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The Partition Function of Andrews and Stanley and Al-Salam-Chihara Polynomials

Masao Ishikawa and Jiang Zeng

Abstract. For any partition λ let $\omega(\lambda)$ denote the four parameter weight

$$\omega(\lambda)=a^{\sum_{i\geq 1}\lceil\lambda_{2i-1}/2\rceil}b^{\sum_{i\geq 1}\lfloor\lambda_{2i-1}/2\rfloor}c^{\sum_{i\geq 1}\lceil\lambda_{2i}/2\rceil}d^{\sum_{i\geq 1}\lfloor\lambda_{2i}/2\rfloor},$$

and let $\ell(\lambda)$ be the length of λ. We show that the generating function $\sum\omega(\lambda)z^{\ell(\lambda)}$, where the sum runs over all ordinary (resp. strict) partitions with parts each $\leq N$, can be expressed by the Al-Salam-Chihara polynomials. As a corollary we prove G.E. Andrews' result by specializing some parameters and C. Boulet's results when $N\to+\infty$. In the last section we study the weighted sum $\sum\omega(\lambda)z^{\ell(\lambda)}P_\lambda(x)$ where $P_\lambda(x)$ is Schur's P-function and the sum runs over all strict partitions.

Résumé.

Pour toute partition λ on définit $\omega(\lambda)$ comme la fonction poids de quatre paramètres

$$\omega(\lambda)=a^{\sum_{i\geq 1}\lceil\lambda_{2i-1}/2\rceil}b^{\sum_{i\geq 1}\lfloor\lambda_{2i-1}/2\rfloor}c^{\sum_{i\geq 1}\lceil\lambda_{2i}/2\rceil}d^{\sum_{i\geq 1}\lfloor\lambda_{2i}/2\rfloor},$$

et désigne $\ell(\lambda)$ la longueur de λ. On démontre que la fonction génératrice $\sum\omega(\lambda)z^{\ell(\lambda)}$, où la somme porte sur toutes les partitions ordinaires (resp. strictes) avec chaque part $\leq N$, peut s'exprimer par les polynômes d'Al-Salam-Chihara. Comme corollaire on en déduit un résultat de G.E. Andrews en spécialisant certain parametres et ceux de C. Boulet quand $N\to+\infty$. Dans la dernière section on étudie la somme pondérée $\sum\omega(\lambda)z^{\ell(\lambda)}P_\lambda(x)$ où $P_\lambda(x)$ est la P-fonction de Schur et la somme porte sur toutes les partitions strictes.

1. Introduction

Let λ be an integer partition and λ' its conjugate. Let $\mathcal{O}(\lambda)$ denote the number of odd parts of λ and $|\lambda|$ the sum of its parts. R. Stanley ([**13**]) has shown that if $t(n)$ denotes the number of partitions λ of n for which $\mathcal{O}(\lambda)\equiv\mathcal{O}(\lambda')\pmod 4$, then

$$t(n)=\frac{1}{2}\left(p(n)+f(n)\right),$$

where $p(n)$ is the total number of partitions of n, and

$$\sum_{n=0}^{\infty}f(n)q^n=\prod_{i\geq 1}\frac{(1+q^{2i-1})}{(1-q^{4i})(1+q^{4i-2})}.$$

In [**1**] G.E. Andrews has computed the generating function of ordinary partitions λ with parts each less than or equal to N, with respect to the weight $z^{\mathcal{O}(\lambda)}y^{\mathcal{O}(\lambda')}q^{|\lambda|}$. We should note that in [**12**] A. Sills has given a combinatorial proof of this result, and in [**14**] A. Yee has generalized this result to the generating function of ordinary partitions of parts$\leq N$ and length $\leq M$.

As a generalization of this weight, we consider the following four parameter weight. Let a, b, c and d be commuting indeterminates. Define the following weight functions $\omega(\lambda)$ on the set of all partitions,

$$\omega(\lambda)=a^{\sum_{i\geq 1}\lceil\lambda_{2i-1}/2\rceil}b^{\sum_{i\geq 1}\lfloor\lambda_{2i-1}/2\rfloor}c^{\sum_{i\geq 1}\lceil\lambda_{2i}/2\rceil}d^{\sum_{i\geq 1}\lfloor\lambda_{2i}/2\rfloor}, \tag{1.1}$$

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where $\lceil x \rceil$ (resp. $\lfloor x \rfloor$) stands for the smallest (resp. largest) integer greater (resp. less) than or equal to x for a given real number x. For example, if $\lambda = (5,4,4,1)$ then $\omega(\lambda)$ is the product of the entries in the following diagram for λ.

$$\begin{array}{|c|c|c|c|c|} \hline a & b & a & b & a \\ \hline c & d & c & d & \\ \cline{1-4} a & b & a & b & \\ \cline{1-4} c & & & & \\ \cline{1-1} \end{array}$$

In [**3**] C. Boulet has obtained results on the generating functions for the weights $\omega(\lambda)$ when λ runs over all ordinary partitions and when λ runs over all strict partitions

In this paper we consider a refinement of these results, i.e., the generating functions for the weights $\omega(\lambda) z^{\ell(\lambda)}$, where $\ell(\lambda)$ is the length of λ, when λ runs over all ordinary partitions with parts each $\leq N$ and when λ runs over all strict partitions with parts each $\leq N$, and show that they are related to the basic hypergeometric series, i.e. the Al-Salam-Chihara polynomials (see Theorem 3.4 and Theorem 4.3).

In the last section we show the weighted sum $\sum \omega(\mu) z^{\ell(\mu)} P_\mu(x)$ of Schur's P-functions $P_\mu(x)$ (when $z = 2$, this equals the weighted sum $\sum \omega(\mu) Q_\mu(x)$ of Schur's Q-functions $Q_\mu(x)$) can be expressed by a Pfaffian where μ runs over all strict partitions (with parts each $\leq N$).

