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Central Delannoy numbers, Legendre polynomials, and a balanced join operation preserving the Cohen-Macaulay property

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Abstract. We introduce a new join operation on colored simplicial complexes that preserves the Cohen-Macaulay property. An example of this operation puts the connection between the central Delannoy numbers and Legendre polynomials in a wider context.

Résumé. Nous introduisons une nouvelle opération qui joint des complexes simpliciaux équilibrés d'une telle manière que la propriété de Cohen-Macaulay est preservée. Une exemple de cette opération remette la rélation entre les nombres Delannoy centraux et les polynomiaux de Legendre dans un contexte plus large.

Introduction

The Delannoy numbers, introduced by Henri Delannoy [**7**] more than a hundred years ago, became recently subject of renewed interest, mostly in connection with lattice path enumeration problems. It was also noted for more than half a century, that a somewhat mysterious connection exists between the central Delannoy numbers and Legendre polynomials. This relation was mostly dismissed as a "coincidence" since the Legendre polynomials do not seem to appear otherwise in connection with lattice path enumeration questions.

In our work we attempt to lift a corner of the shroud covering this mystery. First we observe that a variant of table A049600 in the On-Line Encyclopedia of Integer Sequences [**13**] embeds the central Delannoy numbers into another, asymmetric table, and the entries of this table may be expressed by a generalization of the Legendre polynomial substitution formula: the non-diagonal entries are connected to Jacobi polynomials. Then we show that the lattice path enumeration problem associated to these asymmetric Delannoy numbers is naturally identifiable with a 2-colored lattice path enumeration problem (Section 2). This variant helps represent each asymmetric Delannoy number as the number of facets in the *balanced join* of a simplex and the order complex of a fairly transparent poset which we call a Jacobi poset. The balanced join operation takes two balanced simplicial complexes colored with the same set of colors as its input and yields a balanced simplicial complex colored with the same set of colors as its output. It is introduced in Section 3, which also describes the Jacobi posets.

The balanced join operation we were lead to introduce turns out to be fairly interesting by its own merit. According to a famous result of Stanley [**14**], the h-vector of a balanced Cohen-Macaulay is the f-vector of another colored complex. (The converse, and the generalization to flag numbers was shown by Björner, Frankl, and Stanley [**2**].) Since the proof is algebraic, it is usually hard to construct the colored complex explicitly. Using the balanced join operation, we may construct balanced simplicial complexes as the balanced join of a balanced complex and a simplex such that the h-vector of the join is the f-vector of the original colored complex. This applies even if the balanced join does not have the Cohen-Macaulay property. Our main result is Theorem 4.3, stating that the balanced join of two balanced Cohen-Macaulay simplicial complexes is Cohen-Macaulay.

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In Section 5 we return to the Jacobi posets introduced in Section 3 and prove that their order complex is Cohen-Macaulay, thus our main result is applicable to the example that inspired it. The proof consists of providing an EL-labeling from which the the Cohen-Macaulay property follows by the results of Björner and Wachs [**3**] and [**4**]. By the results of Björner, Frankl, and Stanley [**2**] the flag h-vector of a Jacobi poset is the flag f-vector of a colored complex. We find this colored complex as the order complex of a *strict direct product* of two chains. We define the strict direct product of two posets by requiring a strict inequality in both coordinates.

Since removing the top and bottom elements from a Jacobi poset yields a "half-strict" direct product of two chains, there is another potentially interesting bivariate operation looming on the horizon. In Section 6 we introduce a right-strict direct product on posets that allows to assign to a pair (P, Q) of an arbitrary poset P and a graded poset Q a graded poset of the same rank as Q. There is reason to suspect that this product too, preserves the Cohen-Macaulay property, as we can show that the flag h-vector of a right-strict product is positive if the order complex of P has a positive h-vector, and Q has a positive flag h-vector.

In the concluding Section 7 we point out the impossibility of two seemingly plausible generalizations, and highlight the question in commutative algebra that arises when we try to generalize our main result, Theorem 4.3.

The journey taken will hopefully convince more mathematicians that Delannoy numbers are interesting, since they lead to some interesting results and questions in commutative algebra and algebraic combinatorics.

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1. Preliminaries

1.1. Delannoy numbers. The *Delannoy array* $(d_{i,j} : i, j \in \mathbb{Z})$ was introduced by Henri Delannoy [**7**] in the nineteenth century. This array may be defined by the recursion formula

$$d_{i,j} = d_{i-1,j} + d_{i,j-1} + d_{i-1,j-1} \tag{1.1}$$

with the conditions $d_{0,0} = 1$ and $d_{i,j} = 0$ if $i < 0$ or $j < 0$. For $i, j \geq 0$ the number $d_{i,j}$ represents the number of lattice walks from $(0, 0)$ to (i, j) with steps $(1, 0)$, $(0, 1)$, and $(1, 1)$ The significance of these numbers is explained within a historic context in the paper "Why Delannoy numbers?" [**1**] by Banderier and Schwer. The diagonal elements $(d_{n,n} : n \geq 0)$ in this array are the *(central) Delannoy numbers* (A001850 of Sloane [**13**]). These numbers are known through the books of Comtet [**6**] and Stanley [**16**], but it is Sulanke's paper [**17**] that gives the most complete list of all known uses of Delannoy numbers (a total of 29 configurations). For more information and a detailed bibliography we refer the reader to the above mentioned sources.

1.2. Balanced simplicial complexes and the Cohen Macaulay property. A *simplicial complex* $\triangle$ on the vertex set V is a family of subsets of V, such that $\{v\} \in \triangle$ for all $v \in V$ and every subset of a $\sigma \in \triangle$ belongs to $\triangle$. An element $\sigma \in \triangle$ is a *face* and $|\sigma| - 1$ is its *dimension.* The dimension of $\triangle$ is the maximum of the dimensions of its faces. A maximal face is a *facet* and $\triangle$ is *pure* if all its facets have the same dimension. The number of i-dimensional faces is denoted by f_i. An equivalent encoding of the *f-vector* $(f_{-1}, \ldots, f_{n-1})$ of an $(n-1)$-dimensional simplicial complex is its h-vector *h-vector* $(h_0, \ldots, h_n)$ given by $h_i = \sum_{j=0}^{i} (-1)^{i-j} \binom{n-j}{i-j} f_{j-1}$. An $(n-1)$-dimensional simplicial complex $\triangle$ is *balanced* if its vertices may be colored using n colors such that every face has all its vertices colored differently. (See [**15**, 4.1 Definition].[1]) It is always assumed that a fixed coloring is part of the structure of a balanced complex. For such a complex we may refine the notions of f-vector and h-vector, as follows. Assume we use the set of colors $\{1, 2, \ldots, n\}$. For any $S \subseteq \{1, 2, \ldots, n\}$ let f_S be the number of faces whose vertices are colored exactly with the colors from S. The vector $(f_S : S \subseteq \{1, 2, \ldots, n\})$ is called the *flag f-vector* of the colored complex. The *flag h-vector* is then the vector $(h_S : S \subseteq \{1, 2, \ldots, n\})$ whose entries are given by

$$h_S = \sum_{T \subseteq S} (-1)^{|S \setminus T|} f_T.$$

[1]In the original definition of a balanced complex (occurring in [**14**]) it was also assumed that the complex is pure, but as it was observed by Stanley in [**15**, §III.4], "there is no real reason for this restriction".

