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Macdonald polynomials at roots of unity

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Abstract. The aim of this note is to give some factorisation formulas for different versions of the Macdonald polynomials when the parameter t is specialized at roots of unity, generalizing those given in [**LLT1**] for Hall-Littlewood functions.

Résumé. Le but de cette note est de donner quelques formules de factorisations pour différentes versions des polynômes de Macdonald lorsque le paramètre t est spécialisé aux racines de l'unité. Ces formules généralisent celles données dans [**LLT1**] pour les fonctions de Hall-Littlewood.

1. Introduction

In [**LLT1**], Lascoux, Leclerc and Thibon give some factorisation formulas for the specialization of the parameter q at roots of unity for Hall-Littlewood functions. They also give a corollary of these formulas in terms of cyclic characters of the symmetric group. In this note, we give a generalization of these specializations for different versions of the Macdonald polynomials. We obtain similar formulas in terms of plethystic substitutions and cyclic characters. We also give in the last section a congruence for (q,t)-Kostka polynomials indexed by rectangles using Schur functions in the alphabet constituted by the powers of the parameter t. We will mainly follow the notations of [M].

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2. Preliminaries

For a partition $\lambda = (\lambda_1, \dots, \lambda_n)$, we write $l(\lambda)$ its length, $|\lambda|$ its weight, $m_i(\lambda)$ the multiplicity of the part of length i and λ' its conjugate partition. Let q and t be two indeterminates and $F = \mathbb{Q}(q,t)$. Let Λ_F be the ring of symmetric functions over the field F. Let us denote by $\langle \cdot, \cdot \rangle_{q,t}$ the inner product on Λ_F defined on the power sums products by

$$\langle \; p_\lambda(x) \; , \; p_\mu(x) \; \rangle_{q,t} = \delta_{\lambda\mu} z_\lambda(q,t),$$

where

$$z_\lambda(q,t) = \prod_{i\geq 1} (m_i)! \; i^{m_i(\lambda)} \; \prod_{i=1}^{l(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}.$$

The special case $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{\substack{q=0\\t=0}}$ is the usual inner product.

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Let $\{P_\lambda(x;q,t)\}_\lambda$ be the family of Macdonald polynomials obtained by orthogonalization of the Schur basis with respect to the inner product $\langle\cdot,\cdot\rangle_{q,t}$. Define a normalization of these functions by

$$Q_\lambda(x;q,t) = \frac{1}{\langle P_\lambda(x;q,t), P_\lambda(x;q,t)\rangle_{q,t}} P_\lambda(x;q,t).$$

It is clear from the previous definitions that the families $\{P_\lambda\}_\lambda$ and $\{Q_\mu\}_\mu$ are dual to each other with respect to the inner product $\langle\cdot,\cdot\rangle_{q,t}$ (c.f., [M, I, Section 4] and [M, VI, (2.7)]).

Proposition 2.1. *Let $x=(x_1,x_2,\dots)$ and $y=(y_1,y_2,\dots)$ be two sets of variables. The Macdonald polynomials $\{P_\lambda\}_\lambda$ and $\{Q_\lambda\}_\lambda$ satisfy the following Cauchy formula*

$$\sum_\lambda P_\lambda(x;q,t)Q_\lambda(y;q,t) = \prod_{i,j}\frac{(tx_iy_j;q)_\infty}{(x_iy_j;q)_\infty},$$

where $(a;q)_\infty$ is defined to be the infinite product $\prod_{r\geq 0}(1-aq^r)$.

Let $f(x)\in\Lambda_F$ be a symmetric function in the variables $x=(x_1,x_2,\dots)$. We consider the following algebra homomorphism

$$\begin{aligned} \tilde{}:\Lambda_F &\longrightarrow \Lambda_F \\ f(x) &\longmapsto \tilde{f}(x) = f\left(\frac{x}{1-t}\right). \end{aligned}$$

The images of the powersum functions $(p_k)_k \geq 1$ by this morphism are

$$\forall k\geq 1,\quad \tilde{p_k}(x) = \frac{1}{1-t^k}p_k(x).$$

We also define the algebra morphism

$$\begin{aligned} ':\Lambda_F &\longrightarrow \Lambda_F \\ f(x) &\longmapsto f'(x) = f\left(\frac{1-q}{1-t}x\right). \end{aligned}$$

The images of the powersum functions are

$$\forall k\geq 1,\quad p_k^{'}(x) = \frac{1-q^k}{1-t^k}p_k(x).$$

Let us consider the following modified version of the Macdonal polynomial

$$Q'_\mu(x;q,t) = Q_\mu\left(\frac{1-q}{1-t}x;q,t\right).$$

We can see that the set $\{Q'_\mu\}_\mu$ is the dual basis of $\{P_\lambda\}_\lambda$ with respect to the usual inner product.

Proposition 2.2. *Let $x=(x_1,x_2,\dots)$ and $y=(y_1,y_2,\dots)$ be two sets of variables. The Macdonald polynomials $\{P_\lambda\}_\lambda$ and $\{Q_\lambda^{'}\}_\lambda$ satisfy the following Cauchy formula*

$$\sum_\lambda P_\lambda(x;q,t)Q'_\lambda(y;q,t) = \prod_{i,j}\frac{1}{1-x_iy_j}.$$

Let $J_\mu(x;q,t)$ be the symmetric function with two parameters defined by

$$J_\mu(x;q,t) = c_\mu(q,t)P_\mu(x;q,t) = c'_\mu(q,t)Q_\mu(x;q,t), \tag{2.1}$$

where

$$c_\mu(q,t) = \prod_{s\in\mu}(1-q^{a(s)}t^{l(s)+1}) \quad\text{and}\quad c'_\mu(q,t) = \prod_{s\in\mu}(1-q^{a(s)+1}t^{l(s)}).$$

The symmetric function $J_\mu(x;q,t)$ is called the integral form of $P_\mu(x;q,t)$ or $Q_\mu(x;q,t)$ [M, VI, Section 8]. Using this integral form, we can define an other modified version of the Macdonald polynomial and the (q,t)-Kostka polynomials $K_{\lambda,\mu}(q,t)$ by

$$\tilde{J}_\mu(x;q,t) = J_\mu\left(\frac{x}{1-t};q,t\right) = \sum_\lambda K_{\lambda,\mu}(q,t)s_\lambda.$$

Haiman, Haglund and Loehr consider a modified version of $\tilde{J}_\mu(x;q,t)$ and introduce others (q,t)-Kostka polynomials $\tilde{K}_{\lambda,\mu}(q,t)$ by

$$\tilde{H}_\mu(x;q,t) = t^{n(\mu)}\tilde{J}_\mu(x;q,t^{-1}) = \sum_\lambda \tilde{K}_{\lambda,\mu}(q,t)s_\lambda$$

where $n(\mu) = \sum_i (i-1)\mu_i$. In [**HHL**], they give a combinatorial interpretation of this modified version expanded on monomials by introducing a notion of major index and inversion on arbitrary fillings of μ by integers.

