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New results on the combinatorial invariance of Kazhdan-Lusztig polynomials

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Abstract. We prove that the Kazhdan-Lusztig polynomials are combinatorial invariants for intervals up to length 8 in Coxeter groups of type **A** and up to length 6 in Coxeter groups of type **B** and **D**. As a consequence of our methods, we also obtain a complete classification, up to isomorphism, of Bruhat intervals of length 7 in type **A** and of length 5 in types **B** and **D**, which are not lattices.

Résumé. On montre que les polynômes de Kazhdan-Lusztig sont invariants combinatoires pour les intervaux de longueur jusqu'à 8 pour les groupes de Coxeter de type **A** et de longueur jusqu'à 6 pour les groupes de Coxeter de type **B** et **D**. Comme conséquence de nos méthodes, on obtient aussi une classification complète, à isomorphisme près, des intervaux de Bruhat de longueur 7 pour le type **A** et de longueur 5 pour les types **B** et **D**, qui ne sont pas des réseaux.

1. Introduction

In [**12**] Kazhdan and Lusztig defined, for every Coxeter group W, a family of polynomials, indexed by pairs of elements of W, which have become known as the Kazhdan-Lusztig polynomials of W. They are related to the algebraic geometry and topology of Schubert varieties, and also play a crucial role in representation theory (see, e.g., [**7**, Chapter 7], [**1**, Chapter 5]). In order to prove the existence of these polynomials, Kazhdan and Lusztig used another family of polynomials which arise from the multiplicative structure of the Hecke algebra associated with W. These are known as the R-polynomials of W. Lusztig's and Dyer's combinatorial invariance conjecture states that the Kazhdan-Lusztig polynomial associated with a pair (x, y) supposedly only depends on the poset structure of the Bruhat interval $[x, y]$. The conjecture is equivalent to the same statement for the R-polynomials and it is known to hold for intervals up to length 4. In [**10**] we proved that the conjecture is true for intervals of length 5 and 6 in Coxeter groups of type **A**.

In this paper, we establish the conjecture for intervals of length 7 and 8 in Coxeter groups of type **A** and for those of length 5 and 6 in Coxeter groups of type **B** and **D**. We use the combinatorial descriptions of such groups in terms of (signed) permutations (see, e.g., [**1**, Chapter 8]). One of the main tools is an extension of the notion of diagram of a pair, introduced for the symmetric group by Kassel et al. in [**11**] and developed in [**9**], to the groups of signed permutations. The main idea behind the proof is that of determining certain subsets of pairs of (signed) permutations, which somehow "summarize" the behaviour of all the pairs. The combinatorial invariance is then proved by enumerating all the pairs in these sets, with the assistance of Maple computation, and for each of them determining the poset structure of the associated interval and computing the corresponding R-polynomial. As a consequence of our methods, we also obtain a complete classification, up to isomorphism, of Bruhat intervals of length 7 in type **A** and of length 5 in types **B** and **D**, which are not lattices (see [**2, 3, 6, 10**] for previous classification results).

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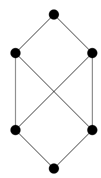
2. Preliminaries

Let $\mathbf{N} = \{1, 2, \dots\}$ and $\mathbf{Z}$ be the set of integers. For $n, m \in \mathbf{Z}$, with $n \leq m$, let $[n, m] = \{n, n+1, \dots, m\}$. For $n \in \mathbf{N}$, let $[n] = [1, n]$, $[-n] = [-n, -1]$ and $[\pm n] = [n] \cup [-n]$. We refer to [**13**] for general poset theory. Given a poset P, we denote by $\lhd$ the covering relation. Given $x, y \in P$, with $x < y$, we set $[x, y] = \{z \in P : x \leq z \leq y\}$, and call it an *interval* of P. We denote by $-P$ the poset *dual* to P, that is, the poset having the same elements of P but the reverse order.

We refer to [**1**] for basic notions about Coxeter groups. Given a Coxeter group W, with set of generators S, the set of *reflections* of W is $T = \{wsw^{-1} : w \in W,\ s \in S\}$. Given $x \in W$, the *length* of x, denoted by $\ell(x)$, is the minimal k such that x is the product of k generators. The *Bruhat graph* of W, denoted by $BG(W)$ is the directed graph having W as vertex set and such that there is an edge $x \to y$ if and only if $y = xt$, with $t \in T$, and $\ell(x) < \ell(y)$. If this happens, we label the edge (x, y) by the reflection t and write $x \xrightarrow{t} y$. A *Bruhat path* is a (directed) path in the Bruhat graph of W. The *Bruhat order* of W is the partial order induced by $BG(W)$: given $x, y \in W$, $x \leq y$ in the Bruhat order if and only if there is a Bruhat path from x to y. Every Coxeter group W, partially ordered by the Bruhat order, is a graded poset with rank function given by the length. For $x, y \in W$, with $x < y$, we set $\ell(x, y) = \ell(y) - \ell(x)$ and call it the *length* of the pair (x, y). In [**9**] we introduced the *absolute length* of the pair (x, y), denoted by $a\ell(x, y)$, which is the (directed) distance from x to y in $BG(W)$. If $\ell(x, y) = 3$, then it is known that $x \xrightarrow{t} y$ if and only if the interval $[x, y]$ is isomorphic to the 2-crown, that is, the poset whose Hasse diagram is the following:



Finally, if W is finite then it has a maximum, denoted by w_0. The maps $x \mapsto x^{-1}$ and $x \mapsto w_0 x w_0$ are automorphisms of the Bruhat order, while the maps $x \mapsto x w_0$ and $x \mapsto w_0 x$ are antiautomorphisms.

We refer to [**1**, §5.2] for basic notions about reflection orderings, which are total orderings on the set T of reflections with certain properties. We only recall that, if W is finite and $s_1 s_2 \dots s_m$ is a reduced decomposition of w_0, then a possible reflection ordering is $t_1 \prec t_2 \prec \cdots \prec t_m$, where $t_i = s_m \dots s_{i+1} s_i s_{i+1} \dots s_m$, for all $i \in [m]$. Moreover, all reflection orderings are obtained in this way (see [**1**, Exercise 5.20]).