A fundamental theorem on balanced simplicial complexes is Stanley's result [**14**, Corollary 4.5].

THEOREM 1.1 (Stanley). *The h-vector of a balanced Cohen-Macaulay simplicial complex is the f-vector of some other simplicial complex.*

The definition of the Cohen-Macaulay property is fairly involved, we refer the reader to Stanley [**15**]. To prove our main result, we use Reisner's criterion [**15**, Chapter II, Corollary 4.1] which characterizes Cohen-Macaulay simplicial complexes in terms of the homology groups of each *link*. The link $\mathrm{lk}_{\triangle}(\tau)$ of a face $\tau \in \triangle$ is defined by

$$\mathrm{lk}_{\triangle}(\tau) := \{\sigma \in \triangle : \sigma \cap \tau = \emptyset, \sigma \cup \tau \in \triangle\}.$$

The homology used is *simplicial homology* [**15**, Chapter 0, Section 4]. An oriented j-simplex in $\triangle$ is a j-face $\sigma = \{v_0, \ldots, v_j\} \in \triangle$, enriched with an equivalence class of orderings, two orderings being equivalent if they differ by an even permutation of vertices. We write $[v_0, v_1, \ldots, v_j]$ for the oriented simplex associated to the equivalence class of the linear order $v_0 < \ldots < v_j$. The k-module $C_j(\triangle)$ (for $j = -1, \ldots, \dim(\triangle)$, where k is a field) is then the free k-module generated by all oriented j-simplices modulo the relations $[\sigma_1] + [\sigma_2] = 0$ whenever $[\sigma_1]$ and $[\sigma_2]$ are different oriented simplices corresponding to the j-simplex. These modules, together with the boundary maps $\partial_j : C_j(\triangle) \to C_{j-1}(\triangle)$, given by

$$\partial_j[v_0, \ldots, v_j] = \sum_{i=0}^{j} (-1)^i [v_0, \ldots, \widehat{v}_i, \ldots, v_j]$$

form the *oriented chain complex* of $\triangle$. (As usual, $\widehat{v}_i$ indicates omitting v_i.) Reisner's criterion is then the following.

THEOREM 1.2 (Reisner). *The simplicial complex $\triangle$ is Cohen-Macaulay over k if and only if for all $\sigma \in \triangle$ and $i < \dim \mathrm{lk}_{\triangle}(\sigma)$ we have $\widetilde{H}_i(\mathrm{lk}_{\triangle}(\sigma), k) = 0$. Here $\widetilde{H}_i$ denotes the i-th reduced homology group of the appropriate oriented chain complex.*

Rephrasing results of his work with Björner and Frankl [**2**], Stanley refined Theorem 1.1 to flag numbers as follows [**15**, Chapter III, Theorem 4.6].

THEOREM 1.3 (Björner-Frankl-Stanley). *A vector $(\beta_S : S \subseteq \{1, 2, \ldots, n\})$ is the flag h-vector of some $(n-1)$-dimensional balanced Cohen-Macaulay simplicial complex if and only if it is the flag f-vector of some other colored simplicial complex.*

REMARK 1.4. *Although Stanley uses the term "balanced" twice in his statement* [**15**, Chapter III, Theorem 4.6]*, it is clear from his proof that the second complex only needs to be colored with the same color set as the first. The number of colors thus used may exceed the size of the largest face in the second complex. For example, an $(n-1)$-simplex is balanced and Cohen-Macaulay, all entries in its flag f-vector are 1's. The flag h-entries are all zero except for $h_{\emptyset} = 1$. Thus the second complex must have only one face, the empty set. This complex may be trivially colored using n colors (without actually using any of them).*

An important example of a balanced simplicial complex is the *order complex* $\triangle(P \setminus \{\widehat{0}, \widehat{1}\})$ of a *graded partially ordered set* P. The order complex $\triangle(Q)$ of any poset Q is the simplicial complex on the vertex set Q whose faces are the chains of Q. A poset is graded if it has a unique minimum $\widehat{0}$, a unique maximum $\widehat{1}$, and a rank function ρ. Since all saturated chains of P have the same cardinality, $\triangle(P \setminus \{\widehat{0}, \widehat{1}\})$ is pure, and coloring every element with its rank makes $\triangle(P \setminus \{\widehat{0}, \widehat{1}\})$ balanced.

2. Central Delannoy numbers and Legendre polynomials

The following connection between the central Delannoy numbers and Legendre polynomials has been known for at least half a century [**8**], [**10**], [**11**]:

$$d_{n,n} = P_n(3), \tag{2.1}$$

where $P_n(x)$ is the n-th Legendre polynomial. To date there seems to be a consensus that this link is not very relevant. Banderier and Schwer [**1**] note that there is no "natural" correspondence between Legendre polynomials and the original lattice path enumeration problem associated to the Delannoy array, while Sulanke [**17**] states that "the definition of Legendre polynomials does not appear to foster any combinatorial interpretation leading to enumeration".

Without disagreeing with these statements concerning the *original* lattice path-enumeration problem, in this section we point out the existence of a *modified* lattice path enumeration problem whose solution yields a modified Delannoy array $\widetilde{d}_{m,n}$ satisfying $\widetilde{d}_{n,n} = d_{n,n}$ and

$$\widetilde{d}_{m,n} = P_n^{(0,m-n)}(3) \quad \text{for } m \geq n. \tag{2.2}$$

Here $P_n^{(\alpha,\beta)}(x)$ is the n-th Jacobi polynomial of type (α, β) defined by

$$P_n^{(\alpha,\beta)}(x) = (-2)^{-n}(n!)^{-1}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\left((1-x)^{n+\alpha}(1+x)^{n+\beta}\right).$$

Since the polynomials $P_n^{(0,0)}(x)$ are the Legendre polynomials [**5**, Chapter V, (2.2)], substituting $m = n = 0$ into (2.2) yields (2.1). In section 3 we present a face-enumeration problem associated to Jacobi polynomials that is related our modified lattice path enumeration problem.

The lattice path enumeration problem in question is essentially identical to the one of A049600 in the On-Line Encyclopedia of Integer Sequences [**13**].