REMARK 2.1. *Let μ and ρ be partitions of the same weight. We have*

$$X^\mu_\rho(q,t) = \langle\ \tilde{J}_\mu(q,t)\ ,\ p_\rho(x)\ \rangle,$$

where $X^\mu_\rho(q,t)$ is the Green polynomial with two variables, defined by

$$X^\mu_\rho(q,t) = \sum_\lambda \chi^\lambda_\rho K_{\lambda\mu}(q,t).$$

Here χ^λ_ρ is the value of the irreducible character of the symmetric group corresponding to the partition λ on the conjugancy class indexed by ρ.

3. Plethystic formula

We recall the definitions of some combinatorial quantities associated to a cell $s = (i,j)$ of a given partition. The arm length $a(s)$, arm-colength $a^{'}(s)$, leg length $l(s)$ and leg-colength $l^{'}(s)$ are respectively the number of cells at the east, at the west, at the south and at the north of the cell s (cf [M, VI, (6.14)])

$$a(s) = \lambda_i - j \quad , \quad a^{'}(s) = j-1,$$
$$l(s) = \lambda^{'}_j - i \quad , \quad l^{'}(s) = i-1.$$

We call plethysm of a symmetric function g by a powersum p_n, the following operation

$$p_n \circ g = g(x_1^n, x_2^n, \ldots).$$

As the powersums generate Λ_F, the operation $f \circ g$ is naturaly defined for any symmetric functions f and g (see [M, I, 8] for more details).

In this section, we shall show a plethystic formula for Macdonald polynomials when the second parameter t is specialized at primitive roots of unity.

PROPOSITION 3.1. ([M,VI, (6.11')]) *Let l be a positive integer and λ a partition such that $l(\lambda) \leq l$. The Macdonald polynomials $P_\lambda(x;t,q)$ on the alphabet $x_i = t^i$ for $0 \leq i \leq l-1$ and $x_i = 0$ for all $i \geq l$ can be written*

$$P_\lambda(1,t,\ldots,t^{l-1};q,t) = t^{n(\lambda)} \prod_{s\in\lambda} \frac{1-q^{a^{'}(s)}t^{l-l^{'}(s)}}{1-q^{a(s)}t^{l(s)+1}}. \tag{3.1}$$

PROPOSITION 3.2. *Let l be a positive integer and λ a partition such that $l(\lambda) \leq l$. For ζ a primitive l-th root of unity, the Macdonald polynomial P_λ satisfy the following specialization*

$$P_\lambda(1,\zeta,\zeta^2,\ldots,\zeta^{l-1};q,\zeta) = \begin{cases} (-1)^{(l-1)r} & \text{if } \lambda = (r^l) \text{ for some } r \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. Supplying zeros at the end of λ, we consider the partition λ as a sequence of length exactly equal to l, i.e., $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ for $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l \geq 0$. The multiplicity of 0 in λ is $m_0 = l - l(\lambda)$. We will denote by $\varphi_r(t)$ the polynomial

$$\varphi_r(t) = (1-t)(1-t^2)\ldots(1-t^r).$$

Let

$$f(t) = \frac{(1-t^l)(1-t^{l-1})\cdots(1-t^{l-l(\lambda)})(1-t^{l-l(\lambda)-1})\cdots(1-t^2)(1-t)}{\varphi_{m_0}(t)\varphi_{m_1}(t)\varphi_{m_2}(t)\cdots\cdots}$$

be the product of factors of the form $1-q^0t^\alpha$ for some $\alpha > 0$ in the formula (3.1). If we suppose that $f(\zeta) \neq 0$, the factor $1-t^l$ should be contained in one of $\varphi_{m_i}(t)$. This means that there exists $i \geq 0$ such that $m_i \geq l$. Since we consider λ as a sequence of length exactly l, this implies the condition $m_r = l$ for some $r \geq 0$. Thus, if $P_\lambda(1,\zeta,\zeta^2,\ldots,\zeta^{l-1};q,\zeta) \neq 0$, the shape of λ should be (r^l).

Suppose now that $\lambda = (r^l)$. By Proposition 3.1, it follows that

$$\begin{aligned}
P_\lambda(1,\zeta,\zeta^2,\ldots,\zeta^{l-1};q,\zeta) &= \zeta^{n(\lambda)} \prod_{s\in\lambda} \frac{1-q^{a'(s)}\zeta^{l-l'(s)}}{1-q^{a(s)}\zeta^{1+l(s)}} \\
&= \zeta^{n(\lambda)} \prod_{(i,j)\in\lambda} \frac{1-q^{j-1}\zeta^{l-(i-1)}}{1-q^{r-j}\zeta^{l-i+1}} \\
&= \zeta^{n(\lambda)} \prod_{i=1}^{l}\prod_{j=1}^{r} \frac{1-q^{j-1}\zeta^{l-i+1}}{1-q^{r-j}\zeta^{l-i+1}}
\end{aligned}$$