We follow [**1**, Chapter 5] for the definition of R-polynomials and Kazhdan-Lusztig polynomials of W. There exists a unique family of polynomials $\{R_{x,y}(q)\}_{x,y \in W} \subseteq \mathbf{Z}[q]$ satisfying the following conditions:

(i) $R_{x,y}(q) = 0$, if $x \not\leq y$;
(ii) $R_{x,y}(q) = 1$, if $x = y$;
(iii) if $x < y$ and $s \in S$ is such that $ys \lhd y$ then

$$R_{x,y}(q) = \begin{cases} R_{xs,ys}(q), & \text{if } \ xs \lhd x, \\ qR_{xs,ys}(q) + (q-1)R_{x,ys}(q), & \text{if } \ xs \rhd x. \end{cases}$$

These are known as the *R-polynomials* of W. The existence of such a family is a consequence of the invertibility of certain basis elements of the Hecke algebra $\mathcal{H}$ of W and is proved in [**7**, §§7.4, 7.5]. Then, there exists a unique family of polynomials $\{P_{x,y}(q)\}_{x,y \in W} \subseteq \mathbf{Z}[q]$ satisfying the following conditions:

(i) $P_{x,y}(q) = 0$, if $x \not\leq y$;
(ii) $P_{x,y}(q) = 1$, if $x = y$;
(iii) if $x < y$ then $\deg(P_{x,y}(q)) < \ell(x, y)/2$ and

$$q^{\ell(x,y)} P_{x,y}\left(q^{-1}\right) - P_{x,y}(q) = \sum_{x < z \leq z} R_{x,z}(q)\, P_{z,y}(q).$$

These are known as the *Kazhdan-Lusztig polynomials* of W. The existence of such a family is proved in [**7**, §§7.9, 7.10, 7.11]. We also need the following property of the R-polynomials (see [**1**, Exercise 5.11]):

$$\sum_{x \leq z \leq y} (-1)^{\ell(x,z)} R_{x,z}(q) R_{z,y}(q) = 0. \tag{1}$$

Finally, there exists a unique family of polynomials $\{\widetilde{R}_{x,y}(q)\}_{x,y\in W} \in \mathbf{Z}_{\geq 0}[q]$ such that

$$R_{x,y}(q) = q^{\ell(x,y)/2}\widetilde{R}_{x,y}(q^{1/2} - q^{-1/2})$$

for all $x, y \in W$. These are known as the *$\widetilde{R}$-polynomials* of W and their coefficients have a nice combinatorial interpretation in terms of reflection orderings. Given $x, y \in W$, with $x < y$, we denote by $BP(x,y)$ the set of all Bruhat paths from x to y. The *length* of $\Delta = (x_0, x_1, \dots, x_k) \in BP(x,y)$, denoted by $|\Delta|$, is the number k of its edges. Let $\prec$ be a fixed reflection ordering on the set T of reflections. A path $\Delta = (x_0, x_1, \dots, x_k) \in BP(x,y)$, with

$$x_0 \xrightarrow{t_1} x_1 \xrightarrow{t_2} \cdots \xrightarrow{t_k} x_k,$$

is said to be *increasing* with respect to $\prec$ if $t_1 \prec t_2 \prec \cdots \prec t_k$. We denote by $BP^{\prec}(x,y)$ the set of all paths in $BP(x,y)$ which are increasing with respect to $\prec$. Then, we have the following (see [**1**, Theorem 5.3.4]):

$$\widetilde{R}_{x,y}(q) = \sum_{\Delta\in BP^{\prec}(x,y)} q^{|\Delta|}. \tag{2}$$

More precisely, set $\ell = \ell(x,y)$ and $a\ell = a\ell(x,y)$, the following holds (see [**4**] and [**9**, Corollary 2.6]):

$$\widetilde{R}_{x,y}(q) = q^{\ell} + c_{\ell-2}\, q^{\ell-2} + \cdots + c_{a\ell+2}\, q^{a\ell+2} + c_{a\ell}\, q^{a\ell}, \tag{3}$$

where $c_k = |\{\Delta \in BP^{\prec}(x,y) : |\Delta| = k\}| \geq 1$, for all $k \in [a\ell, \ell-2]$, with $k \equiv \ell \pmod 2$. Finally, by results in [**3**] and [**5**], we have that the absolute length of a pair is a combinatorial invariant, that is, $a\ell(x,y)$ only depends on the poset structure of the interval $[x,y]$.

We now briefly recall some basic facts about Bruhat order in classical Weyl groups, that is, Coxeter groups of type **A**, **B** and **D**, following [**1**, Chapter 8]. We denote by S_n the *symmetric group* over n elements. To denote a permutation $x \in S_n$ we use the *one-line notation*: we write $x = x_1x_2\dots x_n$ to mean that $x(i) = x_i$ for all $i \in [n]$. The symmetric group S_n is a Coxeter group of type $\mathbf{A}_{n-1}$, with generators given by the simple transpositions $(i, i+1)$, for $i \in [n-1]$. We recall that, given $x \in S_n$, a *free rise* of x is a pair $(i,j) \in \mathbf{N}^2$, with $i < j$ and $x(i) < x(j)$, such that there is no $k \in \mathbf{N}$, with $i < k < j$ and $x(i) < x(k) < x(j)$. Given $x, y \in S_n$, then $x \lhd y$ in the Bruhat order if and only if $y = x(i,j)$, where (i,j) is a free rise of x. Following [**8**], if this happen we write $y = ct_{(i,j)}(x)$ and $x = ict_{(i,j)}(y)$, where ct stands for *covering transformation* and ict for *inverse covering transformation*.

We denote by B_n the *hyperoctahedral group*, defined by

$$B_n = \{x : [\pm n] \to [\pm n] : x \text{ is a bijection, } x(-i) = -x(i) \text{ for all } i \in [n]\}.$$

and call its elements *signed permutations*. To denote a signed permutation $x \in B_n$ we use the *window notation*: we write $x = [x_1, x_2, \dots, x_n]$, to mean that $x(i) = x_i$ for all $i \in [n]$ (the images of the negative entries are then uniquely determined). We also denote x by the sequence $|x_1|\,|x_2|\,\dots\,|x_n|$, with the negative entries underlined. For example, $\underline{3}\,\underline{2}\,1$ denotes the signed permutation $[-3, -2, 1]$. As a set of generators for B_n, we take $S = \{s_0, s_1, \dots, s_{n-1}\}$, where $s_0 = (1, -1)$ and $s_i = (i, i+1)(-i, -i-1)$ for all $i \in [n-1]$. The hyperoctahedral group B_n, with this set of generators, is a Coxeter group of type $\mathbf{B}_n$. Let $x \in B_n$. A rise (i,j) of x is *central* if $(0,0) \in [i,j] \times [x(i), x(j)]$. A central rise (i,j) of x is *symmetric* if $j = -i$. Then, we have the following characterization of the covering relation in the Bruhat order of B_n (see [**8**, Theorem 5.5]). Let $x, y \in B_n$. Then $x \lhd y$ if and only if either (i) $y = x(i,j)(-i,-j)$, where (i,j) is a noncentral free rise of x, or (ii) $y = x(i,j)$, where (i,j) is a central symmetric free rise of x. In both cases we write $y = ct_{(i,j)}(x)$ and $x = ict_{(i,j)}(y)$. The maximum of B_n is $w_0 = \underline{1}\,\underline{2}\,\dots\,\underline{n}$.

