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An algorithm to describe bijections involving Dyck paths

Yvan Le Borgne

Abstract. We use an algorithm to define bijections involving Dyck paths. This algorithm is parametrized by rewriting rules and is similar to the derivation of a word in a context-free grammar. The bijections are variations of a classical one which is based on the insertion of a peak in the last descent. A systematic study of the algorithms parametrized by a single rewriting rule leads to 6 bijections, taking into account a trivial symmetry. We obtain 6 classical or new parameters on Dyck paths, which are distributed as the length of the last descent. We have a description for 5 of these parameters. We present additional bijections appearing in several combinatorial contexts that can be defined by generalizations of the initial algorithm.

Résumé. On utilise un algorithme pour définir des bijections impliquant les chemins de Dyck. Cet algorithme est paramétré par des règles de réécriture et est proche de la dérivation d'un mot dans une grammaire algébrique. Les bijections sont des variations de la construction classique des chemins de Dyck par l'insertion d'un pic dans la dernière descente. Une étude systématique des algorithmes paramétrés par une seule règle de réécriture permet d'identifier essentiellement 6 bijections. De chacune de ces bijections on déduit un paramètres classique ou nouveau dont la distribution est identique à celle de la longueur de la dernière descente. On donne une description de 5 de ces 6 paramètres. On présente d'autres bijections définissables par des généralisations de cet algorithme et utilisées dans divers contextes combinatoires.

Introduction

The Catalan numbers $(\frac{1}{2n+1}\binom{2n+1}{n})_{n\geq 0} = 1, 1, 2, 5, 14, 42, 129, \ldots$ define a sequence which occurs as the counting sequence of more than one hundred classes of combinatorial objects: ordered trees, binary trees, triangulations of polygons, Dyck paths ... (see Exercises 6.19 and 6.25 in [**8**] and its periodic update on the web). This sequence is also the expansion of an algebraic power series $C(t)$ that satisfies the functional equation $C(t) = 1 + tC(t)^2$. This equation is usually reflected on these combinatorial classes as a recursive decomposition of any object into two independent and smaller objects of the same class, if any smaller. Providing such a decomposition as regards a class usually proves that the counting sequence of this class is the Catalan sequence. Another way to fix this counting sequence consists of defining a bijection, which preserves the size of objects, between this class and another one counted by the Catalan sequence (see [**2**] for a more general discussion). Once the counting sequence has been computed, there often remain enumerative and open problems about the class: one wants to take into account not only the size of the objects, but also additional parameters. For example, in the case of two additional parameters, we need to obtain some information on the generating function

$$C(t; u, v) = \sum_{n,i,j\geq 0} c_n(i, j)u^i v^j t^n$$

where $c_n(i, j)$ is the number of objects of size n for which these two parameters equal i and j respectively. The usual decomposition of the objects may not fit well with the additional parameters. In such a case, we have to find either a new decomposition or a bijection that translates these objects and their parameters into other objects with more tractable parameters. In the literature, this kind of problem motivates many bijections between the various combinatorial interpretations of the Catalan sequence.

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The main aim of this extended abstract is not to solve a particular enumerative problem, but to propose a common framework describing some of these bijections. In an enumerative context we require that these bijections induce one-to-one maps when restricted to objects of any fixed size. Thus, there are 1!1!2!5!14!42! different restrictions of bijections to objects of size less than 5 between two classes counted by the Catalan sequence. But bijections of practical use in combinatorics are not so arbitrary. We want to define a subset $\mathcal{B}$ of these bijections satisfying the following (informal) condition. The set $\mathcal{B}$ should be *expressive*: it contains many bijections already present in the literature and many parameters with classical distributions are preserved or translated. The bijections of $\mathcal{B}$ should admit a *uniform description*, without too many *ad hoc* definitions. We formalize the notion of uniform description for maps between two classes $\mathcal{C}$ and $\mathcal{D}$. A uniform description will be a ordered pair $(\mathcal{P}, F)$ where $\mathcal{P}$ is a set whose elements are called "programs", and F is a map between $\mathcal{P} \times \mathcal{C}$ and $\mathcal{D}$ such that for any $p \in \mathcal{P}$ and $c \in \mathcal{C}$, the size of c is the size of $F(p, c) \in \mathcal{D}$. We denote by $\mathcal{M}$ the set of maps $\{F(p, .)|p \in \mathcal{P}\}$ that is the set of the partial evaluations of F on its first argument. By definition, $\mathcal{B}$ is the set of the one-to-one maps in $\mathcal{M}$. With this uniform description, we could formulate some additional wishes. For any program $p \in \mathcal{P}$, we could check if $F(p, .)$ is a bijection. Given a partial map m between a finite subset $C \subset \mathcal{C}$ and $\mathcal{D}$, we could efficiently compute, if any, at least one (or all) of the programs p compatible on C with m. Given finite subsets C and D of $\mathcal{C}$ and $\mathcal{D}$ respectively and two families of parameters on $\mathcal{C}$ and $\mathcal{D}$ respectively, we could also efficiently compute, if any, the programs in $\mathcal{P}$ that translate the size and the additional parameters when restricted to C and D. This is the end of the dream. Our modest attempt in this extended abstract is a relatively expressive set, as regards the length of the last descent and the area of Dyck paths. The description and the proofs are also relatively uniform. The last two properties (identify a description or guess a bijection preserving parameters) are not even discussed here.

In this extended abstract, we restrict the study to maps between two classes counted by the Catalan sequence: almost decreasing sequences and Dyck paths/words which stand for the classes $\mathcal{C}$ and $\mathcal{D}$ respectively. The latter class is very often used as the image set of a bijection, to prove that a class admits an algebraic recursive decomposition. The former class appears in a classical recursive step-by-step construction of Dyck paths, obtained by inserting a new peak in the last descent. In Section 1 we propose a first uniform description $(\mathcal{P}, F)$ for maps between almost decreasing sequences and Dyck paths. These maps are variations of the step-by-step construction of Dyck paths. The definition of F is an algorithm similar to the derivation of a word in a context-free grammar. This algorithm is parametrized by certain rewriting rules called insertion modes. $\mathcal{P}$ is the set of these insertion modes. In Section 2, we study all maps that are defined by the algorithm parametrized by a single insertion mode. Among the $210 = |\mathcal{P}|$ possibilities, we prove that 32 code bijections. Actually, some of these bijections are identical, and we only obtain $12 = |\mathcal{B}|$ distinct bijections. We deduce from this study classical and new parameters that have the same distribution on Dyck paths as the length of the last descent. Another such a systematic study was made in [**9**], but the approach was to define a set of parameters with the appropriate (Narayana) distribution, then to find bijections, whereas here, we define a set of bijections and then identify the parameters. In Section 3, we present generalizations of the algorithm that allow us to describe relevant bijections in several combinatorial contexts: a description of the Haiman statistic on Dyck paths and a combinatorial interpretation of the calculations involved in the kernel method.

This work summarizes a chapter of the author's PhD thesis [**7**].

1. An insertion algorithm

In this section, we define almost decreasing sequences and Dyck paths. We recall a classical bijection between these objects. Then we introduce some labelings of Dyck paths and some rewriting rules for these labels, which we respectively call Dyck buildings and insertion mode. Finally we present an algorithm parametrized by a single insertion mode that generalizes the classical bijection. This allow us to define the uniform description $(\mathcal{P}, F)$ systematically studied in Section 2.