For each i, it is easy to see that

$$\prod_{j=1}^{r} \frac{1-q^{j-1}\zeta^{l-i+1}}{1-q^{r-j}\zeta^{l-i+1}} = 1.$$

Hence, we obtain

$$P_\lambda(1,\zeta,\zeta^2,\ldots,\zeta^{l-1};q,\zeta) = \zeta^{n(\lambda)},$$

and it follows immediately from the definition of $n(\lambda)$ that

$$\zeta^{n(\lambda)} = \zeta^{l(l-1))r/2} = (-1)^{(l-1)r}.$$

□

Theorem 3.1. *Let l and r be two positive integers and ζ a primitive l-th root of unity. The Macdonald polynomials $Q'_{(r^l)}(x;q,t)$ satisfy the following specialization formula at $t=\zeta$*

$$Q'_{(r^l)}(x;q,\zeta) = (-1)^{(l-1)r}(p_l \circ h_r)(x).$$

Proof. Recall that

$$\sum_\lambda P_\lambda(x;q,t)Q'_\lambda(y;q,t) = \prod_{i,j}\frac{1}{1-x_iy_j}.$$

If we let $x_i = \zeta^{i-1}$ for $i = 1,2,\ldots,l$ and $x_i = 0$ for $i > l$ and $t = \zeta$, we obtain

$$\sum_\lambda P_\lambda(1,\zeta,\zeta^2,\ldots,\zeta^{l-1};q,\zeta)Q'_\lambda(y;q,\zeta) = \prod_{j\geq 1}\prod_{i=1}^{l}\frac{1}{1-\zeta^{i-1}y_j}. \tag{3.2}$$

By Proposition 3.2, the left hand side of (3.2) is equal to

$$\sum_{r\geq 0}(-1)^{(r-1)l}Q'_{(r^l)}(y;q,\zeta).$$

Since $\prod_{i=1}^{l}(1-\zeta^{i-1}t) = 1-t^l$, the right hand side of (3.2) coincides with

$$\sum_{r\geq 0} h_r(y^l),$$

where y^l denotes the set of variables $(y_1^l, y_2^l, \cdots)$. Comparing the degrees, we can conclude that

$$Q'_{(r^l)}(y;q,\zeta) = (-1)^{(l-1)r}h_r(y^l) = (-1)^{(l-1)r}(p_l\circ h_r)(y).$$

□

Example 3.2. *For $\lambda = (222)$ and $l = 3$, we can compute*

$$\begin{aligned}
Q'_{(222)}(x;q,e^{\frac{2i\pi}{3}}) &= -s_{321}+s_{33}+s_{411}-s_{51}+s_6+s_{222} \\
&= p_3\circ h_2(x).
\end{aligned}$$

In order to give similar formula for the modified versions of the integral form of the Macdonald polynomials, we give a formula for the specialization of the constant $c'_{(r^l)}(t,q)$ at t a primitive l-th root of unity.

LEMMA 3.3. *Let l and r be two positive integers and ζ a l-th primitive root of unity. The normalization constant satisfies the following specialization at $t=\zeta$*

$$c'_{(r^l)}(q,\zeta)=\prod_{i=1}^{r}(1-q^{il}).$$

PROOF. Recall the definition of the normalization constant

$$c'_{(r^l)}(q,t)=\prod_{s\in\mu}(1-q^{a(s)+1}t^{l(s)})=\prod_{i=1}^{r}\prod_{j=1}^{l}(1-q^{r-i+1}t^j)=\prod_{i=1}^{r}\prod_{j=1}^{l}(1-q^it^j).$$

Specializing t at ζ a l-th primitive root of unity, we obtain

$$c'_{(r^l)}(q,\zeta)=\prod_{i=1}^{r}\prod_{j=1}^{l}(1-q^i\zeta^j)=\prod_{i=1}^{r}(1-q^{il}).$$

$\square$

COROLLARY 3.1. *With the same notation as in Theorem 3.1, the modified integral form of the Macdonald polynomials $\tilde{J}_\mu(x;q,t)$ satisfy a similar formula at $t=\zeta$, a primitive l-th root of unity*

$$\tilde{J}_{(r^l)}(x;q,\zeta) \quad = \quad (-1)^{(l-1)r}\prod_{i=1}^{r}(1-q^{il})\ p_l\circ h_r\left(\frac{x}{1-q}\right).$$

PROOF. Using the definition (2.1) of the integral form of the Macdonald polynomials

$$\begin{aligned}\tilde{J}_{(r^l)}(x;q,t)=J_{(r^l)}\left(\frac{x}{1-t};q,t\right) &= c'_{(r^l)}(q,t)Q'_{(r^l)}\left(\frac{x}{1-q};q,t\right)\\ &= c'_{(r^l)}(q,t)\left(Q'_{(r^l)}(\ .\ ;q,t)\circ\frac{1}{1-q}p_1\right)(x).\end{aligned}$$

By specializing the previous egality at a primitive l-th root of unity ζ, we obtain with Theorem 3.1 and the associativity of the plethysm

$$\begin{aligned}\tilde{J}_{(r^l)}(x;q,\zeta) &= c'_{(r^l)}(q,\zeta)(-1)^{r(l-1)}\left((p_l\circ h_r)\circ\frac{1}{1-q}p_1\right)(x)\\ &= c'_{(r^l)}(q,\zeta)(-1)^{r(l-1)}p_l\circ h_r\left(\frac{x}{1-q}\right).\end{aligned}$$

Using the formula of Lemma 3.3, we obtain the formula. $\square$

COROLLARY 3.2. *With the same notations than in Theorem 3.1, the modified Macdonald polynomials $\tilde{H}_\mu(x;q,t)$ satisfy the following specialization at $t=\zeta$, a primitive l-th root of unity*

$$\tilde{H}_{(r^l)}(x;q,\zeta)=\prod_{i=1}^{r}(1-q^{il})\ p_l\circ h_r\left(\frac{x}{1-q}\right).$$

PROOF. The result follows from corollary 3.1 and $\zeta^{n(r^l)}=\zeta^{rl(l-1)/2}=(-1)^{(l-1)r}$. $\square$

EXAMPLE 3.4. *For $\lambda=(222)$ and $l=3$, we can compute*

$$\begin{aligned}\tilde{J}_{(222)}(x;q,e^{\frac{2i\pi}{3}}) &= q^3(s_{111111}-s_{21111}+s_{3111})-(q^3+1)(s_{321}-s_{33}-s_{222})+s_{411}-s_{51}+s_6\\ &= (1-q^3)(1-q^6)\ p_3\circ h_2\left(\frac{x}{1-q}\right).\end{aligned}$$