We denote by D_n the *even-signed permutation group*, defined by

$$D_n = \{x \in B_n : neg(x) \text{ is even}\}.$$

Notation and terminology are inherited from the hyperoctahedral group. As a set of generators for D_n, we take $S = \{s_0, s_1, \dots, s_{n-1}\}$, where $s_0 = (1, -2)(-1, 2)$ and $s_i = (i, i+1)(-i, -i-1)$ for all $i \in [n-1]$. The even-signed permutation group D_n, with this set of generators, is a Coxeter group of type $\mathbf{D}_n$. Let $x \in D_n$. A central rise (i,j) of x is *semi-free* if $\{k \in [i,j] : x(k) \in [x(i), x(j)]\} = \{i, -j, j\}$. Then, for $x, y \in D_n$, we have (see [**8**, Theorem 6.7]) $x \lhd y$ if and only if $y = x(i,j)(-i,-j)$, where (i,j) is (i) a noncentral free rise of x, or (ii) a central nonsymmetric free rise of x, or (iii) a central semi-free rise of x. In all cases we write $y = ct_{(i,j)}(x)$ and $x = ict_{(i,j)}(y)$. The maximum of D_n is $w_0 = \underline{1}\,\underline{2}\,\dots\,\underline{n}$ if n is even, $1\,\underline{2}\,\dots\,\underline{n}$ if n is odd.

3. Main tools

3.1. Diagram of a pair of (signed) permutations. Let $W \in \{S_n, B_n, D_n\}$. For convenience, we set $\langle n \rangle = [n]$ if $W = S_n$ and $\langle n \rangle = [\pm n]$ if $W \in \{B_n, D_n\}$. The *diagram* of a (signed) permutation $x \in W$ is the subset of $\mathbf{Z}^2$ defined by

$$\mathrm{Diag}(x) = \{(i, x(i)) : i \in \langle n \rangle\}.$$

For $x \in W$ and $(h,k) \in \langle n \rangle^2$, we set

$$x[h,k] = |\{i \in \langle n \rangle : i \leq h,\ x(i) \geq k\}| \tag{4}$$

and given $x, y \in W$ and $(h,k) \in \langle n \rangle^2$, we set

$$(x,y)[h,k] = y(h,k) - x(h,k) \tag{5}$$

There are well-known characterizations of the Bruhat order in S_n and B_n (see [**1**, Theorems 2.1.5, 8.1.8]), which can be stated as follows: if $W \in \{S_n, B_n\}$ and $x, y \in W$ then

$$x \leq y \quad \Leftrightarrow \quad (x,y)[h,k] \geq 0, \quad \text{for all } (h,k) \in \langle n \rangle^2 .$$

See [**1**, Theorem 8.2.8] for a combinatorial characterization of the Bruhat order relation in D_n. Here we only recall that if $x, y \in D_n$ then only one implication is true:

$$x \leq y \quad \Rightarrow \quad (x,y)[h,k] \geq 0, \quad \text{for all } (h,k) \in \langle n \rangle^2 .$$

For our purposes, it is convenient to extend the definitions given in (4) and (5) to every $(h,k) \in \mathbf{R}^2$. We call the mapping $(h,k) \mapsto (x,y)[h,k]$ the *multiplicity mapping* of the pair (x,y). Then, the *diagram* of the pair (x,y) is the collection of: (i) the diagram of x, (ii) the diagram of y and (iii) the multiplicity mapping of (x,y). From the preceding considerations, if $x \leq y$, then the values of this mapping are always non-negative. In this case, we pictorially represent the diagram of a pair (x,y) with the following convention: the diagrams of x and y are denoted by black and white dots, respectively, and the mapping $(h,k) \mapsto (x,y)[h,k]$ is represented by colouring the preimages of different positive integers with different levels of grey, with the rule that a lighter grey corresponds to a lower integer. Examples for the symmetric group can be found in [**9**]. In Figure 1, the diagram of (x,y), where $x = 2341$ and $y = 3\underline{4}\underline{2}\underline{1} \in B_4$, is illustrated. Note that, although $x, y \in D_4$, we have $x \not\leq y$ in D_4, since condition (ii) of [**1**, Theorem 8.2.8] fails for $(a,b) = (2,1)$. Figure 2 shows the diagram of (x,y), where $x = 1342$ and $y = 3\underline{4}1\underline{2} \in D_4$. Now, $x \leq y$ in D_4.

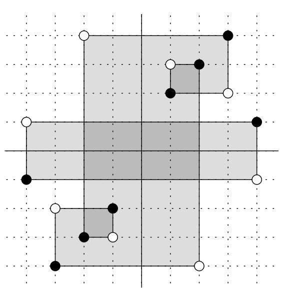


Figure 1: Diagram of a pair in B_n.

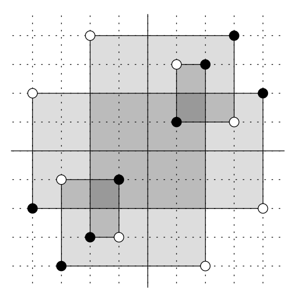


Figure 2: Diagram of a pair in D_n.

The *support* of (x,y) is

$$\Omega(x,y) = \{(h,k) \in \mathbf{R}^2 : (x,y)[h,k] > 0\}$$

and the *support index set* of (x,y) is

$$I_\Omega(x,y) = \{i \in \langle n \rangle : (i, x(i)) \in \overline{\Omega(x,y)}\},$$

where $\overline{\Omega(x,y)}$ denotes the (topological) closure of the set $\Omega(x,y)$. A pair $(x,y) \in W^2$, with $x < y$, is said to have *full support* if $I_\Omega(x,y) = \langle n \rangle$. For instance, both the pairs in Figures 1 and 2 have full support.

3.2. Computing $\widetilde{R}$-polynomials. In [**9**] we described an algorithm for computing $\widetilde{R}$-polynomials in the symmetric group. Following a similar strategy, $\widetilde{R}$-polynomials can be efficiently computed in the groups of signed permutations starting from equation (2), by choosing convenient reflection orderings.