Definition 2.1. For any $(m, n) \in \mathbb{N} \times \mathbb{N}$ let us denote by $\widetilde{d}_{m,n}$ the number of lattice paths from $(0, 0)$ to $(m, n+1)$ having steps $(x, y) \in \mathbb{N} \times \mathbb{P}$. (Here $\mathbb{P}$ denotes the set of positive integers.) We call the numbers $\widetilde{d}_{m,n}$ $(m, n \geq 0)$ the *asymmetric Delannoy numbers.*

$\widetilde{d}_{m,n} :=$

$m \backslash n$	0	1	2	3	4
0	1	2	4	8	16
1	1	3	8	20	48
2	1	4	13	38	104
3	1	5	19	63	192
4	1	6	26	96	321

Table 1. The asymmetric Delannoy numbers $\widetilde{d}_{m,n}$ for $0 \leq m, n \leq 4$.

It is immediate from our definition that $\widetilde{d}_{m,n} = T(n+1, m)$ for the array T given in A049600. As a consequence we get $\widetilde{d}_{n,n} = T(n+1, n)$ which is the central Delannoy number $d_{n,n}$, as noted in A049600. Compared to A049600, we shifted the rows up by 1 to move the central Delannoy numbers to the main diagonal, and then we reflected the resulting table to its main diagonal, since this will allow picturing the partially ordered sets in section 3 the "usual" way, i.e., with the larger elements being above the smaller ones. As an immediate consequence of the definition we obtain the following:

Lemma 2.2. *The asymmetric Delannoy numbers satisfy*

$$\widetilde{d}_{m,n} = \sum_{j=0}^{n} \binom{n}{j}\binom{m+j}{j}.$$

It is worth noting that substituting $n = m$ into Lemma 2.2 yields a well-known representation of the central Delannoy number $d_{n,n}$. (See, Sulanke [**17**, Example 1].) We may also easily verify (2.2), as follows. A Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ with nonnegative integer parameters α, β may be also given in the form

$$P_n^{(\alpha,\beta)}(x) = \sum_j \binom{n+\alpha+\beta+j}{j}\binom{n+\alpha}{j+\alpha}\left(\frac{x-1}{2}\right)^j \tag{2.3}$$

See, e.g., Wilf [Chapter 4, Exercise 15 (b)][**18**]. Substituting $\alpha = 0$ we obtain

$$P_n^{(0,\beta)}(x) = \sum_j \binom{n+\beta+j}{j}\binom{n}{j}\left(\frac{x-1}{2}\right)^j, \tag{2.4}$$

from which (2.2) follows by setting $\beta = m - n$ and $x = 3$.

Corollary 2.3. *The asymmetric Delannoy numbers satisfy (2.2).*

REMARK 2.4. *It is usually required that $\alpha, \beta > -1$ in the definition of the Jacobi polynomials "for integrability purposes", cf. Chihara* [**5**, Chapter V., section 2.]. *That said, using (2.4) we may extend the definition of $P_n^{(0,\beta)}(x)$ to any integer $\beta \geq -n$. Using this extended definition, we may state (2.2) for any $m, n \geq 0$.*

A combinatorial interpretation of (2.2) will be facilitated by the following reinterpretation of our lattice path enumeration problem.

PROPOSITION 2.1. *For any $(m,n) \in \mathbb{N} \times \mathbb{N}$ the number $\widetilde{d}_{m,n}$ also enumerates all 2-colored lattice paths from $(0,0)$ to (m,n) satisfying the following:*

(*i*) *Each step is either a blue $(0,1)$ or a red $(x,y) \in \mathbb{N} \times \mathbb{N} \setminus \{(0,0)\}$.*
(*ii*) *At least one of any two consecutive steps is a blue $(0,1)$.*

PROOF. It is easy to verify directly that the number of all 2-colored lattice paths from $(0,0)$ to (m,n) with the above properties is $\sum_{j=0}^{n} \binom{n}{j}\binom{m+j}{j}$, and so we get $\widetilde{d}_{m,n}$ by Lemma 2.2. (j is the number of blue steps). But only a little more effort is necessary to find a fairly plausible bijection between the corresponding sets of lattice paths. Consider first a lattice path from $(0,0)$ to $(m,n+1)$ satisfying Definition 2.1. Replace

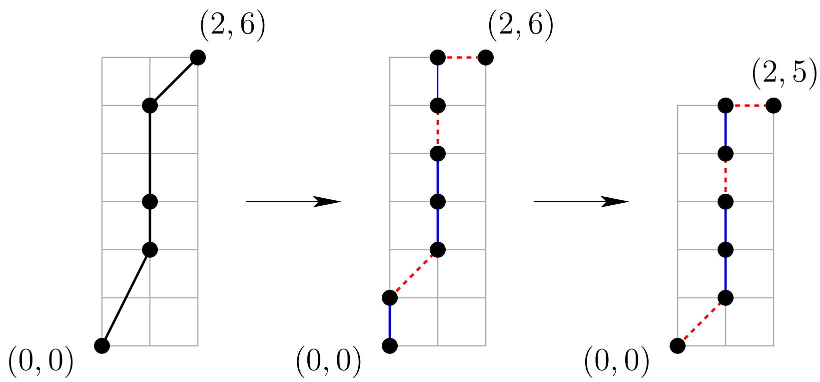

FIGURE 1. Transforming a lattice path into a 2-colored lattice path.

each step $(x,y) \in \mathbb{N} \times \mathbb{P}$ with one or two colored steps as follows. If $(x,y) \neq (0,1)$ then replace it with a blue $(0,1)$ followed by a red $(x, y-1)$. If $(x,y) = (0,1)$ then replace it with a blue $(0,1)$. The resulting 2-colored lattice path from $(0,0)$ to $(m,n+1)$ satisfies conditions (i) and (ii), moreover it starts always with a blue $(0,1)$. Remove this first blue step and shift the colored lattice path down by 1 unit. We obtain a 2-colored lattice path from $(0,0)$ to (m,n) satisfying the conditions of our proposition. Fig. 1 shows the two stages of such a transformation. (In the picture, $m = 2$ and $n = 5$. Red edges are marked with dashed lines.)

Finding the inverse of this transformation is easy. Given a valid 2-colored lattice path from $(0,0)$ to (m,n), let us first shift the path up by 1 and prepend a blue $(0,1)$ step from $(0,0)$ to $(0,1)$. Thus we obtain a 2-colored lattice path from $(0,0)$ to (m,n) satisfying (i), while condition (ii) may be strengthened to stating that every red step is preceded by a blue $(0,1)$ step. Replace each red step (x,y) and the blue step preceding it with a single colorless step $(x, y+1)$. After this, remove the blue color of the remaining $(0,1)$-steps.

We leave it to the reader to verify the fact that the two operations described above are inverses of each other. □

Using the equivalent definition of Proposition 2.1 it is easy to verify the following additional property of the asymmetric Delannoy numbers.