2. Preliminaries

A q-shifted factorial is defined by

$$(a;q)_0 = 1, \qquad (a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}), \qquad n = 1,2,\dots.$$

We also define $(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k)$. Since products of q-shifted factorials occur very often, to simplify them we shall use the compact notations

$$\begin{aligned}(a_1,\dots,a_m;q)_n &= (a_1;q)_n \cdots (a_m;q)_n,\\ (a_1,\dots,a_m;q)_\infty &= (a_1;q)_\infty \cdots (a_m;q)_\infty.\end{aligned}$$

We define an ${}_{r+1}\phi_r$ *basic hypergeometric series* by

$${}_{r+1}\phi_r\left(\begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, \dots, b_r \end{matrix}; q, z\right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots, a_{r+1}; q)_n}{(q, b_1, \dots, b_r; q)_n} z^n.$$

The *Al-Salam-Chihara polynomial* $Q_n(x) = Q_n(x;\alpha,\beta|q)$ is, by definition,

$$\begin{aligned} Q_n(x;\alpha,\beta|q) &= \frac{(\alpha\beta;q)_n}{\alpha^n}\, {}_3\phi_2\left(\begin{matrix} q^{-n}, \alpha u, \alpha u^{-1} \\ \alpha\beta, 0 \end{matrix}; q, q\right), \\ &= (\alpha u;q)_n u^{-n}\, {}_2\phi_1\left(\begin{matrix} q^{-n}, \beta u^{-1} \\ \alpha^{-1} q^{-n+1} u^{-1} \end{matrix}; q, \alpha^{-1} q u\right), \\ &= (\beta u^{-1};q)_n u^{n}\, {}_2\phi_1\left(\begin{matrix} q^{-n}, \alpha u \\ \beta^{-1} q^{-n+1} u \end{matrix}; q, \beta^{-1} q u^{-1}\right), \end{aligned}$$

where $x = \frac{u+u^{-1}}{2}$ (see [**6**] p.80). This is a specialization of the Askey-Wilson polynomials (see [**2**]), and satisfies the three-term recurrence relation

$$2xQ_n(x) = Q_{n+1}(x) + (\alpha+\beta)q^n Q_n(x) + (1-q^n)(1-\alpha\beta q^{n-1})Q_{n-1}(x), \tag{2.1}$$

with $Q_{-1}(x) = 0$, $Q_0(x) = 1$.

We also consider a more general recurrence relation:

$$2x\widetilde{Q}_n(x) = \widetilde{Q}_{n+1}(x) + (\alpha+\beta)tq^n \widetilde{Q}_n(x) + (1-tq^n)(1-t\alpha\beta q^{n-1})\widetilde{Q}_{n-1}(x), \tag{2.2}$$

which we call the *associated Al-Salam-Chihara recurrence relation.* Put

$$\widetilde{Q}_n^{(1)}(x) = u^{-n}\,(t\alpha u;q)_n\,{}_2\phi_1\left(\begin{matrix} t^{-1}q^{-n},\beta u^{-1} \\ t^{-1}\alpha^{-1}q^{-n+1}u^{-1}\end{matrix};\,q,\alpha^{-1}qu\right), \tag{2.3}$$

$$\widetilde{Q}_n^{(2)}(x) = u^{n}\,\frac{(tq;q)_n(t\alpha\beta;q)_n}{(t\beta uq;q)_n}\,{}_2\phi_1\left(\begin{matrix} tq^{n+1},\alpha^{-1}qu \\ t\beta q^{n+1}u\end{matrix};\,q,\alpha u\right), \tag{2.4}$$

where $x = \frac{u+u^{-1}}{2}$. In [**5**], Ismail and Rahman have presented two linearly independent solutions of the associated Askey-Wilson recurrence equation (see also [**4**]). By specializing the parameters, we conclude that $\widetilde{Q}_n^{(1)}(x)$ and $\widetilde{Q}_n^{(2)}(x)$ are two linearly independent solutions of the associated Al-Salam-Chihara equation (2.2) (see [**4**, p.203]). Here, we use this fact and omit the proof. The series (2.3) and (2.4) are convergent if we assume $|u| < 1$ and $|q| < |\alpha| < 1$ (see [**4**, p.204]).

Let

$$W_n = \widetilde{Q}_n^{(1)}(x)\widetilde{Q}_{n-1}^{(2)}(x) - \widetilde{Q}_{n-1}^{(1)}(x)\widetilde{Q}_n^{(2)}(x) \tag{2.5}$$

denote the Casorati determinant of the equation (2.2). Then we obtain

$$W_1 = \frac{u^{-1}(t\alpha u,\beta u;q)_\infty}{(\alpha u,t\beta uq;q)_\infty}. \tag{2.6}$$

In the following sections we need to find a polynomial solution of the recurrence equation (2.2) which satisfies a given initial condition, say $\widetilde{Q}_0(x) = \widetilde{Q}_0$ and $\widetilde{Q}_1(x) = \widetilde{Q}_1$. Since $\widetilde{Q}_n^{(1)}(x)$ and $\widetilde{Q}_n^{(2)}(x)$ are linearly independent solutions of (2.2), this $\widetilde{Q}_n(x)$ can be written as a linear combination of these functions, say

$$\widetilde{Q}_n(x) = C_1\,\widetilde{Q}_n^{(1)}(x) + C_2\,\widetilde{Q}_n^{(2)}(x).$$

If we substitute the initial condition $\widetilde{Q}_0(x) = \widetilde{Q}_0$ and $\widetilde{Q}_1(x) = \widetilde{Q}_1$ into this equation and solve the linear equation, then we conclude that