We recall that, if we set $T_1 = \{(i,j)(-i,-j) : i \in [-n], j \in [\pm(-i-1)]\}$ and $T_2 = \{(i,-i) : i \in [-n]\}$, then, the set of reflections in D_n is T_1 (see, e.g., [**1**, Prop. 8.1.5]) and the set of reflections in B_n is $T_1 \cup T_2$ (see, e.g., [**1**, Prop. 8.2.5]). In both B_n and D_n we identify the reflection $(i,j)(-i,-j) \in T_1$, where $i \in [-n]$ and $j \in [\pm(-i-1)]$, with the pair (i,j). Then, we have the following.

Proposition 3.1. *A possible reflection ordering in D_n is the lexicographic order between pairs. And a possible reflection ordering in B_n is the same as in D_n, with the reflection $(i,-i)$ inserted between $(i,-1)$ and $(i,1)$, for all $i \in [-n,-2]$, and $(-1,1)$ inserted as the last one.*

Proof. They arise from appropriate choices of a reduce decomposition of the maximum element w_0. □

For example, a reflection ordering in D_4 is

$$\begin{gathered}(-4,-3) \prec (-4,-2) \prec (-4,-1) \prec (-4,1) \prec (-4,2) \prec (-4,3) \prec \\ (-3,-2) \prec (-3,-1) \prec (-3,1) \prec (-3,2) \prec \\ (-2,-1) \prec (-2,1),\end{gathered}$$

and a reflection odering in B_4 is

$$\begin{gathered}(-4,-3) \prec (-4,-2) \prec (-4,-1) \prec (-4,4) \prec (-4,1) \prec (-4,2) \prec (-4,3) \prec \\ (-3,-2) \prec (-3,-1) \prec (-3,3) \prec (-3,1) \prec (-3,2) \prec \\ (-2,-1) \prec (-2,2) \prec (-2,1) \prec \\ (-1,1).\end{gathered}$$

3.3. Symmetries. Let W be any finite Coxeter group and let w_0 be its maximum. We define the following equivalence relations between pairs $(x,y) \in W^2$, with $x < y$:

$$\begin{aligned}(x_1,y_1) \sim^+ (x,y) \quad &\Leftrightarrow \quad (x_1,y_1) \in \{(x,y),(x^{-1},y^{-1}),(w_0xw_0,w_0yw_0),(w_0x^{-1}w_0,w_0y^{-1}w_0)\} \\ (x_1,y_1) \sim^- (x,y) \quad &\Leftrightarrow \quad (x_1,y_1) \in \{(yw_0,xw_0),(w_0y,w_0x),(y^{-1}w_0,x^{-1}w_0),(w_0y^{-1},w_0x^{-1})\} \\ (x_1,y_1) \sim (x,y) \quad &\Leftrightarrow \quad (x_1,y_1) \sim^+ (x,y) \quad \text{or} \quad (x_1,y_1) \sim^- (x,y)\end{aligned}$$

Then, it is known that

$$\begin{aligned}(x_1,y_1) \sim^+ (x,y) \quad &\Rightarrow \quad [x_1,y_1] \cong [x,y] \\ (x_1,y_1) \sim^- (x,y) \quad &\Rightarrow \quad [x_1,y_1] \cong -[x,y]\end{aligned}$$

Moreover (see, e.g., [**1**, Exercise 4.10]) we have

$$(x_1,y_1) \sim (x,y) \quad \Rightarrow \quad \widetilde{R}_{x_1,y_1}(q) = \widetilde{R}_{x,y}(q).$$

In classical Weyl groups, if $(x_1,y_1) \sim (x,y)$ then the diagram of (x_1,y_1) is obtained from that of (x,y) by a certain reflection, as described for the symmetric group in [**10**, Figure 2]. The only exception is the case $W = D_n$ and n odd when, for example, $xw_0 = [x(1), -x(2), \ldots, -x(n)]$. Then, in order to generate all possible intervals and $\widetilde{R}$-polynomials, we will consider diagrams up to these symmetries.

3.4. Odd signed permutation poset. In the remainder of the paper we will act on diagrams by "deleting" or "inserting" dots. In the groups of type **D**, this would not always be allowed, because of the restriction on the parity of the number of negative entries. In this subsection we present a way of bypassing this problem. We start with defining the *odd-signed permutation set*:

$$D_n^{\text{odd}} = \{x \in B_n : neg(x) \text{ is odd}\}.$$

Although D_n^{odd} is not a group, we can still define on it the Bruhat order as in D_n, giving the same characterization of the covering relation (see the end of Section 2). More precisely, given $x,y \in D_n^{\text{odd}}$, we say that $x \lhd y$ in the *Bruhat order* if and only if $y = x(i,j)(-i,-j)$, where (i,j) is (i) a noncentral free rise of x, or (ii) a central nonsymmetric free rise of x, or (iii) a central semifree rise of x. Then, we have the following.

Proposition 3.2. *The map $\varphi : D_n \to D_n^{\text{odd}}$ defined by*

$$[x_1, x_2, \ldots, x_n] \overset{\varphi}{\longmapsto} [-x_1, x_2, \ldots, x_n]$$

is an isomorphism of posets.

By Proposition 3.2, whose proof is omitted, working with the posets D_n and D_n^{odd} is essentially the same thing. From now on, we will denote the even-signed permutation group by D_n^{even} and we will write $x, y \in D_n$ to mean either $x, y \in D_n^{\text{even}}$ or $x, y \in D_n^{\text{odd}}$, with the only requirement that $neg(x) \equiv neg(y) \pmod 2$.

3.5. Simplifications. Let $W \in \{S_n, B_n, D_n\}$. An *index set* is a subset $I \subseteq \langle n \rangle$, such that $I = -I$ if $W \in \{B_n, D_n\}$. Let $x \in W$ and I be an index set. We denote by $x\,|_I$ the (signed) permutation whose diagram is obtained from that of x, by considering only the dots corresponding to the indices in I, removing the others, and renumbering the remaining indices and values. We call $x\,|_I$ the *subpermutation* of x induced by I. We start with noting that all the information about the poset structure of $[x, y]$ and about the $\widetilde{R}$-polynomial associated is contained in the support of (x, y).

Proposition 3.3. *Let $x, y \in W$, with $x < y$. Set $x_\Omega = x\,\big|_{I_\Omega(x,y)}$ and $y_\Omega = y\,\big|_{I_\Omega(x,y)}$. Then*

(i) $[x, y] \cong [x_\Omega, y_\Omega]$;

(ii) $\widetilde{R}_{x_\Omega, y_\Omega}(q) = \widetilde{R}_{x,y}(q)$.