LEMMA 2.5. *The asymmetric Delannoy numbers satisfy the recursion formula*

$$\widetilde{d}_{m,n} = \widetilde{d}_{0,0} + \sum_{i=0}^{m} \sum_{j=0}^{n-1} \widetilde{d}_{i,j} \quad \textit{for all } m, n \geq 0.$$

Here the second sum is empty if $n = 0$.

As a consequence any entry in Table 1 may be obtained by adding 1 to the sum of the entries in the preceding columns, up to the row of the selected entry.

COROLLARY 2.6. *The asymmetric Delannoy numbers satisfy the recursion formula*

$$\widetilde{d}_{m,n} = 2 \cdot \widetilde{d}_{m,n-1} + \widetilde{d}_{m-1,n} - \widetilde{d}_{m-1,n-1}.$$

3. Jacobi posets and balanced joins

The blue steps of a valid 2-colored path introduced in Proposition 2.1 form increasing chains in a partially ordered set. In this section we investigate this partial order.

DEFINITION 3.1. Given any integer β and $n \geq \max(0, -\beta)$, we call the *Jacobi poset P_n^β of type β and rank $n+1$* the following graded poset.

(*i*) For each $q \in \{1, \ldots, n\}$, P_n^β has $n + \beta + 1$ elements of rank q, they are labeled $(0, q)$, $(1, q)$, ..., $(n + \beta, q)$.

(*ii*) Given (p, q) and (p', q') in $P_n^\beta \setminus \{\widehat{0}, \widehat{1}\}$ we set $(p, q) < (p', q')$ iff. $p \leq p'$ and $q < q'$.

(We also require $\widehat{0}$ to be the minimum element and $\widehat{1}$ to be the maximum element.)

We may think of the elements of $P_n^\beta \setminus \{\widehat{0}, \widehat{1}\}$ as the endpoints of all possible blue $(0, 1)$ steps when we enumerate all valid 2-colored lattice paths from $(0, 0)$ to $(n + \beta, n)$. We have $(p, q) < (p', q')$ if and only of if there is a valid 2-colored lattice path containing both $(p, q - 1) - (p, q)$ and $(p', q' - 1) - (p', q')$ as blue steps, such that the first blue step precedes the second in the path. Fig. 2 represents the Jacobi poset P_5^{-3}, which may be associated to enumerating the valid 2-colored lattice paths from $(0, 0)$ to $(2, 5)$. In the picture of the poset we marked the elements corresponding to the blue edges with empty circles. Given any valid

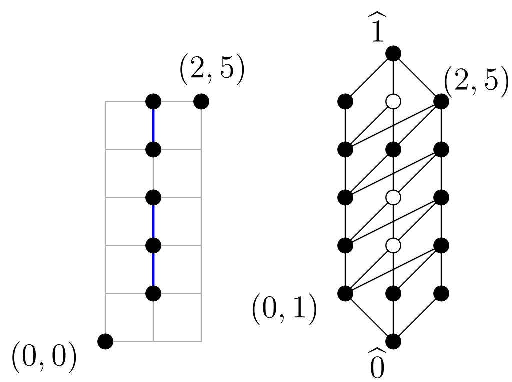

FIGURE 2. The Jacobi poset P_5^{-3} and the partial chain encoding the lattice path in Fig.1

2-colored path from $(0, 0)$ to $(n + \beta, n)$, the set of its blue edges correspond to a partial chain in $P_n^\beta \setminus \{\widehat{0}, \widehat{1}\}$ and, conversely, any partial chain of $P_n^\beta \setminus \{\widehat{0}, \widehat{1}\}$ encodes a set of blue edges that may be uniquely completed to a valid 2-colored path by adding the appropriate red edges.

Obviously, the face numbers of the order complex of $P_n^\beta \setminus \{\widehat{0}, \widehat{1}\}$ satisfy

$$f_{j-1}\left(\triangle\left(P_n^\beta \setminus \{\widehat{0}, \widehat{1}\}\right)\right) = \binom{n}{j}\binom{n + \beta + j}{j}. \tag{3.1}$$

As a consequence of this equality and (2.4) we obtain that

$$\sum_{j=0}^{n} f_{j-1}\left(\triangle\left(P_n^\beta \setminus \{\widehat{0}, \widehat{1}\}\right)\right) \cdot \left(\frac{x-1}{2}\right)^j = P^{(0,\beta)}(x) \quad \text{for } \beta \geq 0. \tag{3.2}$$

Note that, for negative values of β, (3.2) still holds in the extended sense of Remark 2.4. As another consequence of (3.2), the asymmetric Delannoy number $\widetilde{d}_{m,n}$ equals the number of all partial chains (including the empty chain) in the Jacobi poset $P_n^{m-n} \setminus \{\widehat{0}, \widehat{1}\}$. This fact is also "visually obvious" in terms of the enumeration problem presented in Proposition 2.1, since any valid 2-colored lattice path may be uniquely reconstructed from its blue steps. This visualization inspires the following definition.

Definition 3.2. Let $\triangle_1$ and $\triangle_2$ be pure balanced simplicial complexes of the same dimension. Let us fix a pair of colorings $\lambda = (\lambda_1, \lambda_2)$ such that λ_i colors the vertices of $\triangle_i$ $(i = 1, 2)$ in a balanced way, and the set of colors is the same in both colorings. We call the simplicial complex

$$\triangle_1 *_\lambda \triangle_2 := \{\sigma \cup \tau : \sigma \in \triangle_1, \tau \in \triangle_2, \lambda_1(\sigma) \cap \lambda_2(\tau) = \emptyset\}$$

the *balanced join* of $\triangle_1$ and $\triangle_2$ with respect to λ.

Example 3.3. *Let $\triangle_1$ and $\triangle_2$ be both $(n-1)$-dimensional simplices. These have essentially one balanced coloring and, independently of the choice of $\lambda = (\lambda_1, \lambda_2)$, the free join $\triangle_1 *_\lambda \triangle_2$ is isomorphic to the boundary complex of an n-dimensional cross-polytope.*

Using the notion of the balanced join we may express the relation between the asymmetric Delannoy numbers and Jacobi posets as follows.

Theorem 3.4. *Given $m, n \geq 0$, let λ_1 be the coloring of the order complex of $P_n^{m-n} \setminus \{\widehat{0}, \widehat{1}\}$ induced by the rank function, and let λ_2 be any balanced coloring of an $(n-1)$-dimensional simplex $\triangle^{n-1}$ with the color set $\{1, 2, \ldots, n\}$, Then, for $\lambda = (\lambda_1, \lambda_2)$, the asymmetric Delannoy number $\widetilde{d}_{m,n}$ is the number of facets in the balanced join $\triangle \left(P_n^{m-n} \setminus \{\widehat{0}, \widehat{1}\}\right) *_\lambda \triangle^{n-1}$.*

In fact, Theorem 3.4 may be generalized to any graded poset P of rank $n+1$ as follows.