REMARK 3.5. *As the Madonald polynomials $P_{(r^l)}(x;q,t)$ indexed by rectangles satisfy the following specialization at $t=\zeta$, a primitive l-th root of unity,*

$$\left.\frac{1}{\langle P_{(r^l)}(x;q,t)\ ,\ P_{(r^l)}(x;q,t)\rangle_{q,t}}\right|_{t=\zeta}=0$$

we obtain the following specializations

$$Q_{(r^l)}(x;q,\zeta) = 0 \quad \textit{and} \quad J_{(r^l)}(x;q,\zeta) = 0.$$

4. Pieri formula at roots of unity

In order to prove the factorization formulas, we prepare an auxiliary result (Proposition 4.1) on the coefficients of Pieri formula at root of unity (cf [M, VI, (6.24 ii)])

$$Q'_\mu(x;q,t)g'_r(x;q,t) = \sum_\lambda \psi_{\lambda/\mu}(q,t)\, Q'_\lambda(x;q,t). \tag{4.1}$$

Let λ and μ be partitions such that λ/μ is a horizontal (r-)strip θ. Let $C_{\lambda/\mu}$ (resp. $R_{\lambda/\mu}$) be the union of columns (resp. rows) of λ that intersects with θ, and $D_{\lambda/\mu} = C_{\lambda/\mu} - R_{\lambda/\mu}$ the set theoretical difference. Then it can be seen from the definition that for each cell s of $D_{\lambda/\mu}$ (resp. $D_{\tilde\lambda/\tilde\mu}$) there exists a unique connected component of θ (resp. $\tilde\theta$), which lies in the same row as s. We denote the corresponding component by θ_s (resp. $\tilde\theta_s$).

Suppose that l and r are positive integers. Set $\tilde\lambda = \lambda \cup (r^l)$ and $\tilde\mu = \mu \cup (r^l)$. We shall consider the difference between $D_{\tilde\lambda/\tilde\mu}$ and $D_{\lambda/\mu}$. It can be seen that there exists a projection

$$p = p_{\lambda/\mu} : D_{\tilde\lambda/\tilde\mu} \longrightarrow D_{\lambda/\mu}.$$

The cardinality of the fiber of each cell $s = (i,j) \in D_{\lambda/\mu}$ is exactly one or two. Let J_s denote the set of second coordinates of the cells in θ_s. If all elements of J_s are all strictly larger than r, the fiber $p^{-1}(s)$ consists of a single element $s = (i,j)$. If all elements of J_s are strictly smaller than r, then the fiber $p^{-1}(s)$ consists of a single element $\tilde s := (i, j+l)$. In the case where J_s contains r, then the fiber $p^{-1}(s)$ consists of exactly two elements $s = (i,j)$ and $\tilde s = (i, j+l)$. For the case where $r \in J_s$, we have the followig lemma, which follows immediately from the definition of the projection $p = p_{\lambda/\mu}$.

LEMMA 4.1. *Let $s = (i,j)$ be a cell of $D_{\lambda/\mu}$ and $\tilde s = (i, j+l)$ be a cell of $D_{\tilde\lambda/\tilde\mu}$ such that $r \in J_s$. The arm length, the arm-colength, the leg length and the leg-colength satisfy the following properties* :

(1) $a_{\tilde\mu}(s) = a_{\tilde\lambda}(\tilde s)$,
(2) $l_{\tilde\mu}(s) - l_{\tilde\lambda}(\tilde s) = l$,
(3) $a_{\tilde\mu}(\tilde s) = a_\mu(s)$,
(4) $l_{\tilde\mu}(\tilde s) = l_\mu(s)$,
(5) $a_{\tilde\lambda}(s) = a_\lambda(s)$,
(6) $l_{\tilde\lambda}(s) - l_\lambda(s) = l$.

PROPOSITION 4.1. *Let λ and μ be two partitions such that $\mu \subset \lambda$ and $\theta = \lambda - \mu$ a horizontal strip. Let r and l be positive integers and ζ a primitive root of unity. Then it follows that*

$$\psi_{\lambda\cup(r^l)/\mu\cup(r^l)}(q,\zeta) = \psi_{\lambda/\mu}(q,\zeta).$$

PROOF. Recall that for a cell s of the partition ν,

$$\psi_{\lambda/\mu}(q,t) = \prod_{s\in D_{\lambda/\mu}} \frac{b_\mu(s)}{b_\lambda(s)},$$

where

$$b_\nu(s) = \frac{1 - q^{a_\nu(s)}t^{l_\nu(s)+1}}{1 - q^{a_\nu(s)+1}t^{l_\nu(s)}}$$

If $s = (i,j) \in D_{\lambda/\mu}$ satisfies the condition $r > j$ for all $j \in J_s$, then the fiber $p^{-1}(s)$ of the projection p is $\{\tilde s = (i, j+l)\}$, and we have $a_\mu(s) = a_{\tilde\mu}(s), a_\lambda(s) = a_{\tilde\lambda}(s)$ and $l_\mu(s) + l = l_{\tilde\mu}(s), l_\lambda(s) + l = l_{\tilde\lambda}(s)$. It is clear from these identities that $b_\mu(s)/b_\lambda(s) = b_{\tilde\mu}(s)/b_{\tilde\lambda}(s)$ at $t = \zeta$ in this case. Suppose that s satisfies $j > r$ for all $j \in J_s$. In this case, the fiber $p^{-1}(s)$ consisits of a single element $\{s = (i,j)\}$, and we have $a_\mu(s) = a_{\tilde\mu}(s)$

and $a_\lambda(s) = a_{\tilde{\lambda}}(s)$ and $l_\mu(s) = l_{\tilde{\mu}}(s)$ and $l_\lambda(s) = l_{\tilde{\lambda}}(s)$. Hence we have $b_\mu(s)/b_\lambda(s) = b_{\tilde{\mu}}(s)/b_{\tilde{\lambda}}(s)$. Consider the case where $r \in J_s$. In this case, the fiber $p^{-1}(s)$ consists of two elements $\{s, \tilde{s}\}$. Let consider