Proof. For the symmetric group, it has been proved in [**9**, Proposition 5.2] and [**10**, Proposition 3.1]. For the groups B_n and D_n, the characterization of the covering relation in terms of rises ensures that the interval $[x, y]$ reflects a process of "unmounting" the diagram of (x, y) similar to that described in [**9**] for the symmetric group and (i) follows. A similar consideration together with equation (2) implies (ii). □

It is useful to introduce the following notion of Ω-*equivalence* between pairs:

$$(x', y') \sim_\Omega (x, y) \quad \Leftrightarrow \quad (x'_\Omega, y'_\Omega) = (x_\Omega, y_\Omega).$$

According to Proposition 3.3, the same interval (up to poset isomorphism) and the same $\widetilde{R}$-polinomial are associated with all the pairs in an Ω-equivalence class.

Now, let $x \in W$ and I be an index set. For $(h, k) \in \mathbf{R}^2$, we set

$$x[h, k]\,|_I = |\{i \in I : i \leq h,\, x(i) \geq k\}|,$$

Let $x, y \in W$ and I be an index set such that $x(I) = y(I)$. For $(h, k) \in \mathbf{R}^2$, we set

$$(x, y)[h, k]\,|_I = y(h, k)\,|_I - x(h, k)\,|_I\,.$$

Then, we set

$$\Omega(x, y)\,|_I = \{(h, k) \in \mathbf{R}^2 : (x, y)[h, k]\,|_I > 0\}$$

Definition 3.4. Let $x, y \in W$, with $x < y$. Let I_1 and I_2 be two index sets, with $I_\Omega(x, y) = I_1 \cup I_2$ and $I_1 \cap I_2 = \emptyset$, such that $x(I_1) = y(I_1)$ and $x(I_2) = y(I_2)$. Set $x_r = x\,|_{I_r}$, $y_r = y\,|_{I_r}$ and $\Omega_r = \Omega(x, y)\,|_{I_r}$, for $r = 1, 2$. Note that, necessarily, $x_1 < y_1$ and $x_2 < y_2$. We say that the pair (x, y) is *trivially decomposable* into the two pairs (x_1, y_1) and (x_2, y_2) if Ω_1 and Ω_2 are either disjoint or if they intersect in a region whose closure does not contain any of the dots of the diagrams of x and y.

For example, the pair $(x, y) \in B_4^2$, whose diagram is shown in Figure 1, is trivially decomposable into the two pairs $(123, 2\underline{3}1) \in B_3^2$ and $(1, \underline{1}) \in B_1^2$. We have the following general result.

Proposition 3.5. *Let $x, y \in W$, with $x < y$, be trivially decomposable into (x_1, y_1) and (x_2, y_2). Then*

(i) $[x, y] \cong [x_1, y_1] \times [x_2, y_2]$;

(ii) $\widetilde{R}_{x,y}(q) = \widetilde{R}_{x_1,y_1}(q) \cdot \widetilde{R}_{x_2,y_2}(q)$.

Proof. For S_n, it has been proved in [**9**, Proposition 2.16] and [**10**, Propositions 3.2, 3.4, 3.5]. For B_n and D_n the proof is similar, since under the hypotheses of the proposition, the process of "unmounting" the diagram of (x, y), that the interval $[x, y]$ reflects, is completely independent for Ω_1 and Ω_2 and (i) follows. A similar consideration together with equation (2) implies (ii). □

3.6. Enlarging an interval. In this subsection we show how it is possible, given an interval $[x, y]$, to obtain all intervals of length $\ell(x, y) + 1$ containing $[x, y]$ as subinterval, in terms of the diagram of (x, y). We start with introducing a notion of "insertion" of a dot in a diagram.

DEFINITION 3.6. Let $x \in S_n$ and $h, k \in [n+1]$. The permutation obtained from x by *inserting the dot* (h, k), denoted by $x^{(h,k)}$, is the only permutation $\widehat{x} \in S_{n+1}$ satisfying (i) $\widehat{x}(h) = k$, (ii) $\widehat{x}\,\big|_{[n+1]\setminus\{h\}} = x$.

Similarly, for $x \in B_n$ (resp. D_n), $h \in [n+1]$ and $k \in [\pm(n+1)]$, the signed permutation obtained from x by *inserting the dot* (h, k), denoted by $x^{(h,k)}$, is the only permutation $\widehat{x} \in B_{n+1}$ (resp. D_{n+1} or D^{odd}_{n+1}, depending on whether $k > 0$ or $k < 0$) satisfying (i) $\widehat{x}(h) = k$ (thus $\widehat{x}(-h) = -k$), (ii) $\widehat{x}\,\big|_{[\pm(n+1)]\setminus\{h,-h\}} = x$.

If we consider the pairs (x, y) whose diagrams are shown in Figures 1 and 2, then the diagrams of the pairs $\big(x^{(3,-3)}, y^{(3,-3)}\big)$ are illustrated in Figures 3 and 4, respectively. Note that the signed permutations $x^{(3,-3)}, y^{(3,-3)}$ shown in Figure 4 belong to D^{odd}_5 and, according to the considerations following Proposition 3.2, we still write $\big(x^{(3,-3)}, y^{(3,-3)}\big) \in D_5$.

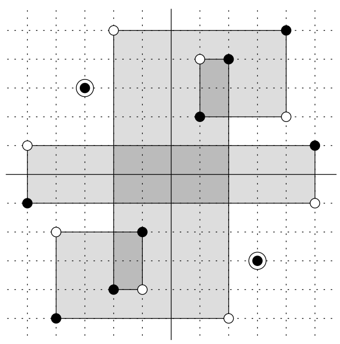


Figure 3: Inserting a dot in B_n.

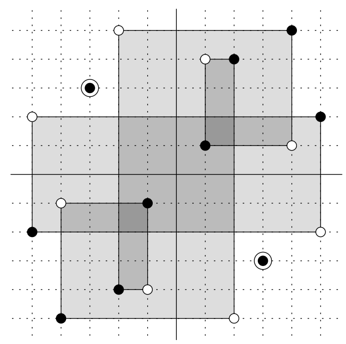


Figure 4: Inserting a dot in D_n.

In particular, we are interested in inserting dots out of the support, as it happens in the diagrams in Figures 3 and 4. In this case we obtain a pair which is in the same Ω-equivalence class as the originary pair. Then, we have the following result, which is an immediate consequence of Proposition 3.3.