Theorem 3.5. *Given any graded poset P of rank $n+1$, let λ_1 be the coloring of the order complex of $P \setminus \{\widehat{0}, \widehat{1}\}$ induced by the rank function, and let λ_2 be any balanced coloring of an $(n-1)$-dimensional simplex $\triangle^{n-1}$ with the color set $\{1, 2, \ldots, n\}$, Then, for $\lambda = (\lambda_1, \lambda_2)$, the number of facets in the balanced join $\triangle \left(P \setminus \{\widehat{0}, \widehat{1}\}\right) *_\lambda \triangle^{n-1}$ equals the total number of partial chains in $P \setminus \{\widehat{0}, \widehat{1}\}$.*

Theorem 3.5 is an immediate consequence of the fact that $\triangle^{n-1}$ must have precisely one vertex of each color, thus any partial chain from $P \setminus \{\widehat{0}, \widehat{1}\}$ may be uniquely complemented to a facet of $\triangle \left(P \setminus \{\widehat{0}, \widehat{1}\}\right) *_\lambda \triangle^{n-1}$ by inserting exactly those vertices of the simplex which are colored by the ranks missed by the partial chain. Theorem 3.4 is then a consequence of Theorem 3.5, Lemma 2.2 and equation (3.1).

4. Properties of the balanced join operation

In this section we take a closer look at the balanced join operation introduced at the end of the previous section. Let us point out first that the operation does depend on the colorings chosen.

Example 4.1. *Consider the "star graph" $\triangle$ shown in Fig. 3. This is a 1-dimensional simplicial complex*

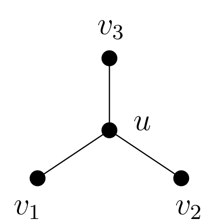

Figure 3. A "star graph" that may be colored in essentially one way

which has essentially one balanced coloring with 2 colors: v_1, v_2, and v_3 must have the same color, and u must have the other color. Yet, when we fix the set $\{1, 2\}$ to be the set of our colors, we have 2 options to chose the color of u (the rest of the coloring is then uniquely determined). Thus in a balanced join $\triangle *_\lambda \triangle$ we may choose $\lambda = (\lambda_1, \lambda_2)$ in such a way that $\lambda_1 \neq \lambda_2$, or we may use the same coloring twice. If $\lambda_1 = \lambda_2$ then the complex $\triangle *_\lambda \triangle$ has $1 \cdot 3 + 3 \cdot 1 = 6$ edges, while in the case $\lambda_1 \neq \lambda_2$ the complex $\triangle *_\lambda \triangle$ has $1 \cdot 1 + 3 \cdot 3 = 10$ edges.

However, the observation made in Example 3.3 may be generalized as follows. Given any balanced simplicial complex $\triangle$ of dimension $(n-1)$, and a balanced coloring λ_1 of it, the balanced join $\triangle *_\lambda \triangle^{n-1}$ with an $(n-1)$-simplex $\triangle^{n-1}$ obviously does not depend on the choice of the its coloring λ_2. Moreover, we have the following fact:

Theorem 4.2. *The flag h-vector of the balanced join $\triangle *_\lambda \triangle^{n-1}$ of a balanced $(n-1)$-dimensional simplicial complex with an $(n-1)$-simplex $\triangle^{n-1}$ satisfies*

$$h_S\left(\triangle *_\lambda \triangle^{n-1}\right) = f_S\left(\triangle\right)$$

for any subset S of the colors used.

Proof. We may assume that the set of colors is $\{1,2,\ldots,n\}$. Any face of $\triangle *_\lambda \triangle^{n-1}$ of color S is a disjoint union $\sigma\cup\tau$ with $\sigma\in\triangle$, $\tau\in\triangle^{n-1}$. The set $T:=\lambda_1(\sigma)$ must be a subset of S, and $\lambda_2(\tau)$ must be equal to $S\setminus T$. There is precisely one $\tau\in\triangle^{n-1}$ with this property, hence we obtain

$$f_S\left(\triangle *_\lambda \triangle^{n-1}\right) = \sum_{T\subseteq S} f_T\left(\triangle\right).$$

The statement now follows by the sieve formula. □

The proof of Theorem 4.2 is almost trivial, but the statement provides an interesting "constructive reason" for a situation that occurs in Theorem 1.3. The proof of Theorem 1.3 is algebraic, and it is usually hard to find a balanced simplicial complex combinatorially whose flag f-vector is the flag h-vector of the given Cohen-Macaulay balanced simplicial complex. An example of a combinatorial explanation in the special case of order complexes of certain distributive lattices is given by Skandera [**12**, Theorem 3.2]. The above construction is fairly "rigid", but it yields also examples of balanced simplicial complexes without the Cohen-Macaulay property. On the other hand, the balanced join preserves the Cohen-Macaulay property if we apply it to Cohen-Macaulay complexes.

Theorem 4.3. *Assume that $\triangle_1$ and $\triangle_2$ are balanced Cohen-Macaulay complexes of dimension $(n-1)$, and that their colorings λ_1 and λ_2 use the same set of colors $\{1,2,\ldots,n\}$. Then the balanced join $\triangle_1 *_\lambda \triangle_2$ is also a Cohen-Macaulay simplicial complex.*

Proof. We prove the Cohen-Macaulay property by induction on the size of $\triangle_1\cup\triangle_2$, using Reisner's Theorem (Theorem 1.2 in the Preliminaries). For that purpose, consider the link of any face $\tau_1\cup\tau_2$ where $\tau_1\in\triangle_1$, $\tau_2\in\triangle_2$ and $\lambda_1(\tau_1)\cap\lambda_2(\tau_2)=\emptyset$. The faces of $\mathrm{lk}_{\triangle_1 *_\lambda \triangle_2}(\tau_1\cup\tau_2)$ are precisely the faces of the form $\sigma_1\cup\sigma_2$ where $\sigma_i\in\mathrm{lk}_{\triangle_i}(\tau_i)$ for $i=1,2$, and the sets of colors $\lambda_1(\sigma_1)$, $\lambda_2(\sigma_2)$, $\lambda_1(\tau_1)$, and $\lambda_2(\tau_2)$ are pairwise disjoint. Using this description it is easy to deduce

$$\mathrm{lk}_{\triangle_1 *_\lambda \triangle_2}(\tau_1\cup\tau_2) = \mathrm{lk}_{\triangle_1}(\tau_1)_{\{1,\ldots,n\}\setminus\lambda_2(\tau_2)} *_\lambda \mathrm{lk}_{\triangle_2}(\tau_2)_{\{1,\ldots,n\}\setminus\lambda_1(\tau_1)}.$$