Let w be a word. By definition, the letter a occurs $|w|_a$ times in w. We denote the empty word by ϵ. A word w over the alphabet $X \equiv \{x, \overline{x}\}$ is a *Dyck word* if $|w|_x = |w|_{\overline{x}}$ and for any prefix u of w, $|u|_x \geq |u|_{\overline{x}}$. The *size* of the Dyck word w is the number $|w|_x$. A *Dyck path* is a walk in the plane, that starts from the origin, is made up of *rises*, *i.e.* steps $(1, 1)$, and *falls*, *i.e.* steps $(1, -1)$, remains above the horizontal axis and finishes on it. The Dyck path related to a Dyck word w is the walk obtained by representing a letter x by a rise, and a letter $\overline{x}$ by a fall, see Figure 1. In the rest of the paper we identify the two notions, denoting

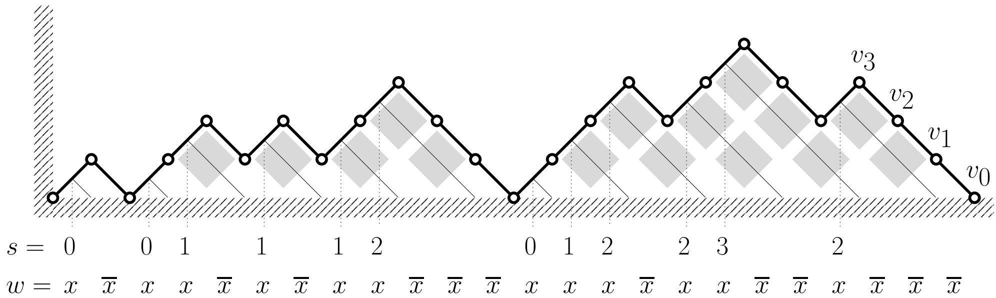

FIGURE 1. A Dyck path with its Dyck word w and its canonical almost decreasing sequence s.

them both w. A *vertex* in a Dyck path w is the origin of the plane or an endpoint of a step in w. In terms of Dyck words, a vertex corresponds to a factorization $w = uv$ where u is the subwalk between the origin and the vertex while v is the remaining subwalk. A *peak* is a vertex preceded by a rise and followed by a fall. A sequence of n non-negative integers $s = (s_k)_{k=1\dots n}$ is an *almost decreasing sequence* if $s_1 = 0$ and for all $k < n$, $s_{k+1} \leq 1 + s_k$. The empty sequence, denoted $\emptyset$, is an almost decreasing sequence.

The *height* of a rise is the ordinate of its starting vertex. We map a Dyck path to the sequence of the heights of its rises which is an almost decreasing sequence. We call this sequence the canonical almost decreasing sequence of this Dyck path since the map is a classical bijection. We recall a recursive step-by-step definition of the reverse of this map that we illustrate in Figure 1. We assume that we have already fixed that s is mapped to w. We want to define the image w_i of the almost decreasing sequence s, i obtained from s by the appending of the non-negative integer i. In Figure 1, the canonical sequence s ends with the value 2. Therefore, the possible values for i are 3, 2, 1, or 0. These are exactly the values not bigger than the height of the rightmost peak of w. This fact is a property called (P) of the bijection. The Dyck path w_i is obtained from w by an insertion of a factor $x\overline{x}$ in v_i, which is the rightmost vertex of w at height i. This insertion induces that the rightmost peak of w_i is the peak in the inserted factor $x\overline{x}$ thus this peak is at height $i+1$. This corresponds to the property (P) for w_i. This leads to the following step-by-step definition of the image of an almost decreasing sequence s of size n: starting from the empty path ϵ, insert a factor $x\overline{x}$ in the rightmost vertex at height s_k for k running from 1 to n.

To generalize this kind of step-by-step definition of a map, we use some labels in the Dyck words to indicate where a rise and a fall should be inserted in the path mapped to s to obtain the word image of s, i. Consider the (infinite) alphabet $L = \bigcup_{0 \leq k} \{k, \overline{k}\}$. The letters $k \in \mathbb{N}$ will be called *rising labels of index k* while the letters $\overline{k}$ will be called *falling labels of index k*. Let $L(N) = \bigcup_{0 \leq k \leq N} \{k, \overline{k}\}$. Given an alphabet A, the *projection* π_B over the alphabet $B \subseteq A$ is the morphism defined on the letters by $\pi_B(a) = a$ if $a \in B$ and $\pi_B(a) = \epsilon$ otherwise. A word w over the alphabet $X \cup L$ is a *Dyck building* if $\pi_X(w)$ is a Dyck word, and there exists a non-negative integer K, the *rank* of w, such that $|w|_k = |w|_{\overline{k}} = 1$ for $k \leq K$ and $|w|_k = |w|_{\overline{k}} = 0$ otherwise. See examples on Figure 3. When a sequence $l_1, l_2 \dots l_i$ of labels occurs between two letters x or $\overline{x}$, we represent them, on the corresponding vertex of the Dyck path, by a stack of labels where l_1 is at the bottom and l_i at the top. In our construction, the labels of index i in a Dyck building indicate where to insert the rise and the fall when we read the value i in the almost decreasing sequence.

A step-by-step definition of these insertions requires updates of the labels during each insertion to prepare the following insertions. We use rewriting rules to describe these updates. The set $G = \{A, \overline{A}, B, \overline{B}\}$ is the alphabet of *generic labels*. A ordered pair $m = (u, v)$ of words over the alphabet $X \cup G$ is an *insertion mode* if all the letters of $X \cup G$ occur exactly once in uv, x occurs in u, $\overline{x}$ occurs in v, A occurs before $\overline{A}$ in uv and B occurs before $\overline{B}$ in uv. For instance, the pair $(BxA\overline{B}, \overline{x}\overline{A})$ is an insertion mode. The substitution of all occurrences of the letter a by the word u in the word w is denoted $w[a := u]$. Two substitutions performed in parallel are denoted by $w[a := u, b := v]$ whereas a sequence of substitutions, first of the occurrences of a and then of the occurrences of b, is denoted $w[a := u][b := v]$. For example, $abc[a := bb, b := c] = bbcc$ and $abc[a := bb][b := c] = cccc$. The *insertion* is a map ρ with three arguments: a Dyck building w, an insertion mode $m = (u, v)$ and a value k in an almost decreasing sequence. This triplet is mapped to the

Dyck building

$$\rho_m^k(w) = w[k := u, \overline{k} := v][A := k, \overline{A} := \overline{k}, B := k+1, \overline{B} := \overline{k+1}]$$

also called the result of the insertion according to m of the value k in w.

Repeating this procedure, with a fixed insertion mode m, for all values of an almost decreasing sequence leads to the algorithm on Figure 2 where comments are enclosed by $/*$ and $*/$.

```
Input: An almost decreasing sequence s = (s_k)_{k=1...n}
Parameter: An insertion mode m
w_0 := 0 0̄; /* Start from the “empty” building containing only labels */
For k from 1 to n do
  w'_k := π_{X∪L(s_k)}(w_{k-1}); /* Erase in w the labels of index greater than s_k */
  w_k := ρ_m^{s_k}(w'_k); /* Insert x and x̄ in w at the location of the labels s_k and s̄_k
                             then update locally the labels of indexes s_k and 1 + s_k */
done;
Output: π_X(w_n); /* Erase all the labels to obtain a Dyck word */
```

Figure 2. Step-by-step algorithm with a single insertion mode

The word output by the algorithm with the almost decreasing sequence s as input and the insertion mode m as parameter is denoted $\Upsilon_m(s)$.

Example 1.1. In Figure 3, we trace the algorithm during the computation of $\Upsilon_{(BA\overline{B}x,\overline{A}\overline{x})}(0,1,1,2,3,1,2)$.