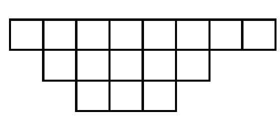
$$\begin{aligned}\widetilde{Q}_n(x) = \frac{u(\alpha u,t\beta uq;q)_\infty}{(t\alpha u,\beta u;q)_\infty}\Big[&\Big\{\widetilde{Q}_1\widetilde{Q}_0^{(2)}(x) - \widetilde{Q}_0\widetilde{Q}_1^{(2)}(x)\Big\}\,\widetilde{Q}_n^{(1)}(x) \\ &+ \Big\{\widetilde{Q}_0\widetilde{Q}_1^{(1)}(x) - \widetilde{Q}_1\widetilde{Q}_0^{(1)}(x)\Big\}\,\widetilde{Q}_n^{(2)}(x)\Big]\end{aligned} \tag{2.7}$$

and

$$\lim_{n\to\infty} u^n\widetilde{Q}_n(x) = \frac{u(t\beta uq,\alpha u;q)_\infty}{(u^2;q)_\infty}\Big\{\widetilde{Q}_1\widetilde{Q}_0^{(2)}(x) - \widetilde{Q}_0\widetilde{Q}_1^{(2)}(x)\Big\}. \tag{2.8}$$

3. Strict Partitions

A partition μ is *strict* if all its parts are distinct. One represents the associated shifted diagram of μ as a diagram in which the ith row from the top has been shifted to the right by i places so that the first column becomes a diagonal. A strict partition can be written uniquely in the form $\mu = (\mu_1, \ldots, \mu_{2n})$ where n is an non-negative integer and $\mu_1 > \mu_2 > \cdots > \mu_{2n} \geq 0$. The *length* $\ell(\mu)$ is, by definition, the number of nonzero parts of μ. We define the weight function $\omega(\mu)$ exactly the same as in (1.1). For example, if $\mu = (8, 5, 3)$, then $\ell(\mu) = 3$, $\omega(\mu) = a^6b^5c^3d^2$ and its shifted diagram is as follows.



Let

$$\Psi_N = \Psi_N(a,b,c,d;z) = \sum \omega(\mu) z^{\ell(\mu)}, \tag{3.1}$$

where the sum is over all strict partitions μ such that each part of μ is less than or equal to N. For example, we have

$$\begin{aligned}
\Psi_0 &= 1,\\
\Psi_1 &= 1 + az,\\
\Psi_2 &= 1 + a(1+b)z + abcz^2,\\
\Psi_3 &= 1 + a(1+b+ab)z + abc(1+a+ad)z^2 + a^3bcdz^3.
\end{aligned}$$

In fact, the only strict partition such that $\ell(\mu) = 0$ is $\emptyset$, the strict partitions μ such that $\ell(\mu) = 1$ and $\mu_1 \le 3$ are the following three:

$$\begin{array}{|c|}\hline a\\\hline\end{array} \qquad \begin{array}{|c|c|}\hline a & b\\\hline\end{array} \qquad \begin{array}{|c|c|c|}\hline a & b & a\\\hline\end{array}\,,$$

the strict partitions μ such that $\ell(\mu) = 2$ and $\mu_1 \le 3$ are the following three:

$$\begin{array}{cc}\hline \multicolumn{1}{|c|}{a} & \multicolumn{1}{c|}{b}\\\hline & \multicolumn{1}{|c|}{c}\\\cline{2-2}\end{array} \qquad \begin{array}{ccc}\hline \multicolumn{1}{|c|}{a} & \multicolumn{1}{c|}{b} & \multicolumn{1}{c|}{a}\\\hline & \multicolumn{1}{|c|}{c} & \\\cline{2-2}\end{array} \qquad \begin{array}{ccc}\hline \multicolumn{1}{|c|}{a} & \multicolumn{1}{c|}{b} & \multicolumn{1}{c|}{a}\\\hline & \multicolumn{1}{|c|}{c} & \multicolumn{1}{c|}{d}\\\cline{2-3}\end{array}\,,$$

and the strict partition μ such that $\ell(\mu) = 3$ and $\mu_1 \le 3$ is the following one:

$$\begin{array}{ccc}\hline \multicolumn{1}{|c|}{a} & \multicolumn{1}{c|}{b} & \multicolumn{1}{c|}{a}\\\hline & \multicolumn{1}{|c|}{c} & \multicolumn{1}{c|}{d}\\\cline{2-3} & & \multicolumn{1}{|c|}{a}\\\cline{3-3}\end{array}\,.$$

The sum of the weights of these strict partitions is equal to Ψ_3. In this section we always assume $|a|, |b|, |c|, |d| < 1$. One of the main results of this section is that the *even index* terms and the *odd index* terms of Ψ_N respectively satisfy the associated Al-Salam-Chihara recurrence relation:

Theorem 3.1. Set $q = abcd$. Let $\Psi_N = \Psi_N(a, b, c, d; z)$ be as in (3.1) and put $X_N = \Psi_{2N}$ and $Y_N = \Psi_{2N+1}$. Then X_N and Y_N satisfy

$$\begin{aligned}
X_{N+1} = \left\{1 + ab + a(1+bc)z^2q^N\right\} X_N &\\
- ab(1 - z^2q^N)(1 - acz^2q^{N-1})X_{N-1}, & \qquad (3.2)
\end{aligned}$$

$$\begin{aligned}
Y_{N+1} = \left\{1 + ab + abc(1+ad)z^2q^N\right\} Y_N &\\
- ab(1 - z^2q^N)(1 - acz^2q^N)Y_{N-1}, & \qquad (3.3)
\end{aligned}$$

where $X_0 = 1$, $Y_0 = 1 + az$, $X_1 = 1 + a(1 + b)z + abcz^2$ and

$$Y_1 = 1 + a(1 + b + ab)z + abc(1 + a + ad)z^2 + a^3bcdz^3.$$

Especially, if we put $X'_N = (ab)^{-\frac{N}{2}} X_N$ and $Y'_N = (ab)^{-\frac{N}{2}} Y_N$, then X'_N and Y'_N satisfy