Here the simplicial complexes $\mathrm{lk}_{\triangle_1}(\tau_1)_{\{1,\ldots,n\}\setminus\lambda_2(\tau_2)}$ and $\mathrm{lk}_{\triangle_2}(\tau_2)_{\{1,\ldots,n\}\setminus\lambda_1(\tau_1)}$ are both balanced and the appropriate restrictions of λ_1 resp. λ_2 color both with the same color set $\{1,\ldots,n\}\setminus(\lambda_1(\tau_1)\cup\lambda_2(\tau_2))$. By Reisner's theorem, the link of every face in a Cohen-Macaulay complex is Cohen-Macaulay. According to Stanley's theorem [**15**, Chapter III, Theorem 4.5], every rank-selected subcomplex of a balanced Cohen-Macaulay complex is Cohen-Macaulay. Thus, whenever at least one of τ_1 and τ_2 is not the empty set, we may apply our induction hypothesis to the balanced join of $\mathrm{lk}_{\triangle_1}(\tau_1)_{\{1,\ldots,n\}\setminus\lambda_2(\tau_2)}$ and $\mathrm{lk}_{\triangle_2}(\tau_2)_{\{1,\ldots,n\}\setminus\lambda_1(\tau_1)}$.

We are left to prove Reisner's criterion for the reduced homology groups of the oriented chain complex associated to $\triangle_1 *_\lambda \triangle_2$. Assume by way of contradiction that there exist an $i<n-1$ and a linear combination $\underline{c}=\sum_{j,k}\alpha_{j,k}\cdot[\sigma_j\cup\tau_k]\in C_i(\triangle_1 *_\lambda \triangle_2)$ that belongs to $\mathrm{Ker}(\partial_i)$ but not to $\mathrm{Im}(\partial_{i+1})$. Here we may assume that each σ_j belongs to $\triangle_1$, each τ_k belongs to $\triangle_2$, and that no two of these sets are the same. Furthermore, we agree that in the oriented simplices we always list the elements of the face from $\triangle_1$ before the elements of the face from $\triangle_2$, hence "putting square brackets around a union of such faces" will not cause any confusion. Finally, for each fixed j at least one scalar $\alpha_{j,k}$ is not zero (otherwise σ_j is superfluous) and for each fixed k at least one $\alpha_{j,k}$ is not zero (otherwise τ_k is superfluous). Assume that our counterexample is smallest in the sense that the maximum of $|\tau_k|$ is as small as possible.

W.l.o.g. we may assume that τ_1 is of maximum size and thus it not contained in any other τ_k. Applying ∂_i to $\underline{c}$ yields a linear combination of oriented simplices, whose underlying simplices are of the form $\sigma_j\cup\tau_1\setminus\{u\}$ where $u\in\sigma_j\cup\tau_1$. Consider among these oriented simplices the ones whose underlying simplex contains τ_1. Because of the maximality of τ_1, none of these may arise by removing some element from a $\sigma_j\cup\tau_k$ with $k\neq 1$. Hence the projection of $\partial_i(\underline{c})$ onto the vector space generated by the oriented simplices containing τ_1 may be obtained from $\partial_{i-|\tau_1|}\left(\sum_j\alpha_{j,1}\cdot[\sigma_j]\right)\in C_{i-|\tau_1|-1}(\triangle_1)$ by sending each $[\sigma']\in C_{i-|\tau_1|-1}(\triangle_1)$ into $[\sigma'\cup\tau_1]\in C_{i-1}(\triangle_1 *_\lambda \triangle_2)$. Since $\partial_i(\underline{c})=0$, we obtain that $\sum_j\alpha_{j,1}\cdot[\sigma_j]\in\mathrm{Ker}(\partial_{i-|\tau_1|})$ in the oriented chain complex associated to $\triangle_1$. Here $i - |\tau_1| \leq i < n-1$, hence applying Reisner's criterion to $\triangle_1$ yields $\sum_j \alpha_{j,1} \cdot [\sigma_j] \in \operatorname{Im}(\partial_{i+1-|\tau_1|})$. Assume

$$\sum_j \alpha_{j,1} \cdot [\sigma_j] = \partial_{i+1-|\tau_1|} \left(\sum_t \beta_t \cdot [\sigma'_t] \right)$$

holds in $C(\triangle_1)$, and consider

$$\underline{c'} = \sum_t \beta_t \cdot [\sigma'_t \cup \tau_1] \in C_{i+1} \left(\triangle_1 *_\lambda \triangle_2 \right).$$

Subtracting $\partial_{i+1}(\underline{c'})$ from $\underline{c}$ removes all terms of the form $\alpha_{j,1} \cdot [\sigma_j \cup \tau_1]$ and introduces only new terms of the form $\alpha \cdot [\sigma' \cup \tau']$, where $\sigma' \in \triangle_1$, $\tau' \in \triangle_2$, and τ' is a proper subset of τ_1. Hence we reduced the number of τ_k's of maximum size in our counterexample. Repeating the same argument finitely many times we arrive at a counterexample in which the maximum size of all τ_k's is smaller than in the original one. We obtain a contradiction unless there is only one τ_k in $\underline{c}$, namely $\tau_1 = \emptyset$. However, for elements of the form $\sum_{j,1} \alpha_{j,1}[\sigma_j \cup \emptyset]$ where $\sigma_j \in \triangle_1$, the effect of the boundary map is described with the same formulas in the oriented chain complex associated to $\triangle_1$ and in the oriented chain complex associated to $\triangle_1 *_\lambda \triangle_2$. Applying Reisner's theorem to $\triangle_1$ yields a contradiction. □

5. Jacobi posets and balanced Cohen-Macaulay complexes

Theorem 4.3 is applicable to $\triangle\left(P_n^{m-n} \setminus \{\widehat{0}, \widehat{1}\}\right) *_\lambda \triangle^{n-1}$ because of the following statement.

Proposition 5.1. *The order complex $\triangle(P_n^\beta \setminus \{\widehat{0}, \widehat{1}\})$ associated to the Jacobi poset P_n^β is Cohen-Macaulay.*

Proof. Let us label each cover relation $(p, q) \prec (p', q+1)$ with $n + \beta - p'$ and each cover relation $\widehat{0} \prec (p, 1)$ with $n + \beta - p$. Finally, let us label each cover relation $(p, n) \prec \widehat{1}$ with 0.

The resulting labeling is an *EL-labeling*, as defined by Björner and Wachs [**4**, Definition 2.1] (these labelings were first introduced in [**3**]). (We omit the proof that we get an EL-labeling, for brevity's sake.)