In terms of words :

$w_0 = 0\bar{0}$

$\longrightarrow^0 \quad w_1 = 10\bar{1}x\bar{0}\bar{x}$

$\longrightarrow^1 \quad w_2 = 21\bar{2}x0\bar{1}\bar{x}x0\bar{x}$

$\longrightarrow^1 \quad w_3 = 21\bar{2}xx0\bar{1}\bar{x}\bar{x}x\bar{0}\bar{x}$

$\longrightarrow^2 \quad w_4 = 32\bar{3}x1\bar{2}\bar{x}xx0\bar{1}\bar{x}\bar{x}x\bar{0}\bar{x}$

$\longrightarrow^3 \quad w_5 = 43\bar{4}x2\bar{3}\bar{x}x1\bar{2}\bar{x}xx0\bar{1}\bar{x}\bar{x}x\bar{0}\bar{x}$

$\longrightarrow^1 \quad w_6 = x\bar{x}x21\bar{2}x\bar{x}xx0\bar{1}\bar{x}\bar{x}\bar{x}x\bar{0}\bar{x}$

$\longrightarrow^2 \quad w_7 = x\bar{x}x32\bar{3}x1\bar{2}\bar{x}x\bar{x}xx0\bar{1}\bar{x}\bar{x}\bar{x}x\bar{0}\bar{x}$

Output: $x\bar{x}xx\bar{x}x\bar{x}xx\bar{x}\bar{x}\bar{x}x\bar{x}$

In terms of paths:

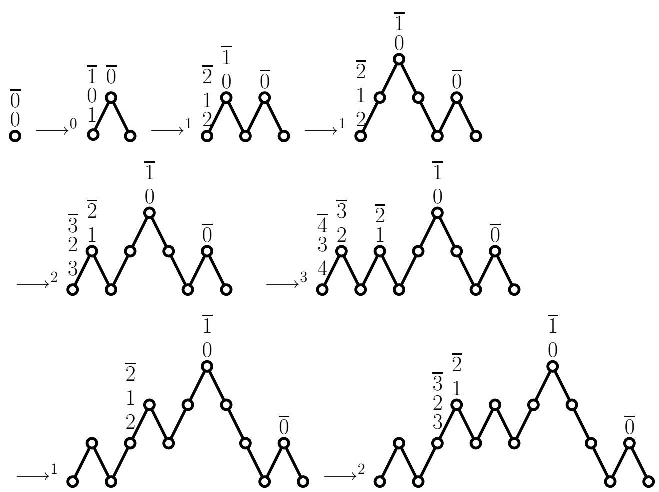

Figure 3. An example of uniform insertion according to $(BA\overline{B}x, \overline{A}\overline{x})$

First, we check that this algorithm has an expected behavior:

Lemma 1.2. *For any insertion mode m, the transformation Υ_m maps almost decreasing sequences of size n to Dyck words of size n.*

Proof. For the smallest objects, $\Upsilon_m(\emptyset) = \epsilon$ and $\Upsilon_m(0) = x\bar{x}$. For longer sequences there is an invariant in this algorithm: after k loops ($k \geq 1$), w is a Dyck building of rank $s_k + 1$ in which $\pi_X(w)$ is a Dyck word of size k and any rising label i appears before the falling label $\bar{i}$. The constraints of order on the generic labels in the definition of insertion modes implies that this invariant is preserved.

□

If $\mathcal{P}$ denotes the set of insertion modes and F is defined by $F(m,s) = \Upsilon_m(s)$, Lemma 1.2 proves that the ordered pair $(\mathcal{P}, F)$ is a uniform description.

2. A systematic study of insertion modes

There are only finitely many programs in the uniform description $(\mathcal{P}, F)$ defined in Section 1. We count them. Then we show that certain transformations on insertion mode produces maps $\Upsilon_m = F(m,.)$ that are either equal, or equivalent through a very simple involution. Then, with the help of the computer, we list counter-examples for maps Υ_m that are not one-to-one, and, with a pen and a paper, we prove that the remaining maps are bijections. Thus we fix the set $\mathcal{B}$ of 12 bijections defined by this uniform description. A trivial symmetry on Dyck paths relates each bijection to another one, so we obtain only 6 significantly distinct bijections. Each bijection in $\mathcal{B}$ is related to a parameter on Dyck path with the same distribution as the length of the last descent. We provide a description for 5 of these 6 parameters which are sometimes classical sometimes new.

2.1. Relations between insertion modes. If we consider the comma as a letter in an insertion mode, there are seven different letters in an insertion mode. Since we independently impose that A appears before $\overline{A}$, B before $\overline{B}$ and $x, \boxed{,}, \overline{x}$ in this order, there are $7!/(2!*2!*3!) = 210 = |\mathcal{P}|$ insertion modes.

2.1.1. *Insertion modes defining the same maps.* We define three relations of equivalence on insertion modes. Let $m_1 = (u_1, v_1)$ and $m_2 = (u_2, v_2)$ be two insertion modes. These modes are *rising-equivalent*, $m_1 \equiv_{\text{rise}} m_2$, if there exists $Y \in \{A, B\}$ such that $m_1 = (Yx, v)$ and $m_2 = (xY, v)$. They are *falling-equivalent*, $m_1 \equiv_{\text{fall}} m_2$, if there exists $\overline{Y} \in \{\overline{A}, \overline{B}\}$ such that $m_1 = (u, \overline{Y}\overline{x})$ and $m_2 = (u, \overline{x}\overline{Y})$. These two modes are *peak-equivalent*, $m_1 \equiv_{\text{peak}} m_2$, if $u_1v_1 = u_2v_2$ and $A\overline{A}$, $B\overline{B}$ are factors of u_1v_1.

LEMMA 2.1. *Two insertion modes m_1 and m_2 that are either rising, falling or peak-equivalent, define the same map $\Upsilon_{m_1} = \Upsilon_{m_2}$.*

PROOF. Let m_1 and m_2 be two insertion modes, s an almost decreasing sequence of n integers. We denote $(w_k^1)_{k=1\dots n}$, respectively $(w_k^2)_{k=1\dots n}$, the sequence of Dyck buildings $(w_k)_{k=1\dots n}$ obtained in the algorithm when the input is s and the parameter m_1, respectively m_2.

• First we assume that m_1 and m_2 are rising-equivalent ($m_1 \equiv_{\text{rise}} m_2$). The key observation is that when m_1 corresponds to an insertion of a rise at the beginning of a sequence of rises x^j, m_2 corresponds to an insertion of a rise at the end of the same sequence of rises, leading to the same Dyck word. Formally, we define an equivalence $\equiv_r$ over Dyck buildings as the symmetric and transitive closure of the relation $\longrightarrow_r$ that corresponds to the commutation of a rising label i and the rise at its right : $u.x.i.v \longrightarrow_r u.i.x.v$. To prove that $w_k^1 \equiv_r w_k^2$, we will prove the following stronger fact: given the Dyck buildings w and w' such that $w \longrightarrow_r w'$, the insertions of the value i in w and w' according to the modes m_1 or m_2, leads to four equivalent Dyck buildings:

$$\rho^i_{m_1}(\pi_{X\cup L(i)}(w)) \equiv_r \rho^i_{m_2}(\pi_{X\cup L(i)}(w)) \equiv_r \rho^i_{m_1}(\pi_{X\cup L(i)}(w')) \equiv_r \rho^i_{m_2}(\pi_{X\cup L(i)}(w')).$$

Since $w \longrightarrow_r w'$, there exists a label j in w and w' such that $w = u'.j.x.u''$ and $w' = u'.x.j.u''$. We discuss according to the relative values of i and j:

$j > i$ The rising label j is erased by $\pi_{X,L(i)}$ so $\pi_{X,L(i)}(w) = \pi_{X,L(i)}(w')$. The insertion according to m_1 (respectively m_2) leads to $v.(i.x).v'$ (respectively $v.(x.i).v'$). These two buildings are clearly equivalent.