$$\begin{aligned}
\left\{(ab)^{\frac12} + (ab)^{-\frac12}\right\} X'_N = X'_{N+1} - a^{\frac12}b^{-\frac12}(1+bc)z^2q^N X'_N &\\
+ (1 - z^2q^N)(1 - acz^2q^{N-1})X'_{N-1}, & \qquad (3.4)
\end{aligned}$$

$$\begin{aligned}
\left\{(ab)^{\frac12} + (ab)^{-\frac12}\right\} Y'_N = Y'_{N+1} - a^{\frac12}b^{\frac12}c(1+ad)z^2q^N Y'_N &\\
+ (1 - z^2q^N)(1 - a^2bc^2dz^2q^{N-1})Y'_{N-1}, & \qquad (3.5)
\end{aligned}$$

where $X'_0 = 1$, $Y'_0 = 1 + az$, $X'_1 = (ab)^{-\frac12} + a^{\frac12}b^{-\frac12}(1 + b)z + (ab)^{\frac12}cz^2$ and

$$Y'_1 = (ab)^{-\frac12} + a^{\frac12}b^{-\frac12}(1 + b + ab)z + a^{\frac12}b^{\frac12}c(1 + a + ad)z^2 + a^{\frac52}b^{\frac12}cdz^3.$$

One concludes that, when $|a|, |b|, |c|, |d| < 1$, the solutions of (3.2) and (3.3) are expressed by the linear combinations of (2.3) and (2.4) as follows.

Theorem 3.2. Assume $|a|, |b|, |c|, |d| < 1$ and set $q = abcd$. Let $\Psi_N = \Psi_N(a, b, c, d; z)$ be as in (3.1).

(i) Put $X_N = \Psi_{2N}$. Then we have

$$\begin{aligned}X_N = &\frac{(-az^2q,-abc;q)_\infty}{(-a,-abcz^2;q)_\infty}\\ &\times\left\{(s_0^X X_1 - s_1^X X_0)(-abcz^2;q)_N\, {}_2\phi_1\left(\begin{matrix} q^{-N}z^{-2},-b^{-1}\\ -(abc)^{-1}q^{-N+1}z^{-2}\end{matrix};q,-c^{-1}q\right)\right.\\ &\left.+(r_1^X X_0 - r_0^X X_1)(ab)^N\frac{(qz^2,acz^2;q)_N}{(-aqz^2;q)_N}\,{}_2\phi_1\left(\begin{matrix} q^{N+1}z^2,-c^{-1}q\\ -aq^{N+1}z^2\end{matrix};q,-abc\right)\right\},\end{aligned}\tag{3.6}$$

where

$$\begin{aligned}r_0^X &= {}_2\phi_1\left(\begin{matrix} z^{-2},-b^{-1}\\ -(abc)^{-1}z^{-2}q\end{matrix};q,-c^{-1}q\right),\\ s_0^X &= {}_2\phi_1\left(\begin{matrix} z^2q,-c^{-1}q\\ -az^2q\end{matrix};q,-abc\right),\\ r_1^X &= (1+abcz^2)\,{}_2\phi_1\left(\begin{matrix} z^{-2}q^{-1},-b^{-1}\\ -(abc)^{-1}z^{-2}\end{matrix};q,-c^{-1}q\right),\\ s_1^X &= \frac{ab(1-z^2q)(1-acz^2)}{1+az^2q}\,{}_2\phi_1\left(\begin{matrix} z^2q^2,-c^{-1}q\\ -az^2q^2\end{matrix};q,-abc\right).\end{aligned}$$

(ii) Put $Y_N = \Psi_{2N+1}$. Then we have

$$\begin{aligned}Y_N = &\frac{(-a^2bcdz^2q,-abc;q)_\infty}{(-a^2bcd,-abcz^2;q)_\infty}\\ &\times\left\{(s_0^Y Y_1 - s_1^Y Y_0)(-abcz^2;q)_N\, {}_2\phi_1\left(\begin{matrix} q^{-N}z^{-2},-acd\\ -(abc)^{-1}q^{-N+1}z^{-2}\end{matrix};q,-c^{-1}q\right)\right.\\ &\left.+(r_1^Y Y_0 - r_0^Y Y_1)(ab)^N\frac{(qz^2,a^2bc^2dz^2;q)_N}{(-a^2bcdqz^2;q)_N}\,{}_2\phi_1\left(\begin{matrix} q^{N+1}z^2,-c^{-1}q\\ -a^2bcdq^{N+1}z^2\end{matrix};q,-abc\right)\right\},\end{aligned}\tag{3.7}$$

where

$$\begin{aligned}r_0^Y &= {}_2\phi_1\left(\begin{matrix} z^{-2},-acd\\ (-abc)^{-1}qz^{-2}\end{matrix};q,-c^{-1}q\right),\\ r_1^Y &= (1+abcz^2)\,{}_2\phi_1\left(\begin{matrix} q^{-1}z^{-2},-ac\\ -(abc)^{-1}z^{-2}\end{matrix};q,-c^{-1}q\right),\\ s_0^Y &= {}_2\phi_1\left(\begin{matrix} z^2q,-c^{-1}q\\ -a^2bcdz^2q\end{matrix};q,-abc\right),\\ s_1^Y &= \frac{ab(1-z^2q)(1-a^2bc^2dz^2)}{1+a^2bcdz^2q}\,{}_2\phi_1\left(\begin{matrix} z^2q^2,-c^{-1}q\\ -a^2bcdz^2q^2\end{matrix};q,-abc\right).\end{aligned}$$

If we take the limit $N\to\infty$ in (3.6) and (3.7), then by using (2.8), we obtain the following generalization of Boulet's result (see Corollary 3.6).