$$\prod_{u \in p^{-1}(s)} \frac{b_{\tilde{\mu}}(u)}{b_{\tilde{\lambda}}(u)} = \frac{1-q^{a_{\tilde{\mu}}(s)}t^{l_{\tilde{\mu}}(s)+1}}{1-q^{a_{\tilde{\mu}}(s)+1}t^{l_{\tilde{\mu}}(s)}} \frac{1-q^{a_{\tilde{\lambda}}(s)+1}t^{l_{\tilde{\lambda}}(s)}}{1-q^{a_{\tilde{\lambda}}(s)}t^{l_{\tilde{\lambda}}(s)+1}} \frac{1-q^{a_{\tilde{\mu}}(\tilde{s})}t^{l_{\tilde{\mu}}(\tilde{s})+1}}{1-q^{a_{\tilde{\mu}}(\tilde{s})+1}t^{l_{\tilde{\mu}}(\tilde{s})}} \frac{1-q^{a_{\tilde{\lambda}}(\tilde{s})+1}t^{l_{\tilde{\lambda}}(\tilde{s})}}{1-q^{a_{\tilde{\lambda}}(\tilde{s})}t^{l_{\tilde{\lambda}}(\tilde{s})+1}}$$

By items (1) and (2) of Lemma 4.1, it follows that

$$\left\{\frac{1-q^{a_{\tilde{\mu}}(s)}t^{l_{\tilde{\mu}}(s)+1}}{1-q^{a_{\tilde{\mu}}(s)+1}t^{l_{\tilde{\mu}}(s)}}\right\}^{-1}\Bigg|_{t=\zeta} = \frac{1-q^{a_{\tilde{\lambda}}(\tilde{s})+1}t^{l_{\tilde{\lambda}}(\tilde{s})}}{1-q^{a_{\tilde{\lambda}}(\tilde{s})}t^{l_{\tilde{\lambda}}(\tilde{s})+1}}\Bigg|_{t=\zeta}.$$

It also follows from item (3) and (4) of Lemma 4.1,

$$\frac{1-q^{a_{\tilde{\mu}}(\tilde{s})}t^{l_{\tilde{\mu}}(\tilde{s})+1}}{1-q^{a_{\tilde{\mu}}(\tilde{s})+1}t^{l_{\tilde{\mu}}(\tilde{s})}}\Bigg|_{t=\zeta} = \frac{1-q^{a_\mu(s)}t^{l_\mu(s)+1}}{1-q^{a_\mu(s)+1}t^{l_\mu(s)}}\Bigg|_{t=\zeta},$$

and from item (5) and (6),

$$\frac{1-q^{a_{\tilde{\lambda}}(s)+1}t^{l_{\tilde{\lambda}}(s)}}{1-q^{a_{\tilde{\lambda}}(s)}t^{l_{\tilde{\lambda}}(s)+1}}\Bigg|_{t=\zeta} = \frac{1-q^{a_\lambda(s)+1}t^{l_\lambda(s)}}{1-q^{a_\lambda(s)}t^{l_\lambda(s)+1}}\Bigg|_{t=\zeta}.$$

Therefore, it follows that

$$\prod_{u \in p^{-1}(s)} \frac{b_{\tilde{\mu}}(u)}{b_{\tilde{\lambda}}(u)} = \frac{b_\mu(s)}{b_\lambda(s)}.$$

Combining these, the assertion follows. □

5. Factorization formulas

In this section, we shall show factorization formulas for different kinds of Macdonald polynomials at roots of unity.

THEOREM 5.1. *Let l be a positive integer and ζ a primitive l-th root of unity. Let $\mu = (1^{m_1}2^{m_2}\cdots n^{m_n})$ be a partition of a positive integer n. For each i, let $m_i = lq_i + r_i$ with $0 \le r_i \le l-1$ and let $\bar{\mu} = (1^{r_1}2^{r_2}\cdots m^{r_n})$. The function Q'_μ satisfy the following factorisation formula at $t = \zeta$*

$$Q'_\mu(x;q,\zeta) \quad = \quad \left(Q'_{(1^l)}(x;q,\zeta)\right)^{q_1} \left(Q'_{(2^l)}(x;q,\zeta)\right)^{q_2} \cdots \left(Q'_{(n^l)}(x;q,\zeta)\right)^{q_n} Q'_{\bar{\mu}}(x;q,\zeta).$$

PROOF. We shall show that the $\mathbb{C}$-linear map defined by

$$\begin{array}{rcl} f_r : \Lambda_F & \longrightarrow & \Lambda_F \\ Q'_\mu(x;q,\zeta) & \longmapsto & Q'_{\mu\cup(r^l)}(x;q,\zeta) \end{array}$$

is an $\Lambda_{\mathbb{C}(q)}$-linear map. Let ζ be a primitive l-th root of unity. From (3.1), we have

$$Q'_\mu(x;q,\zeta)g'_k(x;q,\zeta) = \sum_\lambda \psi_{\lambda/\mu}(q,\zeta)Q'_\lambda(x;q,\zeta)$$

where the sum is taken over the partitions λ such that $\lambda - \mu$ is a horizontal (k-)strip. Using the result of Proposition 4.1, it follows that

$$\begin{aligned} Q'_{\mu\cup(r^l)}(x;q,\zeta)g'_k(x;q,\zeta) \quad &= \quad \sum_\lambda \psi_{\lambda\cup(r^l)/\mu\cup(r^l)}(q,\zeta)Q'_{\lambda\cup(r^l)}(x;q,\zeta) \\ &= \quad \sum_\lambda \psi_{\lambda/\mu}(q,\zeta)Q'_{\lambda\cup(r^l)}(x;q,\zeta). \end{aligned}$$