$j = i$ We write $w = u.i.x.u'$ and $w' = u.x.i.u'$. We observe that

$$\begin{aligned}\rho^i_{m_1}(\pi_{X\cup L(i)}(w)) = u.(i.x).x.v' &\longrightarrow_r \rho^i_{m_2}(\pi_{X\cup L(i)}(w)) = u.(x.i).x.v'\\ = \rho^i_{m_1}(\pi_{X\cup L(i)}(w')) = u.x.(i.x).v' &\longrightarrow_r \rho^i_{m_2}(\pi_{X\cup L(i)}(w')) = u.x.(x.i).v'\end{aligned}$$

where $v' = \rho^i_{m_1}(u') = \rho^i_{m_2}(u')$ by definition of $\equiv_{\text{rise}}$.

$j < i$ We assume that $w = u.j.x.u'.i.u''$ and $w' = u.x.j.u'.i.u''$. We observe that

$$\begin{aligned}\rho^i_{m_1}(\pi_{X\cup L(i)}(w)) = u.j.x.u'.(i.x).v'' &\longrightarrow_r \rho^i_{m_2}(\pi_{X\cup L(i)}(w)) = u.j.x.u'.(x.i).v''\\ \longrightarrow_r \rho^i_{m_2}(\pi_{X\cup L(i)}(w')) = u.x.j.u'.(x.i).v'' &\longrightarrow_r \rho^i_{m_1}(\pi_{X\cup L(i)}(w')) = u.x.j.u'.(i.x).v''\end{aligned}$$

where $v'' = \rho^i_{m_1}(u'') = \rho^i_{m_2}(u'')$ by the definition of $\equiv_{\text{rise}}$. The careful reader will check that the relative positions of the rising labels i and j do not perturb the proof.

For $k = 0$, the assumption $w_0^1 \equiv_r w_0^2$ is satisfied since $w_0^1 = 0\overline{0} = w_0^2$. By induction, for all $k \leq n$, $w_k^1 \equiv_r w_k^2$ and in particular for $k = n$. Since the projection over the alphabet X is the same for two buildings $\equiv_r$-equivalent :

$$\Upsilon_{m_1}(s) = \pi_X(w_n^1) = \pi_X(w_n^2) = \Upsilon_{m_2}(s).$$

• If $m_1 \equiv_{\text{fall}} m_2$, the proof of $\Upsilon_{m_1} = \Upsilon_{m_2}$ is symmetric to the previous one.

• We assume that $m_1 = (u_1, v_1)$ and $m_2 = (u_2, v_2)$ are peak-equivalent ($m_1 \equiv_{\text{peak}} m_2$). We check by induction on k that $w_k^1 = w_k^2$ and that the labels of index i, if they appear, appear as the factor $i\overline{i}$. This assumption is satisfied for $k = 0$ since $w_0^1 = 0\overline{0} = w_0^2$. We suppose that it is satisfied for $k-1$. Since $s_k\overline{s_k}$ is a factor of $w_{k-1}^1 = w_{k-1}^2$, the rewriting induced by s_k corresponds to the insertion at the same place, in terms of generic labels, of $u_1.v_1$ respectively $u_2.v_2$ which are equal by the definition of $\equiv_{\text{peak}}$. Thus $w_k^1 = w_k^2$. Moreover, $A\overline{A}$ and $B\overline{B}$ are factors of $u_1v_1 = u_2v_2$ so $s_k\overline{s_k}$ and $s_{k+1}\overline{s_{k+1}}$ are factors of $w_k^1 = w_k^2$. For any index $i < s_k$, $i\overline{i}$ remains a factor of $w_k^1 = w_k^2$ since the insertion in $w_{k-1}^1 = w_{k-1}^2$ does not insert anything between the occurrence of i and $\overline{i}$. □

2.1.2. *Two symmetries on insertion modes.* The reflexion according to the vertical axis defines a natural involution over Dyck paths. We generalize this mapping to words over an alphabet $L \cup \overline{L} \cup K$. Let w be a word, the *mirror word* $mir(w)$ of w is recursively defined by $mir(\epsilon) = \epsilon$, for $l \in L$, $mir(l.w') = mir(w').\overline{l}$, for $\overline{l} \in \overline{L}$, $mir(\overline{l}.w') = mir(w').l$ and for $k \in K$, $mir(k.w) = mir(w').k$. Let $m = (u, v)$ be an insertion mode, the *mirror insertion mode* is $mir(m) = (mir(v), mir(u))$. The exchange of the (generic) labels of indexes A and B leads to the notion of *exchanged insertion mode* :

$$exc(m) = (u[A := B, B := A, \overline{A} := \overline{B}, \overline{B} := \overline{A}], v[A := B, B := A, \overline{A} := \overline{B}, \overline{B} := \overline{A}]).$$

Let $s = (s_k)_{k=1\ldots n}$ be a sequence of n integers and $i \in \mathbb{N}$; by definition the sequence $t = s \oplus i$ is such that $t_k = s_k + i$ for all $k = 1 \ldots n$. An almost decreasing sequence s admits a single decomposition

$$s = 0, t_1 \oplus 1, t_2$$

where t_1 and t_2 are almost decreasing sequences. ($t_1 \oplus 1$ is the sequence of integers before the second 0 of s, which is the beginning of t_2, if any.) We use this decomposition to recursively define a map exc over almost decreasing sequences : $exc(\emptyset) = \emptyset$ and $exc(0, t_1 \oplus 1, t_2) = 0, exc(t_2) \oplus 1, exc(t_1)$. An inductive proof shows that this map exc is indeed an involution since

$$exc(exc(s)) = exc(0, exc(t_2) \oplus 1, exc(t_1)) = (0, exc(exc(t_1)) \oplus 1, exc(exc(t_2))) = (0, t_1 \oplus 1, t_2) = s.$$

The canonical bijection between almost decreasing sequences and Dyck paths relates the involution exc to an involution on Dyck paths already considered in [**4**]. These transformations of insertion modes, almost decreasing sequences and Dyck paths define relations between maps Υ_m:

LEMMA 2.2. *For any insertion mode m,*

$$mir \circ \Upsilon_m = \Upsilon_{mir(m)}$$

and

$$\Upsilon_m \circ exc = \Upsilon_{exc(m)}.$$

PROOF. Let s be an almost decreasing sequence of n integers and let $(w_k)_{k=1\ldots n}$ the sequence of buildings constructed by the algorithm computing $\Upsilon_m(s)$.

•$mir \circ \Upsilon_m = \Upsilon_{mir(m)}$: Let $(w'_k)_{k=1\ldots n}$ be the sequence of buildings when the algorithm computes $\Upsilon_{mir(m)}(s)$. We check by induction on k that $w'_k = mir(w_k)$. Thus $mir \circ \Upsilon_m(s) = \Upsilon_{mir(m)}(s)$.

•$\Upsilon_m \circ exc = \Upsilon_{exc(m)}$: Let u be a non-empty word on the alphabet $X \cup G$. We denote by $\Upsilon_m^u(s)$ the result of the algorithm when s is any sequence of non-negative integers, and the initial value w_0 is the word u. Since the labels are not moved in the buildings $(w_k)_{0 \leq k \leq n}$ while not erased, we have $\Upsilon_m(s) = \Upsilon_m^{0\overline{0}}(s) = \Upsilon_m^0(s).\Upsilon_m^{\overline{0}}(s)$. Moreover the insertion mode does not depend on the index of the labels so $\Upsilon_m^0(s) = \Upsilon_m^1(s \oplus 1)$.