Corollary 3.3. Assume $|a|,|b|,|c|,|d|<1$ and set $q=abcd$. Let s_i^X, s_i^Y, X_i, Y_i $(i=0,1)$ be as in the above theorem. Then we have

$$\begin{aligned}\sum_\mu \omega(\mu)z^{\ell(\mu)} &= \frac{(-abc,-az^2q;q)_\infty}{(ab;q)_\infty}(s_0^X X_1 - s_1^X X_0)\\ &= \frac{(-abc,-a^2bcdz^2q;q)_\infty}{(ab;q)_\infty}(s_0^Y Y_1 - s_1^Y Y_0),\end{aligned}\tag{3.8}$$

where the sum runs over all strict partitions.

Especially, by substituting $z=1$ into (3.6) and (3.7), we conclude that the solutions of the recurrence relations (3.4) and (3.5) with the above initial condition are exactly the Al-Salam-Chihara polynomials:

Theorem 3.4. Put $u=\sqrt{ab}$, $x=\frac{u+u^{-1}}{2}$ and $q=abcd$. Let $\Psi_N(a,b,c,d;z)$ be as in (3.1).

(i) The polynomial $\Psi_{2N}(a,b,c,d;1)$ is given by

$$\Psi_{2N}(a,b,c,d;1) = (ab)^{\frac{N}{2}} Q_N(x; -a^{\frac12}b^{\frac12}c, -a^{\frac12}b^{-\frac12}|q),$$

$$= (-a;q)_N\, {}_2\phi_1\left(\begin{matrix} q^{-N}, -c \\ -a^{-1}q^{-N+1}\end{matrix}; q, -bq\right). \tag{3.9}$$

(ii) The polynomial $\Psi_{2N+1}(a,b,c,d;1)$ is given by

$$\Psi_{2N+1}(a,b,c,d;1) = (1+a)(ab)^{\frac{N}{2}} Q_N(x; -a^{\frac12}b^{\frac12}c, -a^{\frac32}b^{\frac12}cd|q)$$

$$= (-a;q)_{N+1}\, {}_2\phi_1\left(\begin{matrix} q^{-N}, -c \\ -a^{-1}q^{-N}\end{matrix}; q, -b\right). \tag{3.10}$$

If we substitute $a = zyq$, $b = z^{-1}yq$, $c = zy^{-1}q$ and $d = z^{-1}y^{-1}q$ into Theorem 3.4, then we immediately obtain the following corollary, which is a strict version of Andrews' result.

Corollary 3.5.

$$\sum_{\substack{\mu \text{ strict partitions} \\ \mu_1 \le 2N}} z^{\mathcal{O}(\mu)} y^{\mathcal{O}(\mu')} q^{|\mu|} = \sum_{j=0}^{N} \begin{bmatrix} N \\ j \end{bmatrix}_{q^4} (-zyq;q^4)_j (-zy^{-1}q;q^4)_{N-j} (yq)^{2N-2j}, \tag{3.11}$$

and

$$\sum_{\substack{\mu \text{ strict partitions} \\ \mu_1 \le 2N+1}} z^{\mathcal{O}(\mu)} y^{\mathcal{O}(\mu')} q^{|\mu|} = \sum_{j=0}^{N} \begin{bmatrix} N \\ j \end{bmatrix}_{q^4} (-zyq;q^4)_{j+1} (-zy^{-1}q;q^4)_{N-j} (yq)^{2N-2j}, \tag{3.12}$$

where

$$\begin{bmatrix} N \\ j \end{bmatrix}_q = \begin{cases} \frac{(1-q^N)(1-q^{N-1})\cdots(1-q^{N-j+1})}{(1-q^j)(1-q^{j-1})\cdots(1-q)}, & \text{for } 0 \le j \le N, \\ 0, & \text{if } j < 0 \text{ and } j > N. \end{cases}$$

If we put $N \to \infty$ in Corollary 3.4, then we immediately obtain the following corollary (cf. Corollary 2 of [**3**]). We can also prove this corollary by setting $z \to 1$ in (3.8).

Corollary 3.6. (Boulet) Let $q = abcd$, then

$$\sum_{\mu} \omega(\mu) = \frac{(-a;q)_\infty (-abc;q)_\infty}{(ab;q)_\infty}, \tag{3.13}$$

where the sum runs over all strict partitions μ.

4. Ordinary Partitions

First we present a generalization of Andrews' result in [**1**]. Let us consider

$$\Phi_N = \Phi_N(a,b,c,d;z) = \sum_{\substack{\lambda \\ \lambda_1 \le N}} \omega(\lambda) z^{\ell(\lambda)}, \tag{4.1}$$

where the sum runs over all partitions λ such that each part of λ is less than or equal to N. For example, the first few terms can be computed directly as follows:

$$\Phi_0 = 1,$$

$$\Phi_1 = \frac{1+az}{1-acz^2},$$

$$\Phi_2 = \frac{1+a(1+b)z+abcz^2}{(1-acz^2)(1-qz^2)},$$

$$\Phi_3 = \frac{1+a(1+b+ab)z+abc(1+a+ad)z^2+a^3bcdz^3}{(1-z^2ac)(1-z^2q)(1-z^2acq)},$$

where $q = abcd$ as before. If one compares these with the first few terms of Ψ_n, one can easily guess the following theorem holds:

THEOREM 4.1. Let N be a non-negative integer, and let $\Phi_N = \Phi_N(a,b,c,d;z)$ be as in (4.1). Then we have

$$\Phi_N(a,b,c,d;z) = \frac{\Psi_N(a,b,c,d;z)}{(z^2q;q)_{\lfloor N/2 \rfloor}(z^2ac;q)_{\lceil N/2 \rceil}}, \tag{4.2}$$

where $\Psi_N = \Psi_N(a,b,c,d;z)$ is the generating function defined in (3.1). Note that Ψ_N is explicitly given in terms of basic hypergeometric functions in Theorem 3.2.