Consequently, for each $r \ge 1$, the multiplication by g_k commutes with the morphism f_r. Since $\{g_k(x;q,\zeta_l)\}_{k\ge1}$ generate the algebra $\Lambda_{\mathbb{C}(q)}$ [M, VI, (2.12)], the map f_r is $\Lambda_{\mathbf{C}(q)}$-linear. This implies that

$$\begin{aligned} \forall F \in \Lambda_{\mathbb{C}_q},\ f_r(F(x)) \quad &= \quad F(x)f_r(1) \\ &= \quad F(x)Q'_{(r^l)}(x;q,\zeta). \end{aligned}$$

□

Corollary 5.1. *With the same notation as in Theorem 3.1, we have*

$$\tilde{J}_\mu(x;q,\zeta) \quad = \quad \left(\tilde{J}_{(1^l)}(x;q,\zeta)\right)^{q_1} \left(\tilde{J}_{(2^l)}(x;q,\zeta)\right)^{q_2} \cdots \left(\tilde{J}_{(n^l)}(x;q,\zeta)\right)^{q_n} \tilde{J}_{\bar{\mu}}(x;q,\zeta).$$

Proof. If we define

$$\Psi_{\lambda/\mu}(q,t) := \psi_{\lambda/\mu}(q,t)\frac{c'_\mu(q,t)}{c'_\lambda(q,t)},$$

then the Pieri formula for the integral form $\tilde{J}_\mu(x;q,t)$ is written as follows

$$\tilde{J}_\mu(x;q,t)\tilde{g_k}(x;q,t) = \sum_\lambda \Psi_{\lambda/\mu}(q,t)\tilde{J}_\lambda(x;q,t),$$

where the sum is over the partitions λ such that $\lambda - \mu$ is a horizontal (k-)strip.
Let a positive integer r be arbitrarily fixed, and $\tilde{\nu}$ denote the partition $\nu \cup (r^l)$. Since we have already shown that $\psi_{\tilde{\lambda}/\tilde{\mu}}(q,\zeta) = \psi_{\lambda/\mu}(q,\zeta)$, it suffices to show that

$$\frac{c'_{\tilde{\mu}}(q,\zeta)}{c'_{\tilde{\lambda}}(q,\zeta)} = \frac{c'_\mu(q,\zeta)}{c'_\lambda(q,\zeta)}.$$

We shall actually show that

$$\frac{c'_{\tilde{\mu}}(q,\zeta)}{c'_\mu(q,\zeta)} = \frac{c'_{\tilde{\lambda}}(q,\zeta)}{c'_\lambda(q,\zeta)}.$$

It follows from the definition that

$$\begin{aligned}\frac{c'_{\tilde{\mu}}(q,t)}{c'_\mu(q,t)} &= \frac{\prod_{s\in\tilde{\mu}}(1-q^{a_{\tilde{\mu}}(s)+1}t^{l_{\tilde{\mu}}(s)})}{\prod_{s\in\mu}(1-q^{a_{\tilde{\mu}}(s)+1}t^{l_{\tilde{\mu}}(s)})} \\ &= \frac{\prod_{\substack{s\in\tilde{\mu}\\ s\notin(r^l)}}(1-q^{a_{\tilde{\mu}}(s)+1}t^{l_{\tilde{\mu}}(s)})}{\prod_{s\in\mu}(1-q^{a_{\tilde{\mu}}(s)+1}t^{l_{\tilde{\mu}}(s)})} \prod_{s\in(r^l)\subset\tilde{\mu}}(1-q^{a_{\tilde{\mu}}(s)+1}t^{l_{\tilde{\mu}}(s)}).\end{aligned}$$

The Young diagram of the partition $\tilde{\mu}$ is the disjoint union of the cells $\{\tilde{s}\in\tilde{\mu}|s\in\mu\}$ and (r^l). For each $s\in\mu$, we have as seen in previous Theorem that $a_{\tilde{\mu}}(\tilde{s}) = a_\mu(s)$, and $l_{\tilde{\mu}}(\tilde{s}) = l_\mu(s)$ or $l_\mu(s)+l$. Hence at $t=\zeta$, we have

$$\frac{c'_\mu(q,\zeta)}{c'_{\tilde{\mu}}(q,\zeta)} = \prod_{s\in(r^l)\subset\tilde{\mu}}(1-q^{a_{\tilde{\mu}}(s)+1}\zeta^{l_{\tilde{\mu}}(s)}). \tag{3.1}$$

$$\frac{c'_\lambda(q,\zeta)}{c'_{\tilde{\lambda}}(q,\zeta)} = \prod_{s\in(r^l)\subset\tilde{\lambda}}(1-q^{a_{\tilde{\lambda}}(s)+1}\zeta^{l_{\tilde{\lambda}}(s)}). \tag{3.2}$$

Although there is a difference between the positions where the block (r^l) is inserted in the Young diagram of μ and λ, (3.1) and (3.2) coincide at $t=\zeta$, since $a_{\tilde{\mu}}(s) = a_{\tilde{\lambda}}(s)$ for each $s\in(r^l)$. Thus we have

$$\frac{c'_\mu(q,\zeta)}{c'_{\tilde{\mu}}(q,\zeta)} = \frac{c'_\lambda(q,\zeta)}{c'_{\tilde{\lambda}}(q,\zeta)}.$$

□

Let $\nu = (\nu_1,\ldots,\nu_p)$ be a partition. For some $l\geq 0$, we denote by ν^l the partition where each part of ν is repeated l times. We can give a more explicit expression for the factorisation formula in the special case where $\mu = \nu^l$.