If a graded poset has an EL-labeling then its order complex is *shellable* by the result of Björner and Wachs [**4**, Proposition 2.3]. Shellable simplicial complexes are Cohen-Macaulay, see Stanley [**15**, Chapter III, Theorem 2.5]. □

Since $\triangle(P_n^\beta \setminus \{\widehat{0}, \widehat{1}\})$ is Cohen-Macaulay and balanced, we may apply Theorem 1.3 to observe that its flag h-vector is the flag f-vector of some balanced simplicial complex. For a reader familiar with EL-labelings it is not difficult to construct such a simplicial complex, by inspecting the "descent sets" arising in the proof of Proposition 5.1. To ease the burden of the reader who is not familiar with EL-labelings, we provide an explicit description of such a balanced simplicial complex, and we verify the equality of the appropriate invariants by explicitly computing them. The simplicial complex to be constructed arises as the order complex of a partially ordered set.

Definition 5.1. Given two partially ordered sets P and Q, we define their *strict direct product* $P \bowtie Q$ as the set $P \times Q$ ordered by the relation $(p, q) < (p', q')$ if $p < p'$ and $q < q'$.

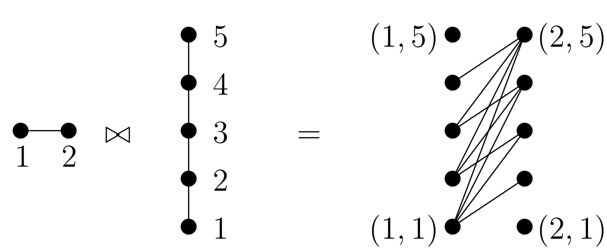

Figure 4. The strict direct product $C_1 \bowtie C_4$

Fig. 4 represents the strict direct product of a chain C_1 of length 1 with a chain C_4 of length 4. We obtain a partially ordered set that is not graded. However, the following statement is obviously true in general.

Lemma 5.2. *Given any pair of posets (P, Q), every admissible coloring of $\triangle(P)$ may be extended to an admissible coloring to $\triangle(P \bowtie Q)$ by coloring each $(p, q) \in P \times Q$ with the color of its first coordinate. The analogous statement is true for the second coordinates.*

Here we call a coloring admissible if the vertices of any face are colored with all different colors. For example, the order complex of the poset shown in Fig. 4 may be colored with 2 colors, by extending the coloring of the chain C_1, or with 5 colors, by extending the coloring of the chain C_5. Using the notion of the direct product, the flag h-vector of the order complex associated to a Jacobi poset may be described as follows.

Proposition 5.2. *The flag h-vector of $\triangle(P_n^\beta \setminus \{\widehat{0}, \widehat{1}\})$, colored by the rank function, equals the flag f-vector of $\triangle(C_{n+\beta-1} \bowtie C_{n-1})$, with respect to the coloring induced by the rank function of the second coordinate.*

Proof. For any $S \subseteq \{1, \ldots, n\}$, choosing a facet of $\triangle(P_n^\beta \setminus \{\widehat{0}, \widehat{1}\})_S$ is equivalent to choosing an $|S|$-element multiset on $\{0, 1, \ldots, n+\beta\}$. Hence we have

$$f_S\left(\triangle(P_n^\beta \setminus \{\widehat{0}, \widehat{1}\})\right) = \binom{n+\beta+|S|}{|S|}. \tag{5.1}$$

Using the identity

$$\binom{n+\beta+s}{s} = \sum_{t=0}^{s} \binom{s}{t} \cdot \binom{n+\beta}{t}$$

it is easy to deduce that the flag h-vector of $\triangle(P_n^\beta \setminus \{\widehat{0}, \widehat{1}\})$ must satisfy

$$h_S\left(\triangle(P_n^\beta \setminus \{\widehat{0}, \widehat{1}\})\right) = \binom{n+\beta}{|S|}. \tag{5.2}$$

Introducing ρ for the rank function $\rho : C_{n-1} \to \{1, \ldots n\}$ (note that the least element has rank one!), consider $\triangle(C_{n+\beta-1} \bowtie C_{n-1})$ with the coloring $\lambda(p, q) = \rho(q)$. For any $S \subseteq \{1, 2, \ldots, n\}$, choosing a saturated chain in $\triangle(C_{n+\beta-1} \bowtie C_{n-1})_S$ involves fixing the second coordinates, and choosing an $|S|$-element subset of a set with $n+\beta$ elements. Hence we have

$$f_S\left(C_{n+\beta-1} \bowtie C_{n-1}\right) = \binom{n+\beta}{|S|} \tag{5.3}$$

as stated. $\square$

It should be noted that the strict direct product associated by Proposition 5.2 to P_5^{-3} (shown in Fig. 2) is $C_1 \bowtie C_4$ (shown in Fig. 4). To summarize our findings: we obtained that the asymmetric Delannoy number $\widetilde{d}_{m,n}$ counts the facets of the balanced Cohen-Macaulay complex $\triangle\left(P_n^{m-n} \setminus \{\widehat{0}, \widehat{1}\}\right) *_\lambda \triangle^{n-1}$. The flag h-vector of this complex is the flag f-vector of the order complex $\triangle(P_n^\beta \setminus \{\widehat{0}, \widehat{1}\})$. This complex is still balanced and Cohen-Macaulay, and its flag h-vector equals the flag f-vector of the colored complex described in Proposition 5.2. No further similar reduction is possible, since the order complex of the strict direct product of two chains is usually not Cohen-Macaulay. For example, the order complex of $C_1 \bowtie C_4$, shown in Fig. 4, is not even connected. The number of colors used also exceeds the size of the largest face. Thus, it appears, this is how far we may get using Theorem 1.3 in reducing the question of enumerating flags in the simplicial complex associated to the asymmetric Delannoy numbers.

6. The right-strict direct product of posets

The connection between the Jacobi posets and the strict direct product of two chains exposed in Proposition 5.2 suggests considering the following definition.

Definition 6.1. Given two partially ordered sets (P, Q) we define their *right-strict direct product* $P \rtimes Q$ to be the set $P \times Q$ partially ordered by the relation $(p, q) < (p', q')$ if $p \leq p'$ and $q < q'$.

The definition of the right-strict direct product is "halfway between" the usual definition of the direct product of posets and the strict direct product. Our interest is motivated by the following observation.

PROPOSITION 6.1. *The partially ordered set* $P_n^{\beta} \setminus \{\widehat{0}, \widehat{1}\}$ *is isomorphic to* $C_{n+\beta} \rtimes C_{n-1}$.

The statement is an immediate consequence of the definitions. The fact that we obtain a graded poset (with the $\widehat{0}$ and the $\widehat{1}$ removed) may be generalized as follows.