First of all, as an immediate corollary of Theorem 4.1 and Corollary 3.3, we obtain the following generalization of Boulet's result.

COROLLARY 4.2. Assume $|a|,|b|,|c|,|d| < 1$ and set $q = abcd$. Let s_i^X, s_i^Y, X_i, Y_i $(i = 0,1)$ be as in Theorem 3.2. Then we have

$$\sum_{\lambda} \omega(\lambda) z^{|\mu|} = \frac{(-abc, -az^2q;q)_\infty}{(ab, acz^2, z^2q;q)_\infty}(s_0^X X_1 - s_1^X X_0) \tag{4.3}$$

where the sum runs over all partitions λ.

Theorem 4.1 and Theorem 3.4 also give the following corollary:

COROLLARY 4.3. Put $x = \frac{(ab)^{\frac{1}{2}}+(ab)^{-\frac{1}{2}}}{2}$ and $q = abcd$. Let $\Phi_N = \Phi_N(a,b,c,d;z)$ be as in (4.1).

(i) The generating function $\Phi_{2N}(a,b,c,d;1)$ is given by

$$\begin{aligned}\Phi_{2N}(a,b,c,d;1) &= \frac{(ab)^{\frac{N}{2}} Q_N(x; -a^{\frac{1}{2}}b^{\frac{1}{2}}c, -a^{\frac{1}{2}}b^{-\frac{1}{2}}|q)}{(q;q)_N(ac;q)_N} \\ &= \frac{(-a;q)_N}{(q;q)_N(ac;q)_N}\, {}_2\phi_1\left(\begin{matrix} q^{-N}, -c \\ -a^{-1}q^{-N+1} \end{matrix}; q, -bq\right).\end{aligned} \tag{4.4}$$

(ii) The generating function $\Phi_{2N}(a,b,c,d;1)$ is given by

$$\begin{aligned}\Phi_{2N+1}(a,b,c,d;1) &= \frac{(1+a)(ab)^{\frac{N}{2}} Q_N(x; -a^{\frac{1}{2}}b^{\frac{1}{2}}c, -a^{\frac{3}{2}}b^{\frac{1}{2}}cd|q)}{(q;q)_N(ac;q)_{N+1}} \\ &= \frac{(-a;q)_{N+1}}{(q;q)_N(ac;q)_{N+1}}\, {}_2\phi_1\left(\begin{matrix} q^{-N}, -c \\ -a^{-1}q^{-N} \end{matrix}; q, -b\right).\end{aligned} \tag{4.5}$$

As before we immediately deduce the following corollary from Corollary 4.3. Let $S_N(n,r,s)$ denote the number of partitions π of n where each part of π is $\leq N$, $\mathcal{O}(\pi) = r$, $\mathcal{O}(\pi') = s$. Then we have the result of Andrews [**1**, Theorem 1].

COROLLARY 4.4. (Andrews)

$$\sum_{n,r,s \geq 0} S_{2N}(n,r,s) q^n z^r y^s = \frac{\sum_{j=0}^N \begin{bmatrix} N \\ j \end{bmatrix}_{q^4} (-zyq;q^4)_j(-zy^{-1}q;q^4)_{N-j}(yq)^{2N-2j}}{(q^4;q^4)_N(z^2q^4;q^4)_N}, \tag{4.6}$$

and

$$\sum_{n,r,s \geq 0} S_{2N+1}(n,r,s) q^n z^r y^s = \frac{\sum_{j=0}^N \begin{bmatrix} N \\ j \end{bmatrix}_{q^4} (-zyq;q^4)_{j+1}(-zy^{-1}q;q^4)_{N-j}(yq)^{2N-2j}}{(q^4;q^4)_N(z^2q^4;q^4)_{N+1}}. \tag{4.7}$$

COROLLARY 4.5. (Boulet) Let $q = abcd$, then

$$\sum_{\lambda \text{ partitions}} \omega(\lambda) = \frac{(-a;q)_\infty(-abc;q)_\infty}{(q;q)_\infty(ab;q)_\infty(ac;q)_\infty}. \tag{4.8}$$

Here the sum runs over all partitions λ (cf. [**3**, Theorem 1]).

First we show the following recurrence equations hold.

Proposition 4.6. Let $\Phi_N = \Phi_N(a,b,c,d;z)$ be as before and $q = abcd$. Then the following recurrences hold for any positive integer N.

$$(1 - z^2 q^N)\Phi_{2N} = (1+b)\Phi_{2N-1} - b\,\Phi_{2N-2}, \tag{4.9}$$

$$(1 - z^2 acq^N)\Phi_{2N+1} = (1+a)\Phi_{2N} - a\Phi_{2N-1}. \tag{4.10}$$

5. A weighted sum of Schur's P-functions

We use the notation $X = X_n = (x_1,\dots,x_n)$ for the finite set of variables $x_1, \dots, x_n$. In [**8**], one of the authors used a Paffian expression of $\sum_\lambda \omega(\lambda)s_\lambda(X)$ to prove Stanley's open problem, where the sum runs over all partitions λ and $s_\lambda(X)$ stands for the Schur function with respect to a partition λ. The aim of this section is to give some determinantial formulas for the weighted sum $\sum \omega(\mu)z^{\ell(\mu)}P_\mu(x)$ where $P_\mu(x)$ is Schur's P-function.