We now check by induction on the length of the almost decreasing sequence s that $\Upsilon_m^0(exc(s)) = \Upsilon_{exc(m)}^0(s)$ and $\Upsilon_m^{\overline{0}}(exc(s)) = \Upsilon_{exc(m)}^{\overline{0}}(s)$. For the empty sequence $\emptyset$, $\Upsilon_m^0(exc(\emptyset)) = \epsilon = \Upsilon_{exc(m)}^0(\emptyset)$ and $\Upsilon_m^{\overline{0}}(exc(\emptyset)) = \epsilon = \Upsilon_{exc(m)}^{\overline{0}}(\emptyset)$. A non-empty sequence s satisfies $s = 0.t_1 \oplus 1.t_2$ and $exc(s) = 0.exc(t_2) \oplus 1.exc(t_1)$. The insertion mode m is written $m = (u, v)$. By definition

$$\Upsilon_m^0(exc(s)) = u[0 := \Upsilon_m^0(exc(t_1)), 1 := \Upsilon_m^1(exc(t_2) \oplus 1), \overline{\ldots}]$$

where $\overline{\ldots}$ denotes an expression for the falling labels similar to the one for rising labels that appears before the comma. Here for example, $\overline{\ldots}$ replace $\overline{0} := \Upsilon^{\overline{0}}_m(exc(t_1)), \overline{1} := \Upsilon^{\overline{1}}_m(exc(t_2) \oplus 1)$. By the induction hypothesis, $\Upsilon^0_m(exc(t_1)) = \Upsilon^0_{exc(m)}(t_1)$ and $\Upsilon^1_m(exc(t_2) \oplus 1) = \Upsilon^0_m(exc(t_2)) = \Upsilon^0_{exc(m)}(t_2)$ and so we have

$$\Upsilon^0_m(exc(s)) = u[0 := \Upsilon^0_{exc(m)}(t_1), 1 := \Upsilon^0_{exc(m)}(t_2), \overline{\ldots}].$$

By definition $exc(m) = (u, v)[A := B, B := A, \overline{\ldots}]$ and so,

$$\Upsilon^0_{exc(m)}(s) = u[0 := 1, 1 := 0, \overline{\ldots}][0 := \Upsilon^0_{exc(m)}(t_2), 1 := \Upsilon^1_{exc(m)}(t_1 \oplus 1), \overline{\ldots}],$$

$$\Upsilon^0_{exc(m)}(s) = u[1 := \Upsilon^0_{exc(m)}(t_2), 0 := \Upsilon^1_{exc(m)}(t_1 \oplus 1), \overline{\ldots}].$$

Since $\Upsilon^1_{exc(m)}(t_1 \oplus 1) = \Upsilon^0_{exc(m)}(t_1)$ we conclude that

$$\Upsilon^0_{exc(m)}(s) = u[1 := \Upsilon^0_{exc(m)}(t_2), 0 := \Upsilon^0_{exc(m)}(t_1), \overline{\ldots}] = \Upsilon^0_m(exc(s)).$$

A symmetric proof holds for $\Upsilon^{\overline{0}}_{exc(m)}(s) = \Upsilon^{\overline{0}}_m(exc(s))$. The two equalities lead to

$$\Upsilon_m(exc(s)) = \Upsilon^0_m(exc(s))\Upsilon^{\overline{0}}_m(exc(s)) = \Upsilon^0_{exc(m)}(s)\Upsilon^{\overline{0}}_{exc(m)}(s) = \Upsilon_{exc(m)}(s).$$

□

2.2. 210 insertion modes. We have written a computer program that computes $\Upsilon_m(s)$ in a sensible amount of time for all insertion modes and all almost decreasing sequences of length at most 11.

2.2.1. *The 178 insertion modes inducing non-injective maps.* For 178 insertion modes the program gives a pair of distinct almost decreasing sequences s and s' such that $\Upsilon_m(s) = \Upsilon_m(s')$. Each of these counter-examples implies that the given Υ_m is not one-to-one. Remarkably, these counter-examples have length at most 3. We do not reproduce them here. The study of these maps may be of combinatorial interest since the generating functions of the Dyck paths in the images seem to be well known (*i.e.* present in the Sloane encyclopedia of integer sequences).

2.2.2. *The 32 insertion modes inducing one-to-one maps.* For the remaining 32 insertion modes, the program shows that there are no counter-examples involving almost decreasing sequences of length shorter than 11. We have to prove "by hand" that these modes induce bijections. Using Lemma 2.1, we identify modes that define the same map. Moreover, Lemma 2.2 indicates modes related by the involutions *mir* on Dyck paths or *exc* on almost decreasing sequences. Finally we obtain a partition of the 32 modes into three classes, see Figure 4.

In each class, the equivalences and symmetries preserve the fact that the mode induces or not a bijection. So we merely have to show that one mode in the class induces a bijection to conclude that all modes in the class induce bijections.

Theorem 2.3. *For any insertion mode m in $\{(xB\overline{B}, \overline{x}A\overline{A}), (AB\overline{B}x, \overline{x}\overline{A}), (BA\overline{B}x, \overline{A}\overline{x})\}$, the map Υ_m is a size-preserving bijection between almost decreasing sequences and Dyck paths.*

The almost decreasing sequence $s = 0, 0, 1, 0, 0, 1$ is mapped to 12 distinct Dyck paths by the 12 bijections. Thus we have the following corollary :

Corollary 2.4. The 32 insertion modes in Figure 4 define 12 different bijections.

Proof. (of Theorem 2.3) All proofs follow the same scheme. For an insertion mode m, the first key element of the proof is a conjectured *labeling map* f_m which maps a Dyck path w into a Dyck building $f_m(w)$. Roughly speaking, the map f_m recovers the labels erased by π_X in the last step of the algorithm. Then we are able to recover from $f_m(w)$ the last value of the almost decreasing sequence and the last two steps inserted in the path during the algorithm. This is only true because the mode $m = (u, v)$ is *locally reversible* : u and v both contain at least one generic label. [1] Thus an induction on the size of the Dyck paths allows us to compute the reverse map of Υ_m.

Given an almost decreasing sequence of n elements we denote $\Upsilon^+_m(s)$ the last building w_n produced by the algorithm at the end of the For-loop. Thus the output is $\Upsilon_m(s) = \pi_X(\Upsilon^+_m(s))$. Our induction hypothesis is that for any Dyck path v of size n, there exists a unique almost decreasing sequence $s = s_1 \ldots s_n$ such that $\Upsilon_m(s) = v$ and $\Upsilon^+_m(s) = f_m(v)$.

[1]The mode $(x, B\overline{B}\overline{x}A\overline{A})$ is an example of a not locally reversible mode.

$$
\begin{array}{|c|}
\hline
\begin{array}{ccccc}
\left\{\begin{array}{ll}
 & (xB\overline{B},\overline{x}A\overline{A}) \\
\equiv_{\text{peak}} & (x,B\overline{B}\overline{x}A\overline{A}) \\
\equiv_{\text{peak}} & (xB,\overline{B}\overline{x}A\overline{A}) \\
\equiv_{\text{rise}} & (Bx,\overline{B}\overline{x}A\overline{A})
\end{array}\right. & \Longleftrightarrow & \text{exc} & \left\{\begin{array}{ll}
 & (xA\overline{A},\overline{x}B\overline{B}) \\
\equiv_{\text{peak}} & (x,A\overline{A}\overline{x}B\overline{B}) \\
\equiv_{\text{peak}} & (xA,\overline{A}\overline{x}B\overline{B}) \\
\equiv_{\text{rise}} & (Ax,\overline{A}\overline{x}B\overline{B})
\end{array}\right. \\
\Updownarrow_{\text{mir}} & & & \Updownarrow_{\text{mir}} \\
\left\{\begin{array}{ll}
 & (A\overline{A}x,B\overline{B}\overline{x}) \\
\equiv_{\text{peak}} & (A\overline{A}xB\overline{B},\overline{x}) \\
\equiv_{\text{peak}} & (A\overline{A}xB,\overline{B}\overline{x}) \\
\equiv_{\text{fall}} & (A\overline{A}xB,\overline{x}\overline{B})
\end{array}\right. & \Longleftrightarrow & \text{exc} & \left\{\begin{array}{ll}
 & (B\overline{B}x,A\overline{A}\overline{x}) \\
\equiv_{\text{peak}} & (B\overline{B}xA\overline{A},\overline{x}) \\
\equiv_{\text{peak}} & (B\overline{B}xA,\overline{A}\overline{x}) \\
\equiv_{\text{fall}} & (B\overline{B}xA,\overline{x}\overline{A})
\end{array}\right.
\end{array} \\
\hline
\end{array}
$$