Corollary 5.2. *Let ν be a partition and l a positive integer. We have the following special cases for the factorisation formulas*

$$Q'_{\nu^l}(X;q,\zeta) \quad = \quad (-1)^{(l-1)|\nu|} p_l \circ h_\nu(x), \tag{5.1}$$

$$\tilde{J}_{\nu^l}(X;q,\zeta) \quad = \quad (-1)^{(l-1)|\nu|} \prod_{j=1}^{l(\nu)}\prod_{i=1}^{\nu_j}(1-q^{il})\ p_l\circ h_\nu\left(\frac{x}{1-q}\right). \tag{5.2}$$

EXAMPLE 5.2. *For $\lambda = (222111)$ and $k = 3$, we can compute*

$$\begin{aligned} Q^{'}_{222111}(x;q,e^{2i\pi/3}) &= -s_{22221} - s_{321111} + s_{3222} + s_{33111} - s_{3321} + 3s_{333} + s_{411111} \\ &\quad -2s_{432} + 2s_{441} - s_{51111} + 2s_{522} - 2s_{54} + s_{6111} - 2s_{621} + 2s_{63} \\ &\quad +s_{711} - s_{81} + s_9 + s_{222111} \\ &= p_3 \circ h_{21}(x). \end{aligned}$$

6. A generalization of the plethystic formula

In this section, using the factorisation formula given in Theorem 5.1, we shall give a generalization of the plethystic formula obtained by specializing Macdonald polynomials at roots of unity in Theorem 3.1. For λ a partition, let consider the following map which is the plethystic substitution by the powersum p_λ

$$\begin{aligned} \Psi_\lambda : \Lambda_F &\longrightarrow \Lambda_F \\ f &\longmapsto p_\lambda \circ f \end{aligned}$$

LEMMA 6.1. *Let λ and μ be two partitions, the maps Ψ_λ and Ψ_μ satisfy the following multiplicative property*

$$\Psi_\lambda(f)\,\Psi_\mu(f) = \Psi_{\lambda\cup\mu}(f).$$

PROPOSITION 6.1. *Let d be an integer such that $d|l$ and ζ_d be a primitive d-th root of unity,*

$$Q^{'}_{(r^l)}(x;q,\zeta_d) = (-1)^{\frac{rl(d-1)}{d}} p_d^{l/d} \circ h_r(x).$$

PROOF. Let d and l be two integers such that d divide l. Let $\mu = (r^l)$ the rectangle partition with parts of length r. Using the factorisation formula described in Theorem 5.1, we can write

$$Q^{'}_{(r^l)}(x;q,\zeta_d) = \left(Q^{'}_{(r^d)}(x;q,\zeta_d)\right)^{l/d}. \tag{6.1}$$

With the specialization formula at root of unity written in Theorem 3.1, we have

$$\begin{aligned} \left(Q^{'}_{(r^d)}(x;q,\zeta_d)\right)^{l/d} &= \left((-1)^{(d-1)r} p_d \circ h_r(x)\right)^{l/d} \\ &= (-1)^{\frac{lr(d-1)}{d}} \left(p_d \circ h_r(x)\right)^{l/d} \end{aligned}$$

Using the Lemma 6.1, we obtain

$$\left(Q^{'}_{(r^d)}(x;q,\zeta_d)\right)^{l/d} = (-1)^{\frac{lr(d-1)}{d}} p_d^{l/d} \circ h_r(x)$$

Finally, we obtain by (6.1)

$$Q^{'}_{(r^l)}(x;q,\zeta_d) = (-1)^{\frac{lr(d-1)}{d}} p_d^{l/d} \circ h_r(x)$$

□

Using the same proof, we can write a similar specialization for integral forms of the Macdonald Polynomials.

COROLLARY 6.1. *The Macdonald polynomials $\tilde{J}_\mu(x;q,t)$ satisfy the same generalization than the $Q'_\mu(x;q,t)$*

$$\tilde{J}_{(r^l)}(x;q,\zeta_d) = (-1)^{\frac{rl(d-1)}{d}} \left(\prod_{i=1}^{r}(1-q^{il})\right)^{l/d} p_d^{l/d} \circ h_r\left(\frac{x}{1-q}\right).$$

Example 6.2. *For* $\lambda = (222222)$ *(i.e* $r = 2$ *and* $l = 6$*) and* $d = 3$ *we can compute*

$$\begin{aligned} Q'_{(222222)}(x; q, e^{2i\pi/3}) &= -s_{322221} + s_{33222} + 2s_{333111} - 2s_{33321} + 2s_{3333} + s_{422211} - 2s_{432111} \\ &\quad + s_{43221} + 2s_{441111} - s_{4422} + 4s_{444} + s_{522111} - 2s_{52221} + s_{53211} - 2s_{54111} \\ &\quad + s_{5421} - 4s_{543} + 3s_{552} - s_{621111} + 2s_{6222} + s_{63111} - 2s_{6321} + 4s_{633} \\ &\quad + s_{6411} - 3s_{651} + 3s_{66} + s_{711111} - 2s_{732} + 2s_{741} - s_{81111} + 2s_{822} - 2s_{84} \\ &\quad + s_{9111} - 2s_{921} + 2s_{93} + s_{\underline{10}11} - s_{\underline{11}1} + s_{\underline{12}} + s_{222222} \\ &= p_3^2 \circ h_2(x) = p_{(33)} \circ h_2(x). \end{aligned}$$

7. Macdonald polynomials at roots of unity and cyclic characters of the symmetric group

In the following, we will denote the symmetric group of order k by $\mathfrak{S}_k$. Let $\Gamma \subset \mathfrak{S}_k$ be a cyclic subgroup generated by an element of order r. As Γ is a commutative subgroup its irreducible representations are one-dimensional vector space. The corresponding maps $(\gamma_j)_{j=0\ldots r-1}$ can be defined by

$$\begin{aligned} \gamma_j : \quad \Gamma &\longrightarrow \quad GL(\mathbb{C}) \;\simeq\; \mathbb{C}^* \\ \tau &\longmapsto \quad \zeta_r^j \end{aligned}$$

where ζ_r is a r-th primitive root of unity (See [**S**] for more details). In [**F**], Foulkes considered the Frobenius characteristic of the representations of $\mathfrak{S}_k$ induced by these irreducible representations and obtained an explicit formula that we will give in the next Proposition.
Let k and n be two positive integers such that $u = (k, d)$ (the greatest common divisor between k and n) and $d = u \cdot m$. Let us define the Ramanujan (or Von Sterneck) sum $c(k, d)$ by

$$c(k, d) = \frac{\mu(m)\phi(d)}{\phi(m)}$$

where μ is the Moebius function and ϕ the Euler totient. The quantity $c(k, d)$ corresponds to the sum of the k-th powers of the primitive d-th roots of unity (the previous expression was first given by Holder, see [**HW**]).