Let A_n denote the skew-symmetric matrix

$$\left(\frac{x_i - x_j}{x_i + x_j}\right)_{1\le i,j\le n}$$

and for each strict partition $\mu = (\mu_1,\dots,\mu_l)$ of length $l \le n$, let Γ_μ denote the $n \times l$ matrix $\left(x_j^{\mu_i}\right)$. Let

$$A_\mu(x_1,\dots,x_n) = \begin{pmatrix} A_n & \Gamma_\mu J_l \\ -J_l {}^t\Gamma_\mu & O_l \end{pmatrix}$$

which is a skew-symmetric matrix of $(n+l)$ rows and columns. Define $\mathrm{Pf}_\mu(x_1,\dots,x_n)$ to be $\mathrm{Pf}\,A_\mu(x_1,\dots,x_n)$ if $n+l$ is even, and to be $\mathrm{Pf}\,A_\mu(x_1,\dots,x_n,0)$ if $n+l$ is odd. By Ex.13, p.267, [**11**], Schur's P-function $P_\mu(x_1,\dots,x_n)$ is defined to be

$$\frac{\mathrm{Pf}_\mu(x_1,\dots,x_n)}{\mathrm{Pf}_\emptyset(x_1,\dots,x_n)},$$

where it is well-known that $\mathrm{Pf}_\emptyset(x_1,\dots,x_n) = \prod_{1\le i<j\le n}\frac{x_i-x_j}{x_i+x_j}$. Meanwhile, by (8.7), p.253, [**11**], Schur's Q-function $Q_\mu(x_1,\dots,x_n)$ is defined to be $2^{\ell(\lambda)}P_\mu(x_1,\dots,x_n)$.

In this section, we consider a weighted sum of Schur's P-functions and Q-functions, i.e.

$$\xi_N(a,b,c,d;X_n) = \sum_{\substack{\mu\\ \mu_1\le N}} \omega(\mu)P_\mu(x_1,\dots,x_n),$$

$$\eta_N(a,b,c,d;X_n) = \sum_{\substack{\mu\\ \mu_1\le N}} \omega(\mu)Q_\mu(x_1,\dots,x_n),$$

where the sums run over all strict partitions μ such that each part of μ is less than or equal to N. More generally, we can unify these problems to finding the following sum:

$$\zeta_N(a,b,c,d;z;X_n) = \sum_{\substack{\mu\\ \mu_1\le N}} \omega(\mu)z^{\ell(\mu)}P_\mu(x_1,\dots,x_n), \tag{5.1}$$

where the sum runs over all strict partitions μ such that each part of μ is less than or equal to N. One of the main results of this section is that $\zeta_N(a,b,c,d;z;X_n)$ can be expressed by a Pfaffian. Further, let us put

$$\zeta(a,b,c,d;z;X_n) = \lim_{N\to\infty}\zeta_N(a,b,c,d;z;X_n) = \sum_\mu \omega(\mu)z^{\ell(\mu)}P_\mu(X_n), \tag{5.2}$$

where the sum runs over all strict partitions. We also write

$$\xi(a,b,c,d;X_n) = \zeta(a,b,c,d;1;X_n) = \sum_\mu \omega(\mu)P_\mu(X_n),$$

where the sum runs over all strict partitions. Then we have the following theorem:

Theorem 5.1. Let n be a positive integer. Then

$$\zeta(a,b,c,d;z;X_n) = \begin{cases} \mathrm{Pf}\,(\gamma_{ij})_{1\le i<j\le n} \,/\, \mathrm{Pf}_\emptyset(X_n) & \text{if } n \text{ is even,} \\ \mathrm{Pf}\,(\gamma_{ij})_{0\le i<j\le n} \,/\, \mathrm{Pf}_\emptyset(X_n) & \text{if } n \text{ is odd,} \end{cases} \tag{5.3}$$

where

$$\gamma_{ij} = \frac{x_i - x_j}{x_i + x_j} + u_{ij} z + v_{ij} z^2 \tag{5.4}$$

with

$$u_{ij} = \frac{a \det \begin{pmatrix} x_i + b x_i^2 & 1 - ab x_i^2 \\ x_j + b x_j^2 & 1 - ab x_j^2 \end{pmatrix}}{(1 - ab x_i^2)(1 - ab x_j^2)}, \tag{5.5}$$

$$v_{ij} = \frac{abc x_i x_j \det \begin{pmatrix} x_i + a x_i^2 & 1 - a(b+d) x_i^2 - abd x_i^3 \\ x_j + a x_j^2 & 1 - a(b+d) x_j^2 - abd x_j^3 \end{pmatrix}}{(1 - ab x_i^2)(1 - ab x_j^2)(1 - abcd x_i^2 x_j^2)}, \tag{5.6}$$

if $1 \le i, j \le n$, and

$$\gamma_{0j} = 1 + \frac{a x_j (1 + b x_j)}{1 - ab x_j^2} z \tag{5.7}$$

if $1 \le j \le n$.

Especially, when $z = 1$, we have

$$\xi(a, b, c, d; X_n) = \begin{cases} \operatorname{Pf}\left(\widetilde{\gamma}_{ij}\right)_{1 \le i < j \le n} / \operatorname{Pf}_{\emptyset}(X_n) & \text{if } n \text{ is even,} \\ \operatorname{Pf}\left(\widetilde{\gamma}_{ij}\right)_{0 \le i < j \le n} / \operatorname{Pf}_{\emptyset}(X_n) & \text{if } n \text{ is odd,} \end{cases} \tag{5.8}$$

where

$$\widetilde{\gamma}_{ij} = \begin{cases} \frac{1 + a x_j}{1 - ab x_j^2} & \text{if } i = 0, \\ \frac{x_i - x_j}{x_i + x_j} + \widetilde{v}_{ij} & \text{if } 1 \le i < j \le n, \end{cases} \quad \textit{with} \tag{5.9}$$

$$\widetilde{v}_{ij} = \frac{a \det \begin{pmatrix} x_i + b x_i^2 & 1 - b(a+c) x_i^2 - abc x_i^3 \\ x_j + b x_j^2 & 1 - b(a+c) x_j^2 - abc x_j^3 \end{pmatrix}}{(1 - ab x_i^2)(1 - ab x_j^2)(1 - abcd x_i^2 x_j^2)}. \tag{5.10}$$