$$
\begin{array}{|c|}
\hline
\begin{array}{ccccc}
\left\{\begin{array}{ll}
 & (AB\overline{B}x,\overline{x}\overline{A}) \\
\equiv_{\text{fall}} & (AB\overline{B}x,\overline{A}\overline{x})
\end{array}\right. & \Longleftrightarrow & \text{exc} & \left\{\begin{array}{ll}
 & (BA\overline{A}x,\overline{x}\overline{B}) \\
\equiv_{\text{fall}} & (BA\overline{A}x,\overline{B}\overline{x})
\end{array}\right. \\
\Updownarrow_{\text{mir}} & & & \Updownarrow_{\text{mir}} \\
\left\{\begin{array}{ll}
 & (Ax,\overline{x}B\overline{BA}) \\
\equiv_{\text{rise}} & (xA,\overline{x}B\overline{BA})
\end{array}\right. & \Longleftrightarrow & \text{exc} & \left\{\begin{array}{ll}
 & (Bx,\overline{x}A\overline{AB}) \\
\equiv_{\text{rise}} & (xB,\overline{x}A\overline{AB})
\end{array}\right.
\end{array} \\
\hline
\end{array}
$$

$$
\begin{array}{|c|}
\hline
\begin{array}{ccccc}
\left\{\begin{array}{ll}
 & (BA\overline{B}x,\overline{A}\overline{x}) \\
\equiv_{\text{fall}} & (BA\overline{B}x,\overline{x}\overline{A})
\end{array}\right. & \Longleftrightarrow & \text{exc} & \left\{\begin{array}{ll}
 & (AB\overline{A}x,\overline{B}\overline{x}) \\
\equiv_{\text{fall}} & (AB\overline{A}x,\overline{x}\overline{B})
\end{array}\right. \\
\Updownarrow_{\text{mir}} & & & \Updownarrow_{\text{mir}} \\
\left\{\begin{array}{ll}
 & (xA,\overline{x}B\overline{AB}) \\
\equiv_{\text{rise}} & (Ax,\overline{x}B\overline{AB})
\end{array}\right. & \Longleftrightarrow & \text{exc} & \left\{\begin{array}{ll}
 & (xB,\overline{x}A\overline{BA}) \\
\equiv_{\text{rise}} & (Bx,\overline{x}A\overline{BA})
\end{array}\right.
\end{array} \\
\hline
\end{array}
$$

FIGURE 4. The 32 insertion modes inducing bijections

Let u be a Dyck path of size $n+1$, we look for $s = s_1, \dots s_n, s_{n+1}$ such that $\Upsilon_m(s) = u$. We want $f_m(u) = \Upsilon_m^+(s)$ and the rank of $\Upsilon_m^+(s_1, \dots s_n, s_{n+1})$ is $s_{n+1} + 1$ thus necessarily $s_{n+1} = r - 1$ where r is the rank of $f_m(u)$. Since m is locally reversible, we identify in $f_m(u)$ the unique rise and fall that may be inserted during the $n + 1^{\text{th}}$ loop. We remove these steps in $f_m(u)$ and consider the projection v over X which is a Dyck path of size n. The induction hypothesis gives us an almost decreasing sequence s_v such that $v = \Upsilon_m(s_v)$ and $\Upsilon_m^+(s_v) = f_m(v)$. The unique possibility for s is $(s_v, r-1)$. It remains to check that it works. During the computation of $\Upsilon_m(s)$ the first n loops are similar to those of the computation of $\Upsilon_m(s_v)$ thus $u_n = v_n = \Upsilon_m^+(v) = f_m(v)$ and so $\Upsilon_m^+(s) = \rho_m^{r-1}(\pi_{X \cup L(r-2)}(f_m(v)))$. The only equality to check is

$$\rho_m^{r-1}(\pi_{X \cup L(r-2)}(f_m(v))) = f_m(u).$$

We prove this identity only for the first insertion mode, the other cases are similar and left to the reader in this summary. We focus on the simplest case: the canonical bijection. We do so to avoid technical details that the reader can find in [**7**] and to bring the essential steps into focus.

The labeling map f_m for the insertion mode $m = (AB\overline{B}x, \overline{x}\overline{A})$ will be denoted f_{descent} : the rank r of $f_{\text{descent}}(u)$ is the height of the rightmost peak in u and, for $i \leq r$, the factors $i\overline{i}$ are inserted in the rightmost vertex of height i to produce $f_{\text{descent}}(u)$ (see Figure 6).

We assume that u is a Dyck word of size $n+1$ and that the induction hypothesis is satisfied for paths of size n. Figure 5 illustrates the proof. From $f_m(u)$, the rank r is the height of the rightmost peak. The path v is obtained by deleting the rightmost rise and the next fall. The reverse operation is the insertion of a factor $x\overline{x}$ in the rightmost vertex V in v at height $r-1$. In $f_m(v)$, this vertex V contains the factor $r-1\overline{r-1}$ implying $u = \Upsilon_m(s_v, r-1)$. Now we check that $f_m(v)$ is converted into $f_m(u)$ when we insert steps according to m and the value $r-1$. Labels of indexes greater than $r-1$ in $f_m(v)$ are erased in $\pi_{X \cup L(k-1)}(f_m(v))$ and by definition there is no label greater than r in $f_m(u)$. According to m, labels of index r appear in the new rightmost peak of u and labels of index $r-1$ in the following vertex, coinciding with the labels in the rightmost vertices at height r and $r-1$ in $f_m(u)$. Since the suffix $\overline{x}^{r-1}$ of v is also a suffix of u after the insertion, labels of index $i < r-1$ coincide with those of $f_m(u)$.

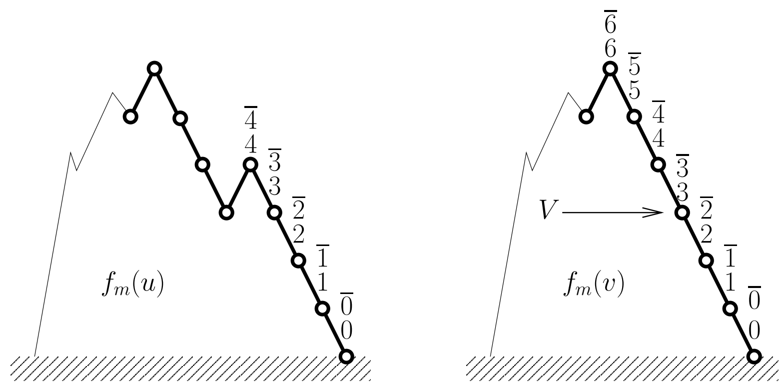

FIGURE 5. $\rho_m^{r-1}(\pi_{X\cup L(r-2)}(f_m(v))) = f_m(u)$ for $m = (xB\overline{B}, \overline{x}A\overline{A})$ and $r = 4$

The labeling map for $(AB\overline{B}x, \overline{x}\overline{A})$ is denoted f_{axis}. The rank r of $f_{\text{axis}}(u)$ is the number of falls ending on the horizontal axis. For $i < r$, the falling label $\overline{i}$ is inserted in the vertex next the $(r-i)^{\text{th}}$ fall ending on the horizontal axis. We insert the factor $01\ldots r\overline{r}$ in the first vertex of the path. See Figure 6.