Proposition 7.1. *Let* τ *be a cyclic permutation of length* k *and* Γ *the maximal cyclic subgroup of* $\mathfrak{S}_k$ *generated by* τ*. Let* j *be a positive integer less than* k*. The Frobenius characteristic of the representation of* $\mathfrak{S}_k$ *induced by the irreducible representation of* Γ*,* $\gamma_j : \tau \longmapsto \zeta_r^j$*, is given by*

$$l_k^{(j)}(x) = \frac{1}{k} \sum_{d|k} c(j, d)\; p_d^{k/d}(x).$$

Example 7.1. For $\mathfrak{S}_6$ and $k = 2$, the corresponding cyclic character $l_6^{(2)}$ can be written

$$\begin{aligned} l_6^{(2)} &= \frac{1}{6}\,(p_{111111} + p_{222} - p_{33} - p_6) \\ &= s_{51} + 2s_{42} + s_{411} + 3s_{321} + 2s_{3111} + s_{222} + s_{2211} + s_{21111}. \end{aligned}$$

Theorem 7.2. *Let* r *and* l *be two positive integers. The specialization of the Macdonald polynomials* $Q'_{(r^l)}(x; q, t)$ *at* $t = \zeta$*, a primitive* l*-th root of unity, is equivalent to*

$$Q'_{(r^l)}(x; q, t) \mod 1 - t^l \quad = \quad \sum_{j=0}^{l-1} t^j \; (l_l^{(j)} \circ h_r)(x).$$

Proof. We will first give a generalization of the Moebius inversion formula due to E. Cohen (see [**C**] for the original work and [**D**] for a simpler proof). Let

$$P(q) = \sum_{k=0}^{n-1} a_k q^k$$

be a polynomial of degree less than $n - 1$ with coefficients a_k in $\mathbb{Z}$. P is said to be even modulo n if

$$(i, n) = (j, n) \implies a_i = a_j.$$

LEMMA 7.3. *The polynomial P is even modulo n if and only if for every divisor d of n, the residue of P modulo the d-th cyclotomic polynomial Φ_d is a constant r_d in $\mathbb{Z}$. In this case, one has*

$$\begin{aligned} (i) \quad a_k &= \frac{1}{n}\sum_{d|n} c(k,d)\ r_d \\ (ii) \quad r_d &= \sum_{t|n} c(n/d,t)\ a_{n/t}. \end{aligned}$$

Let d be an integer such that $d|l$. By expanding $Q'_{(r^l)}(x;q,t)$ (and more generally $Q'_\lambda(x;q,t)$) on the Schur basis, we can define a kind of (q,t)-Kostka polynomials $K'_{\mu,(r^l)}(q,t)$

$$Q'_{(r^l)}(x;q,t) = \sum_\mu K'_{\mu,(r^l)}(q,t)\ s_\mu(x).$$

Let μ be a partition and d an integer such that $d|l$. $P^q_\mu(t) = \sum_{j=0}^{l-1} a_j(q)t^j$ is the residue modulo $1-t^l$ of the (q,t)-Kostka polynomial $K'_{\mu,(r^l)}(q,t)$ if and only if for all ζ_d primitive d-th root of unity

$$P^q_\mu(\zeta_d) = K'_{\mu,(r^l)}(q,\zeta_d).$$

Using Theorem 5.1 , one has

$$P^q_\mu(\zeta_d) = (-1)^{(d-1)rl/d}\ \langle\ p_d^{l/d} \circ h_r(x)\ ,\ s_\mu(x)\ \rangle.$$

So, $P(\zeta_d) \in \mathbb{Z}$ since the entries of the transition matrix between the powersum to the Schur functions are all integers. Using the Lemma 7.3, we obtain

$$\begin{aligned} a_j(q) &= \frac{1}{l}\sum_{d|l} c(j,d)\ \langle\ p_d^{l/d} \circ h_r(x)\ ,\ s_\mu(x)\ \rangle \\ &= \langle\ l_l^{(j)} \circ h_r(x)\ ,\ s_\mu(x)\ \rangle. \end{aligned}$$

□

COROLLARY 7.1. *For two positive integers r and l, the same residue formulas occurs for the modified Macdonald polynomials $\tilde{J}_{(r^l)}(x;q,t)$ and $J'_{(r^l)}(x;q,t)$*

$$\tilde{J}_{(r^l)}(x;q,t) \mod 1-t^l \quad = \prod_{i=1}^{r}(1-q^{il}) \sum_{j=0}^{l-1} t^j\ (l_l^{(j)} \circ h_r)\left(\frac{x}{1-q}\right).$$

8. Congruences for (q,t)-Kostka polynomials

For a given partition λ, let denote by $\tilde{s}^{(q)}_\lambda$ the symmetic function defined as follows

$$\tilde{s}^{(q)}_\lambda(x) = s_\lambda\left(\frac{x}{1-q}\right).$$

We also define on the power sums products the internal product between two symmetric functions. For λ and μ two partitions, we have ([M, I, (7.12)])

$$p_\lambda \star p_\mu = \delta_{\lambda,\mu} z_\lambda p_\lambda.$$

PROPOSITION 8.1. *Let r and l be two positive integers and μ a partition of weight nl. Let denote by $\Phi_l(t)$ the cyclotomic polynomial of order l. The (q,t)-Kostka polynomial $\tilde{K}_{\mu,(r^l)}(q,t)$ satisfy the following congruence*

$$\tilde{K}_{\mu,(r^l)}(q,t) = \prod_{i=1}^{r}(1-q^{il})\ \tilde{s}^{(q)}_\mu(1,t,t^2,\dots,t^{l-1}) \mod \Phi_l(t).$$

And more generally, for all partition ν of weight r

$$\tilde{K}_{\mu,\nu^l}(q,t) = \prod_{j=1}^{l(\nu)}\prod_{i=1}^{\nu_j}(1-q^{il})\ \widetilde{h_{l\nu}\star s_\mu}^{(q)}(1,t,t^2,\dots,t^{l-1}) \mod \Phi_l(t),$$

where $l\nu$ denote the partition $(l\nu_1,\dots,l\nu_p)$.

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