We can generalize this result in the following theorem (Theorem 5.2) using the generalized Vandermonde determinant used in [**9**]. Let n be an non-negative integer, and let $X = (x_1, \dots, x_{2n})$, $Y = (y_1, \dots, y_{2n})$, $A = (a_1, \dots, a_{2n})$ and $B = (b_1, \dots, b_{2n})$ be $2n$-tuples of variables. Let $V^n(X, Y, A)$ denote the $2n \times n$ matrix whose (i, j)th entry is $a_i x_i^{n-j} y_i^{j-1}$ for $1 \le i \le 2n$, $1 \le j \le n$, and let $U^n(X, Y; A, B)$ denote the $2n \times 2n$ matrix $\begin{pmatrix} V^n(X, Y, A) & V^n(X, Y, B) \end{pmatrix}$. For instance if $n = 2$ then $U^2(X, Y; A, B)$ is

$$\begin{pmatrix} a_1 x_1 & a_1 y_1 & b_1 x_1 & b_1 y_1 \\ a_2 x_2 & a_2 y_2 & b_2 x_2 & b_2 y_2 \\ a_3 x_3 & a_3 y_3 & b_3 x_3 & b_3 y_3 \\ a_4 x_4 & a_4 y_4 & b_4 x_4 & b_4 y_4 \end{pmatrix}.$$

Hereafter we use the following notation for n-tuples $X = (x_1, \cdots, x_n)$ and $Y = (y_1, \cdots, y_n)$ of variables:

$$X + Y = (x_1 + y_1, \dots, x_n + y_n), \quad X \cdot Y = (x_1 y_1, \dots, x_n y_n),$$

and, for integers k and l,

$$X^k = (x_1^k, \dots, x_n^k), \quad X^k Y^l = (x_1^k y_1^l, \dots, x_n^k y_n^l).$$

Let $\mathbf{1}$ denote the n-tuple $(1, \dots, 1)$. For any subset $I = \{i_1, \dots, i_r\} \in \binom{[n]}{r}$, let X_I denote the r-tuple $(x_{i_1}, \dots, x_{i_r})$.

Theorem 5.2. *Let $q = abcd$. If n is an even integer, then we have*

$$\xi(a, b, c, d; X_n) = \sum_{r=0}^{n/2} \sum_{I \in \binom{[n]}{2r}} \frac{(-1)^{|I| - \binom{r+1}{2}} a^r q^{\binom{r}{2}}}{\prod_{i \in I} (1 - ab x_i^2)} \prod_{\substack{i, j \in I \\ i < j}} \frac{x_i + x_j}{(x_i - x_j)(1 - q x_i^2 x_j^2)}$$

$$\times \det U^r(X_I^2, \mathbf{1} + q X_I^4, X_I + b X_I^2, \mathbf{1} - b(a+c) X_I^2 - abc X_I^3). \tag{5.11}$$

If n is an odd integer, then we have

$$\xi(a,b,c,d;X_n)=\sum_{m=1}^{n}\frac{1+ax_m}{1-abx_m^2}\sum_{r=0}^{(n-1)/2}\sum_{I\in\binom{[n]\setminus\{m\}}{2r}}\frac{(-1)^{|I|-\binom{r+1}{2}}a^r q^{\binom{r}{2}}}{\prod_{i\in I}(1-abx_i^2)}\prod_{i\in I}\frac{x_m+x_i}{x_m-x_i}$$
$$\times\prod_{\substack{i,j\in I\\ i<j}}\frac{x_i+x_j}{(x_i-x_j)(1-qx_i^2x_j^2)}\cdot\det U^r(X_I^2,\mathbf{1}+qX_I^4,X_I+bX_I^2,\mathbf{1}-b(a+c)X_I^2-abcX_I^3). \tag{5.12}$$

Theorem 5.3. Let $q=abcd$. If n is an even integer, then $\zeta(a,b,c,d;z;X_n)$ is equal to

$$\sum_{r=0}^{n/2}z^{2r}\sum_{I\in\binom{[n]}{2r}}\frac{(-1)^{|I|-\binom{r+1}{2}}(abc)^r q^{\binom{r}{2}}\prod_{i\in I}x_i}{\prod_{i\in I}(1-abx_i^2)}\prod_{\substack{i,j\in I\\ i<j}}\frac{x_i+x_j}{(x_i-x_j)(1-qx_i^2x_j^2)}$$
$$\times\det V^r(X_I^2,\mathbf{1}+qX_I^4,X_I+aX_I^2,\mathbf{1}-a(b+d)X_I^2-abdX_I^3)$$
$$+\sum_{r=0}^{n/2}z^{2r-1}\sum_{I\in\binom{[n]}{2r}}\sum_{\substack{k<l\\ k,l\in I}}\frac{(-1)^{|I|-\binom{r}{2}-1}a^rb^{r-1}c^{r-1}q^{\binom{r-1}{2}}\{1+b(x_k+x_l)+abx_kx_l\}\prod_{i\in I'}x_i}{\prod_{i\in I}(1-abx_i^2)}$$
$$\times\frac{\prod_{\substack{i,j\in I\\ i<j}}(x_i+x_j)\cdot\det V^{r-1}(X_{I'}^2,\mathbf{1}+qX_{I'}^4,X_{I'}+aX_{I'}^2,\mathbf{1}-a(b+d)X_{I'}^2-abdX_{I'}^3)}{\prod_{\substack{i,j\in I'\\ i<j}}(x_i-x_j)(1-qx_i^2x_j^2)}, \tag{5.13}$$

where $I'=I\setminus\{k,l\}$.

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Faculty of Education, Tottori University, Koyama, Tottori, Japan
E-mail address: `ishikawa@fed.tottori-u.ac.jp`

Institut Camille Jordan, Université Claude Bernard Lyon I, 43, boulevard du 11 novembre 1918, 69622 Villeurbanne Cedex, France
E-mail address: `zeng@math.univ-lyon1.fr`