The labeling map for $(AB\overline{B}x, \overline{x}\overline{A})$ is denoted f_{umbrella}. A *descent* in the Dyck path u is a maximal sequence of falls. We associate to each descent of k falls an *umbrella* of size k that is the smallest factor of u containing the k falls of the descent and the k preceding rises in u. The *center* of an umbrella is the peak preceding the k last falls. The *start* is the vertex preceding the leftmost rise of the umbrella. We consider the suffix s_u of u which is the longest concatenation of umbrellas: $s_u = u_r u_{r-1} \ldots u_2 u_1$. Some additional umbrellas may appear as factors of s_u, see the umbrella of center α in Figure 6. The rank r of $f_{\text{umbrella}}(u)$ is the number of umbrellas appearing in the concatenation s_u. The label $\overline{0}$ is inserted in the center of u_1, for $i < r-1$ the factor $i\overline{i+1}$ is inserted in the center of u_{i+1} and the factor $r(r-1)\overline{r}$ in the start of u_r. This produces $f_{\text{umbrella}}(u)$.

□

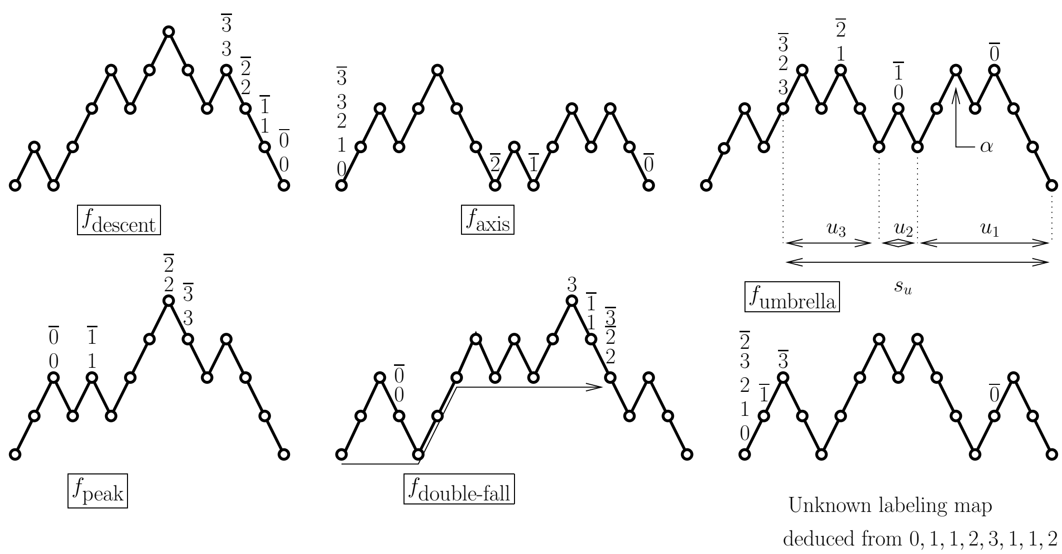

FIGURE 6. Labeling maps for five insertion modes and one unknown map

The labeling map, associated with an insertion mode inducing a bijection, defines a parameter which has the same distribution on Dyck paths as the value of the last element on the almost decreasing sequences. Some of these parameters were known to have the same distribution (in particular the length of the last descent and the number of returns to the horizontal axis). We increase the list of such parameters.

Proposition 2.1. *The following parameters have the same distribution on Dyck paths:*
- *the number of falls in the last descent,*
- *the number of falls ending on the horizontal axis,*
- *the number of umbrellas in the longest suffix which is a concatenation of umbrellas,*
- *the number of peaks before the first double fall,*
- *the number of double falls in the longest increasing prefix .*

Proof. Let u be a Dyck path. Figure 6 provides examples of the definitions of the labelings. The first three labelings already have been defined in the proof of Theorem 2.3.

The number of falls in the last descent, the number of falls ending on the horizontal axis and the number of umbrellas in the longest suffix which is a concatenation of umbrellas are respectively the rank of $f_{\text{descent}}(u)$, $f_{\text{axis}}(u)$ and $f_{\text{umbrella}}(u)$.

The labeling map for the mode $(xA\overline{A}, \overline{x}B\overline{B}) = exc(xB\overline{B}, \overline{x}A\overline{A})$ is denoted $f_{\text{double-fall}}$. The rank r of $f_{\text{double-fall}}(u)$ is the number of peaks in the longest prefix of u that does not contain a *double-fall*, *i.e.* a factor $\overline{x}\overline{x}$. For $i < r$, we insert a factor $i\overline{i}$ in the $(i+1)^{\text{th}}$ peak and the factor $r\overline{r}$ is inserted in the last vertex of the prefix (usually the first double-fall).

The labeling map for the mode $(BA\overline{A}, \overline{x}\overline{B}) = exc(AB\overline{B}x, \overline{x}\overline{A})$ is denoted $f_{\text{double-fall}}$. A *valley* in u is a vertex in the middle of a factor $\overline{x}x$. A prefix of a Dyck path is *increasing* if the height of any vertex is not strictly lower than a valley at its left. The rank r of $f_{\text{double-fall}}(u)$ is the number of double-falls in the longest increasing prefix p_u. We insert in u, for $i < r$, the factor $i\overline{i}$ in the $(i+1)^{\text{th}}$ peak, the label r in the rightmost peak in p_u and the label $\overline{r}$ in the last vertex in p_u.

□

We do not mention here the parameters equivalent up to a vertical reflexion *mir*. We were not able to identify the labeling map of the mode $(AB\overline{A}x, \overline{B}\overline{x})$ even we know, by Lemma 2.2, that it induces a bijection. The last building in Figure 6 is an example of a building produced by this mode.

3. Variations on the algorithm for bijections in several combinatorial contexts

We use modifications of the initial algorithm to define bijections that are relevant in several combinatorial contexts. In this extended abstract we do not emphasize these combinatorial contexts but the alteration of the algorithm. Moreover, we do not prove that the algorithms define the claimed bijections. The interested reader will find detailed proofs and other examples in [**7**].

3.1. Cyclic permutation of the labels. A *cyclic permutation of the label indexes* in a building w of rank k is denoted $Cyc(w)$. It consists of replacing for $0 \leq i \leq k$, the label i, respectively $\overline{i}$ by the label $(i+1 \mod k)$, respectively $\overline{(i+1 \mod k)}$. We generalize the algorithm presented in the Section 1 by performing a cyclic permutation at the end of each For-loop : $w_n := Cyc(w_n)$. We denote by Υ_m^{Cyc} the map defined by this algorithm parametrized by the insertion mode m.

In [**5**], the authors conjectured a formula defined by summation over integer partitions relevant for a problem in algebraic combinatorics. Haiman and Haglund, see [**6**], independently proposed two different pairs of parameter on Dyck paths that interpret this formula by summation over Dyck paths. Both pairs use the *area below the Dyck path*, that is the number of squares between the path and the horizontal axis placed as in Figure 1. We have

$$C(u, v; t) = \sum_w u^{\text{area}(w)} v^{\text{dinv}(w)} t^{\text{size}(w)} = \sum_w u^{\text{area}(w)} v^{\text{bounce}(w)} t^{\text{size}(w)}$$

where in the summations w runs over Dyck paths and $\text{dinv}(w)$, respectively $\text{bounce}(w)$, are the parameters defined by Haiman respectively Haglund. There exists another definition of $C(u, v; t)$ which is clearly symmetric in u and v. An open problem is to find a bijection that explains directly this symmetry in terms of Dyck paths.