COROLLARY 3.7. *Let $x, y \in W$, with $x < y$, and $(h, k) \in [n+1] \times \langle n+1\rangle$, with $(h, k) \notin \overline{\Omega(x^{(h,k)}, y^{(h,k)})}$.*

(i) $[x, y] \cong \big[x^{(h,k)}, y^{(h,k)}\big]$;

(ii) $\widetilde{R}_{x,y}(q) = \widetilde{R}_{x^{(h,k)},y^{(h,k)}}(q)$.

Let $x, y \in W$, with $x < y$. The intervals of length $\ell(x, y) + 1$ containing $[x, y]$ are exactly those of the form $[x, z]$ (if $y \neq w_0$), with $z = ct_{(i,j)}(y)$ and those of the form $[w, y]$ (if $x \neq id$), with $w = ict_{(i,j)}(x)$. In both cases we say that the new pair, (x, z) or (w, y), is *obtained* from (x, y) by

(i) an *external move*, if $\{i, j\} \subseteq \langle n\rangle \setminus I_\Omega(x, y)$;
(ii) an *internal move*, if $\{i, j\} \subseteq I_\Omega(x, y)$;
(iii) an *enlarging move*, if $|\{i, j\} \cap I_\Omega(x, y)| = 1$.

In case (iii), if $\{i, j\}\setminus I_\Omega(x, y) = \{h\}$, then we also say that the enlarging move *uses* the dot $(h, x(h))$. Also, if (x, y) is a pair with full support, (x', y') is any pair Ω-equivalent to (x, y) and (w, z) is obtained from (x', y') by one of the three kinds of moves described, then we say that (w, z) is *obtained* from (x, y) as well.

External moves can be easily managed by the following result.

PROPOSITION 3.8. *Let $x, y \in W$, with $x < y$. Let (w, z) be obtained from (x, y) by an external move and suppose both (x, y) and (w, z) have full support. Then (w, z) is trivially decomposable into (x, y) and a pair (a, b), with $a \lhd b$. In particular* (i) $[w, z] = [x, y] \times \{0, 1\}$; (ii) $\widetilde{R}_{w,z}(q) = q\widetilde{R}_{x,y}(q)$.

We need one last definition.

DEFINITION 3.9. Let $W \in \{S_n, B_n, D_n\}$ and $x, y \in W$, with $x < y$, be such that (x, y) has full support. The *enlarging set* of (x, y), denoted by $\mathrm{Enl}(x, y)$, is the union of all the pairs with full support obtained from (x, y) by internal moves and enlarging moves.

4. Main result

The combinatorial invariance of Kazhdan-Lusztig polynomials for intervals up to a certain length is equivalent to that of the R-polynomials (or their counterpart, the $\widetilde{R}$-polynomials) for the same intervals. We will prove our main result by showing that the $\widetilde{R}$-polynomials are combinatorial invariants. First of all, note that an interval $[x, y]$ does not contain a 2-crown if and only if $a\ell(x, y) = \ell(x, y)$ and, by equation (3), this happens if and only if $\widetilde{R}_{x,y}(q) = q^{\ell(x,y)}$. Thus, we only need to consider intervals containing 2-crowns.

Let $\mathcal{F}_A = \{S_n : n \geq 2\}$, $\mathcal{F}_B = \{B_n : n \geq 1\}$ and $\mathcal{F}_D = \{D_n : n \geq 2\}$.

Definition 4.1. Let $\mathcal{F} \in \{\mathcal{F}_A, \mathcal{F}_B, \mathcal{F}_D\}$. The *essential sets* of $\mathcal{F}$ are recursively defined by

$$ES_3(\mathcal{F}) = \{(x, y) \in W^2 : W \in \mathcal{F},\ [x, y] \text{ is a 2-crown and } (x, y) \text{ has full support}\}/\sim$$

and, for $k \geq 4$

$$ES_k(\mathcal{F}) = \Big[\bigcup \{Enl(x, y) : [(x, y)]_\sim \in ES_{k-1}(\mathcal{F})\}\Big] / \sim .$$

The sets $ES_3(\mathcal{F})$ can be easily determined and they are as follows:

$$ES_3(\mathcal{F}_A) = \{(123, 321)\},$$
$$ES_3(\mathcal{F}_B) = \{(123, 321), (\underline{2}\,\underline{1}3, 3\underline{1}\,\underline{2}), (\underline{3}\,\underline{1}2, 2\underline{1}\,\underline{3}), (\underline{3}\,\underline{2}1, 1\underline{2}\,\underline{3}), (12, 1\underline{2}), (12, \underline{2}\,\underline{1})\},$$
$$ES_3(\mathcal{F}_D) = \{(123, 321), (\underline{2}\,\underline{1}3, 3\underline{1}\,\underline{2}), (\underline{3}\,\underline{1}2, 2\underline{1}\,\underline{3}), (\underline{3}\,\underline{2}1, 1\underline{2}\,\underline{3}), (123, \underline{32}\,\underline{1}), (213, \underline{3}1\,\underline{2}), (312, \underline{21}\,\underline{3})\},$$

where, for simplicity, we have identified every equivalence class with one of its elements.

theorem 4.2. *Let $\mathcal{F} \in \{\mathcal{F}_A, \mathcal{F}_B, \mathcal{F}_D\}$ and $k \geq 3$. The essential set $ES_k(\mathcal{F})$ contains, up to $\sim$, all possible pairs of length k, in Coxeter groups in $\mathcal{F}$, which have full support and are not trivially decomposable, whose corresponding interval $[x, y]$ contains a 2-crown.*

Proof. We proceed by induction on k. For $k = 3$, the result is true by definition. Assume $k \geq 4$. It is easy to prove that all the pairs in $ES_k(\mathcal{F})$ have the required properties. Now, let (x, y) be a pair of length k which has full support and is not trivially decomposable, such that $[x, y]$ contains a 2-crown. We want to show that $[(x, y)]_\sim \in ES_k(\mathcal{F})$. As one can easily check, it is always possible to find an atom z (or a coatom w) of $[x, y]$ such that (z, y) (or (x, w)) is still not trivially decomposable and $[z, y]$ (or $[x, w]$) still contains a 2-crown. Let z be an atom of $[x, y]$ with this properties (the case of a coatom w is similar). Now, let $z_\Omega = z\big|_{I_\Omega(z,y)}$ and $y_\Omega = y\big|_{I_\Omega(z,y)}$. Then, (z_Ω, y_Ω) has length $k - 1$, has full support, is not trivially decomposable and $[z_\Omega, y_\Omega]$ contains a 2-crown. By the induction hypotesis, this implies $[(z_\Omega, y_\Omega)]_\sim \in ES_{k-1}(\mathcal{F})$. Also note that (x, y) is necessarily obtained from (z_Ω, y_Ω) by either an internal move or an enlarging move. Thus, by definition, $(x, y) \in Enl(z_\Omega, y_\Omega)$ and $[(x, y)]_\sim \in ES_k(\mathcal{F})$. □

We can now state and prove the main result of this work.

theorem 4.3. *The Kazhdan-Lusztig polynomials are combinatorial invariants for intervals up to length* 6 *in Coxeter groups of type* **B** *and* **D** *and for intervals up to length* 8 *in Coxeter groups of type* **A**.

