil \supseteq \lambda^{\{1,n\}} \supseteq\cdots\supseteq \lambda^{\{n,n\}} \supseteq \lfloor\frac{\lambda}{n}\rfloor$ and $\sum_{i=1}^n \lambda^{\{i,n\}} = \lambda$. In particular, $\lambda^{\{1,2\}} = \lceil\frac{\lambda}{2}\rceil$ and $\lambda^{\{2,2\}} = \lfloor\frac{\lambda}{2}\rfloor$. If $\frac{\lambda}{n}$ is a partition, i.e., all parts of λ are divisible by n, then $\lambda^{\{i,n\}} = \frac{\lambda}{n}$ for each $1\leq i\leq n$.

Corollary 6.4. *Let $\lambda^{(1)}/\mu^{(1)},\dots,\lambda^{(n)}/\mu^{(n)}$ be n skew shapes, let $\lambda = \lambda^{(1)}+\cdots+\lambda^{(n)}$ and $\mu = \mu^{(1)}+\cdots+\mu^{(n)}$. Then we have $\prod_{i=1}^n s_{\lambda^{\{i,n\}}/\mu^{\{i,n\}}} \geq_s \prod_{i=1}^n s_{\lambda^{(i)}/\mu^{(i)}}$.*

Proof. This claim is obtained from Theorem 6.3 by conjugating the shapes. Indeed, $\left(\bigcup\lambda^{(i)}\right)' = \sum(\lambda^{(i)})'$. □

For a skew shape λ/μ and a positive integer n, define $s^{\langle n\rangle}_{\frac{\lambda}{n}/\frac{\mu}{n}} := \prod_{i=1}^n s_{\lambda^{\{i,n\}}/\mu^{\{i,n\}}}$. In particular, if $\frac{\lambda}{n}$ and $\frac{\mu}{n}$ are partitions, then $s^{\langle n\rangle}_{\frac{\lambda}{n}/\frac{\mu}{n}} = \left(s_{\frac{\lambda}{n}/\frac{\mu}{n}}\right)^n$.

Corollary 6.5. *Let c and d be positive integers and $n = c+d$. Let λ/μ and ν/ρ be two skew shapes. Then $s^{\langle n\rangle}_{\frac{c\lambda+d\nu}{n}/\frac{c\mu+d\rho}{n}} \geq_s s^c_{\lambda/\mu}\, s^d_{\nu/\rho}$.*

Theorem 6.1 is a special case of Corollary 6.5 for $c=d=1$.

Proof. This claim follows from Corollary 6.4 for the sequence of skew shapes that consists of λ/μ repeated c times and ν/ρ repeated d times. □

Corollary 6.5 implies that the map $S:\lambda\mapsto s_\lambda$ from the set of partitions to symmetric functions satisfies the following "Schur log-concavity" property.

Corollary 6.6. *For positive integers c, d and partitions λ, μ such that $\frac{c\lambda+d\mu}{c+d}$ is a partition, we have* $\left(S\left(\frac{c\lambda+d\mu}{c+d}\right)\right)^{c+d} \geq_s S(\lambda)^c S(\mu)^d$.

This notion of Schur log-concavity is inspired by Okounkov's notion of log-concavity; see [**Oko**].

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