If we consider the diagonal of squares below each rise, as in Figure 1, we remark that the area of a path is also the sum of the heights of the rises. Thus the canonical bijection shows that the area is distributed over the Dyck paths as the sum $\sum_{i=1}^{k} s_i$ on the almost decreasing sequences. In fact all the previous bijections Υ_m define parameters on Dyck path distributed as the sum of the value of the almost decreasing sequences but alas no pair of parameters has the same join distribution as $(\text{area}, \text{dinv})$ and $(\text{area}, \text{bounce})$. We show a variation of our algorithm that convert the sum of the s_i into *dinv*. In [**7**], we also provide a labeling map to define a bijection that convert the sum of the s_i into *bounce*. In the future, we plan (hope) to use these algorithms to produce other pairs of parameters whose distribution defines $C(u, v; t)$.

Let w be a Dyck path of size n and let $h = h_1, h_2 \dots h_n$ be the sequence of the height of rises in w. The parameter $dinv(w)$ counts the numbers of pairs (i, j) such that $i < j$ and $h_i \in \{h_j, 1 + h_j\}$.

We will use the labeling map f_{dinv} illustrated in Figure 7. Let k be the number of vertices at maximal height H in w and let l be the number of vertices at height $H - 1$ lying to the right of the rightmost vertex at height H. The rank of $f_{\text{dinv}}(w)$ is $k + l - 1$. For $0 \leq i \leq k - 1$ the factor $i\bar{i}$ is inserted in the $(k - i)^{\text{th}}$ vertex at height H and for $k \leq i \leq k + l - 1$, the factor $i\bar{i}$ is inserted in the $(k + l - i)^{\text{th}}$ vertex at height $H - 1$ lying to the right of the rightmost vertex at height H. In $f_{\text{dinv}}(w)$, an insertion of a rise and a fall in the labels of index k increases the parameter $dinv$ by exactly k.

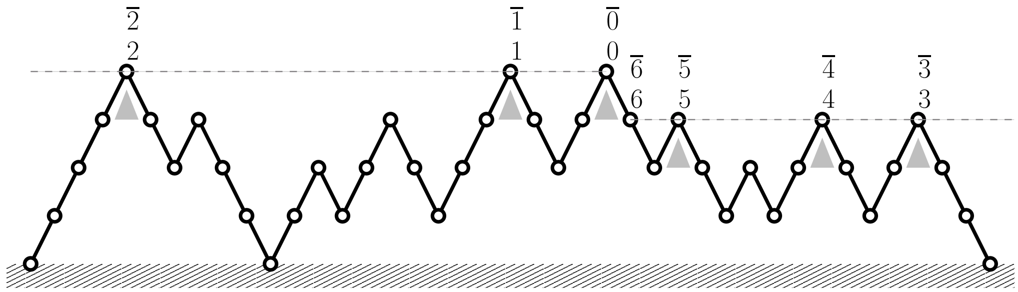

FIGURE 7. Map labeling f_{dinv} for the parameter dinv where $k = 3$ and $l = 4$

PROPOSITION 3.1. *The map* $\Upsilon^{Cyc}_{(xB\overline{B}, \overline{x}A\overline{A})}$ *is a bijection that maps an almost decreasing sequence* s *to a Dyck path* w *such that*

$$\sum_{k=1}^{n} s_k = dinv(w).$$

The proof checks that $f_{\text{dinv}}(w)$ equals $\Upsilon^{Cyc,+}_{(xB\overline{B}, \overline{x}A\overline{A})}(s)$ which is the last building w_n at the end of the For-loop in the generalized algorithm. A similar map was presented in [**1**] in terms of plane trees.

3.2. Insertion depending on the parity of the value. We define an algorithm parametrized by two insertion modes m_1 and m_2. If the value s_k in the almost decreasing sequence is even, we use m_1 to compute w_k otherwise we use m_2. For the example traced on Figure 8 we use in the even case the insertion mode $m_1 = (B\overline{B}x, A\overline{A}\overline{x})$ and in the odd case the mode $m_2 = exc(m_1) = (A\overline{A}x, B\overline{B}\overline{x})$. This defines a bijection denoted $\Upsilon_{[(B\overline{B}x, A\overline{A}\overline{x});(A\overline{A}x, B\overline{B}\overline{x})]}$, that we use in the following context.

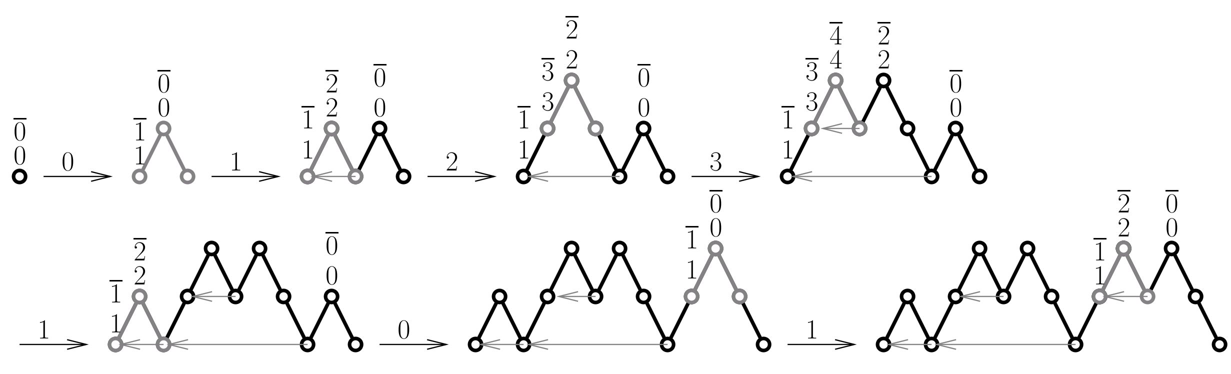

FIGURE 8. The image of $0, 1, 2, 3, 1, 0, 1$ with insertion depending on the parity

In [**7**], we interpret combinatorially the formal manipulations of generating functions involved in the solution of an equation usually used in the kernel method [**3**]. At some point we need a bijection that translates a parameter into another to conclude the interpretation. Here we only present these parameters and the description of the bijection by the extension of the algorithm that distinguish the parity of s_i's.

The first parameter is the *height* of a Dyck path w, that is the maximal value of ordinate of a vertex in w. A *ray* in a Dyck path is a segment with one endpoint a valley, the *source* of the ray, and the other one the preceding vertex of the path that is at the same height. The *ray height of a peak* is the number of rays that cross the vertical segment starting at the peak and finishing on the horizontal axis. The *ray height* of a Dyck path is the maximal height of its peaks. Figure 9 illustrates these definitions.

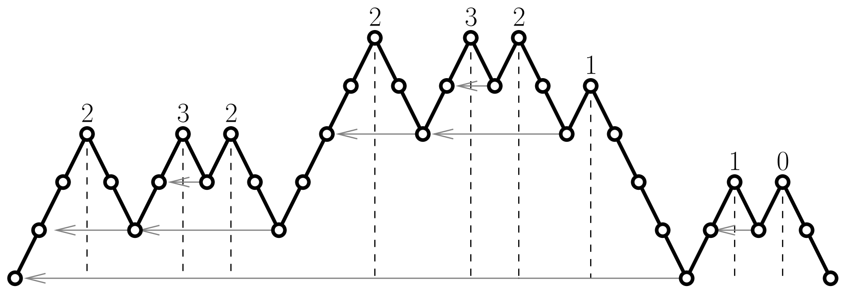

FIGURE 9. A Dyck path of ray height 3

PROPOSITION 3.2. *For any $N \geq 0$, there are as many Dyck paths of height at most $2N+1$ as Dyck paths of ray height at most N.*

The bijection $\Upsilon_{[(B\overline{B}x, A\overline{A}\overline{x});(A\overline{A}x, B\overline{B}\overline{x})]}$ maps almost decreasing sequences whose maximal value is either $2N$ or $2N+1$ to Dyck paths of ray height exactly N.

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CNRS/LaBRI, UNIVERSITY BORDEAUX I, TALENCE, FRANCE
E-mail address: `borgne@labri.fr`