Proof. The combinatorial invariance is known to hold for intervals up to length 4 in all Coxeter groups and in [**10**] it has been established for intervals of length 5 and 6 in the symmetric group. Moreover, by equation (1), if the combinatorial invariance is true for intervals up to a given odd length ℓ, then it is also true for intervals of length $\ell + 1$. So we only need to prove the combinatorial invariance of the $\widetilde{R}$-polynomials for intervals of length 5 in the groups of signed permutations and for those of length 7 in the symmetric group. As already observed, we only need to consider intervals containing 2-crowns. By Proposition 3.3, we may only consider pairs which have full support. Pairs which are trivially decomposable can be managed by Proposition 3.5. Then, by Theorem 4.2, we only need to consider the pairs in the sets $ES_k(\mathcal{F})$.

For the remainder of the proof, we need the assistance of Maple computation. In fact, the essential sets have been generated, according to Definition 4.1, by a Maple program. For the symmetric group it has been done up to length 7 and for the groups of signed permutations up to length 5. The cardinalities of the essential sets are as follows:

k	$\lvert ES_k(\mathcal{F}_A)\rvert$	$\lvert ES_k(\mathcal{F}_B)\rvert$	$\lvert ES_k(\mathcal{F}_D)\rvert$
3	1	6	7
4	4	209	158
5	47	9543	3942
6	913		
7	22400		

For each pair (x, y) in the essential sets $ES_7(\mathcal{F}_A)$, $ES_5(\mathcal{F}_B)$ and $ES_5(\mathcal{F}_D)$, the poset structure of the interval $[x, y]$ has been determined, and the corresponding $\widetilde{R}$-polynomial has been computed, by our own Maple programs, based on algorithms that use the characterizations of the Bruhat order, equation (2) and the reflection orderings mentioned in the previous section. Then, the pairs have been grouped in isomorphism classes, with the help of Stembridge's Maple package for posets [**14**], which includes a fast algorithm for isomorphism testing. Finally, the combinatorial invariance of the $\widetilde{R}$-polynomials for these pairs has been checked. The results of the computation are summarized in Tables 1, 2 and 3, described later.

Note that it may happen that a pair (x, y), which is not trivially decomposable, has a corresponding interval which is reducible as a poset (that is, direct product of smaller posets), say $[x, y] \cong [x, z] \times [z, y]$. Then, consistently with Proposition 3.5, it has to be proved that, whenever this happens, the factorization $\widetilde{R}_{x,y}(q) = \widetilde{R}_{x,z}(q) \cdot \widetilde{R}_{z,y}$ holds. This has also been checked by Maple computation. □

In Tables 1, 2 and 3 (the last one in a short version) all isomorphism types of intervals associated with pairs in the essential sets $ES_5(\mathcal{F}_D)$, $ES_5(\mathcal{F}_B)$ and $ES_7(\mathcal{F}_A)$, respectively, are listed. They are grouped by the value of the $\widetilde{R}$-polynomial and, within each group, they are listed for lexicographically nondecreasing f-vector. For each isomorphism type a representative pair (x, y) is indicated. Self-dual intervals and reducible intervals are marked, and, for each group, the expression of the R-polynomial is also indicated.

Note that some of the reducible intervals associated with trivially decomposable pairs might not have been considered. Nevertheless, this is not the case, since, by Maple computation, it has also been checked that all possible intervals containing 2-crowns that are direct product of smaller intervals belong to one of the isomorphism classes listed in the tables. Moreover, by an unpublished result of Dyer, we have that a Bruhat interval is a lattice if and only if it does not contain a 2-crown. We can conclude that Tables 1, 2 and 3 contain a complete classification, up to isomorphism, of Bruhat intervals which are not lattices, for the respective lengths and types.

The diagrams of the representative pairs are finally depicted in Figures 5, 6 and 7.

type	(x, y)	f-vector	s.d.	red.	$\widetilde{R}_{x,y}(q)$	$R_{x,y}(q)$
1.	$(1\,2\,3,\ 1\,\underline{3}\,\underline{2})$	$(3,5,6,4)$			q^5+2q^3+q	$(q-1)(q^2-q+1)^2$
2.	$(1\,2\,3,\ \underline{2}\,1\,\underline{3})$	$(3,5,5,3)$	√		q^5+2q^3	$(q-1)^3(q^2+1)$
3.	$(1\,2\,3\,4,\ \underline{2}\,\underline{4}\,1\,3)$	$(4,7,7,4)$	√	√		
4.	$(1\,2\,3\,4,\ \underline{1}\,2\,\underline{4}\,3)$	$(4,8,9,5)$		√		
5.	$(1\,2\,3\,4,\ \underline{1}\,4\,\underline{3}\,2)$	$(4,9,10,5)$				
6.	$(2\,1\,3\,4,\ 4\,\underline{3}\,\underline{2}\,1)$	$(4,10,12,6)$				
7.	$(\underline{1}\,\underline{2}\,3\,4,\ 3\,\underline{4}\,\underline{2}\,1)$	$(5,10,10,5)$			q^5+q^3	$(q-1)^3(q^2-q+1)$
8.	$(\underline{1}\,\underline{2}\,3\,4,\ 1\,\underline{4}\,\underline{3}\,2)$	$(5,10,11,6)$				
9.	$(\underline{1}\,2\,\underline{3}\,4,\ 1\,2\,\underline{4}\,\underline{3})$	$(5,11,14,8)$				
10.	$(\underline{1}\,\underline{2}\,3\,4,\ 1\,\underline{4}\,3\,\underline{2})$	$(5,12,13,6)$				
11.	$(\underline{1}\,\underline{3}\,2\,4,\ 1\,\underline{4}\,2\,\underline{3})$	$(5,12,14,7)$				
12.	$(\underline{1}\,4\,\underline{2}\,3,\ 1\,2\,\underline{4}\,\underline{3})$	$(7,15,16,8)$				

TABLE 1. Isomorphism types of pairs in $ES_5(\mathcal{F}_D)$.