PROPOSITION 6.2. *Assume P is an arbitrary poset and Q is a graded poset of rank $n+1$. Then $P \rtimes (Q \setminus \{\widehat{0}, \widehat{1}\})$ may be turned into a graded poset of rank $n+1$ by adding a unique minimum element $\widehat{0}$ and a unique maximum element $\widehat{1}$. The rank function may be taken to be the rank function of Q applied to the second coordinate.*

The question naturally arises: how far can Proposition 5.1 be generalized, under what circumstances can we guarantee that a right-strict direct product of posets has a Cohen-Macaulay order complex?

CONJECTURE 6.2. *If P is a poset with a Cohen-Macaulay order complex and Q is a graded Cohen-Macaulay poset then $P \rtimes (Q \setminus \{\widehat{0}, \widehat{1}\}) \cup \{\widehat{0}, \widehat{1}\}$ is a graded Cohen-Macaulay poset.*

This Conjecture, inspired by Proposition 5.1, is also supported by the following.

THEOREM 6.3. *Assume that P is any poset whose order complex has a non-negative h-vector and that Q is a graded posets with a non-negative flag h-vector. Then the flag h-vector of the graded poset $P \rtimes (Q \setminus \{\widehat{0}, \widehat{1}\}) \cup \{\widehat{0}, \widehat{1}\}$ is non-negative.*

PROOF. Assume that the rank of Q is $n+1$ and that the dimension of $\triangle(P)$ is $(d-1)$. Then, by Proposition 6.2, the rank of $\widetilde{Q} := P \rtimes (Q \setminus \{\widehat{0}, \widehat{1}\}) \cup \{\widehat{0}, \widehat{1}\}$ is also $n+1$. For any $S \subseteq \{1, \ldots, n\}$, the saturated chains in $\widetilde{Q}_S$ are all sets of the form $\{(p_1, q_1), \ldots, (p_{|S|}, q_{|S|})\}$, where $q_1 < \cdots < q_{|S|}$ is a saturated chain in Q_S and $p_1 \leq \cdots \leq p_{|S|}$ is any multichain in P. Thus we obtain

$$f_S(\widetilde{Q}) = f_S(Q) \sum_{j=1}^{\min(d,|S|)} f_{j-1}(P) \binom{j+|S|-j-1}{|S|-j} = f_S(Q) \sum_{j=1}^{\min(d,|S|)} f_{j-1}(P) \binom{|S|-1}{|S|-j} \quad \text{for } S \neq \emptyset.$$

Here, by abuse of notation, we write $f_{j-1}(P)$ as a shorthand for $f_{j-1}(\triangle(P))$. After some straightforward manipulation, which we omit for brevity's sake, we may rewrite these equations as

$$h_S(\widetilde{Q}) = \sum_{T \subseteq S} h_T(Q) \sum_{i=0}^{\min(d,|R|)} h_i(P) \binom{d+|T|-i-1}{|S|-i} \tag{6.1}$$

expressing the $h_S(\widetilde{Q})$'s as non-negative combinations of products of the (flag) h-entries of the original posets. $\square$

7. Concluding remarks

There are two seemingly plausible generalizations that will *not* work.

REMARK 7.1. *It is not possible to generalize the definition of Jacobi posets in such a way that the polynomial on the left hand side of (3.2) became $P^{(\alpha,\beta)}(x)$ for some nonzero α. For any graded poset P of rank $n+1$, substituting $x=1$ into*

$$\sum_{j=0}^{n} f_{j-1}\left(\triangle\left(P \setminus \{\widehat{0}, \widehat{1}\}\right)\right) \cdot \left(\frac{x-1}{2}\right)^j$$

*yields 1, while $P^{(\alpha,\beta)}(1) = \binom{n+\alpha}{\alpha}$ (see Chihara [**5**, Chapter V, (2.9)]) is 1 if and only if $\alpha = 0$.*

REMARK 7.2. *Sequences of symmetric orthogonal polynomials represented in the form*

$$\phi_P(x) := \sum_{j=0}^{n} f_{j-1}\left(\triangle\left(P \setminus \{\widehat{0}, \widehat{1}\}\right)\right) \cdot \left(\frac{x-1}{2}\right)^j$$

*associated to certain Eulerian graded posets P appear in the paper [**9**] of the present author. A polynomial $p(x)$ of degree n is symmetric if it satisfies $P_n(x) = (-1)^n P_n(-x)$. A graded partially ordered set P is Eulerian* *if it satisfies $\sum_{x\leq z\leq y}(-1)^{\rho(x,z)}=0$ for all $[x,y]\subseteq P$ of rank at least 1. If a graded poset P is Eulerian then $\phi_P(x)$ is symmetric. A symmetric Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ satisfies $\alpha=\beta$. In fact, by $P_n^{(\alpha,\beta)}(-x)=(-1)^n\cdot P_n^{(\beta,\alpha)}(-x)$ (see Chihara [**5**, Chapter V, (2.8)]), it must satisfy $P_n^{(\alpha,\beta)}(x)=P_n^{(\beta,\alpha)}(x)$, so $\alpha=\beta$ follows from $P^{(\alpha,\beta)}(1)=\binom{n+\alpha}{\alpha}$ cited in Remark 7.1. By Remark 7.1 we may represent a Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ as $\phi_P(x)$ associated to some poset P only if $\alpha=0$. Therefore the only Jacobi polynomials that could be represented as $\phi_P(x)$ for some Eulerian graded poset P are the Legendre polynomials. It is not difficult to construct such posets for $P_n^{(0,0)}(x)$ for $n\leq 2$. However, for higher values of n we would need graded Eulerian poset of rank $n+1$ with $f_{n-1}=\binom{2n}{n}$ saturated chains, which is not an integer multiple of $2^{\lfloor n/2\rfloor}$ for $n\geq 3$. This makes constructing Eulerian "Legendre posets" of rank higher than 3 impossible, since the number of saturated chains of an Eulerian poset of rank $n+1$ is*

$$f_{\{1,\ldots,n\}}=2^{\lfloor n/2\rfloor}\cdot f_{2\cdot\mathbb{Z}\cap\{1,\ldots,n\}}.$$

This follows from the fact that, in an Eulerian poset, every interval of rank 2 has 4 elements.

The two areas, where the most interesting generalizations seem to be found, are the following. The right-strict direct product, introduced in Section 6, deserves further study. If Conjecture 6.2 turns out to be too hard or false, the proof of Proposition 5.1 may be a hint that preservation of EL-shellability (or a similar property) could be or should be shown instead. The other challenge is to find an algebraic generalization of Theorem 4.3. When $\triangle$ is balanced and colored with n-colors, its face ring is $\mathbb{Z}^n$-graded. The balanced join operation takes the tensor product of two $\mathbb{Z}^n$-graded rings and factors it by all terms of the form $u\otimes v$, where u and v are homogeneous terms of the same multi-degree. It is natural to ask whether such a factor of the tensor product of two $\mathbb{Z}^n$-graded Cohen-Macaulay modules would always have the Cohen Macaulay property.

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