type	(x,y)	f-vector	s.d.	red.	$\widetilde{R}_{x,y}(q)$	$R_{x,y}(q)$
1.	$(1\,2\,3, \underline{3}\,2\,\underline{1})$	$(3,5,6,4)$				
2.	$(\underline{1}\,2\,3, \underline{1}\,2\,\underline{3})$	$(3,6,7,4)$			q^5+2q^3+q	$(q-1)(q^2-q+1)^2$
3.	$(\underline{2}\,1\,3, \underline{2}\,1\,\underline{3})$	$(4,7,7,4)$	$\surd$			
4.	$(\underline{3}\,1\,2, \underline{1}\,\underline{3}\,\underline{2})$	$(3,4,4,3)$	$\surd$	$\surd$		
5.	$(1\,2\,3, 2\,\underline{3}\,1)$	$(3,5,5,3)$	$\surd$			
6.	$(1\,2\,3, \underline{3}\,\underline{2}\,1)$	$(3,5,6,4)$				
7.	$(1\,3\,2, \underline{1}\,2\,\underline{3})$	$(3,5,6,4)$			q^5+2q^3	$(q-1)^3(q^2+1)$
8.	$(\underline{1}\,2\,3, \underline{3}\,\underline{2}\,\underline{1})$	$(3,6,6,3)$	$\surd$			
9.	$(\underline{1}\,2\,3, 2\,\underline{3}\,\underline{1})$	$(3,6,7,4)$				
10.	$(\underline{1}\,3\,2, 1\,\underline{3}\,\underline{2})$	$(4,7,7,4)$	$\surd$			
11.	$(\underline{2}\,1\,3, 1\,\underline{3}\,\underline{2})$	$(4,7,7,4)$				
12.	$(\underline{2}\,1\,3, 2\,\underline{1}\,\underline{3})$	$(4,7,7,4)$	$\surd$	$\surd$		
13.	$(1\,2\,3\,4, 4\,\underline{2}\,1\,3)$	$(4,8,9,5)$		$\surd$		
14.	$(\underline{1}\,2\,3\,4, \underline{3}\,4\,\underline{1}\,2)$	$(4,9,10,5)$				
15.	$(\underline{1}\,2\,4\,3, \underline{4}\,2\,3\,\underline{1})$	$(4,10,12,6)$				
16.	$(2\,1\,3\,4, \underline{4}\,\underline{2}\,1\,3)$	$(5,10,10,5)$	$\surd$	$\surd$		
17.	$(2\,1\,3\,4, 4\,\underline{2}\,3\,1)$	$(5,10,10,5)$				
18.	$(1\,\underline{2}\,3\,4, 1\,\underline{4}\,\underline{2}\,3)$	$(5,10,10,5)$		$\surd$		
19.	$(2\,1\,3\,4, 4\,\underline{3}\,1\,2)$	$(5,10,11,6)$				
20.	$(1\,\underline{3}\,2\,4, 1\,\underline{4}\,\underline{3}\,2)$	$(5,10,11,6)$				
21.	$(3\,1\,2\,4, 4\,\underline{3}\,\underline{1}\,2)$	$(5,11,12,6)$				
22.	$(3\,1\,2\,4, \underline{4}\,\underline{3}\,1\,2)$	$(5,11,12,6)$				
23.	$(\underline{4}\,2\,3\,\underline{1}, \underline{3}\,2\,\underline{1}\,\underline{4})$	$(5,11,12,6)$				
24.	$(\underline{2}\,1\,3\,4, \underline{3}\,4\,\underline{2}\,1)$	$(5,11,12,6)$				
25.	$(1\,\underline{3}\,\underline{2}\,4, 1\,\underline{4}\,\underline{3}\,\underline{2})$	$(5,11,12,6)$				
26.	$(1\,\underline{2}\,3\,4, 4\,\underline{3}\,\underline{2}\,1)$	$(5,11,13,7)$				
27.	$(\underline{1}\,2\,\underline{3}\,4, 1\,2\,\underline{4}\,\underline{3})$	$(5,11,14,8)$				
28.	$(\underline{1}\,\underline{2}\,3\,4, 1\,\underline{4}\,3\,\underline{2})$	$(5,12,13,6)$				
29.	$(\underline{4}\,\underline{1}\,3\,2, \underline{3}\,\underline{1}\,2\,\underline{4})$	$(5,12,13,6)$			q^5+q^3	$(q-1)^3(q^2-q+1)$
30.	$(\underline{2}\,4\,\underline{1}\,3, \underline{2}\,3\,\underline{4}\,\underline{1})$	$(5,12,13,6)$				
31.	$(2\,4\,\underline{1}\,3, 2\,3\,\underline{4}\,\underline{1})$	$(5,12,14,7)$				
32.	$(\underline{1}\,\underline{3}\,2\,4, 1\,\underline{4}\,2\,\underline{3})$	$(5,12,14,7)$				
33.	$(3\,\underline{1}\,2\,4, 4\,\underline{3}\,2\,\underline{1})$	$(6,12,12,6)$	$\surd$			
34.	$(\underline{1}\,3\,2\,4, 4\,\underline{3}\,\underline{1}\,2)$	$(6,12,12,6)$				
35.	$(\underline{2}\,4\,1\,3, 3\,\underline{4}\,\underline{2}\,1)$	$(6,13,13,6)$	$\surd$			
36.	$(1\,\underline{2}\,4\,3, 3\,\underline{4}\,\underline{2}\,1)$	$(6,13,13,6)$				
37.	$(\underline{4}\,3\,2\,1, \underline{3}\,2\,\underline{1}\,\underline{4})$	$(6,13,13,6)$				
38.	$(2\,\underline{3}\,\underline{1}\,4, 2\,\underline{4}\,\underline{3}\,\underline{1})$	$(6,13,13,6)$				
39.	$(\underline{2}\,1\,4\,3, \underline{4}\,1\,3\,\underline{2})$	$(6,13,13,6)$				
40.	$(3\,\underline{2}\,1\,4, 3\,\underline{4}\,\underline{2}\,1)$	$(6,13,13,6)$				
41.	$(4\,3\,\underline{1}\,2, 1\,2\,\underline{4}\,\underline{3})$	$(6,13,14,7)$				
42.	$(1\,4\,\underline{2}\,3, 1\,3\,\underline{4}\,\underline{2})$	$(6,13,14,7)$				
43.	$(3\,\underline{1}\,2\,4, 4\,\underline{3}\,\underline{2}\,1)$	$(6,13,14,7)$				
44.	$(3\,2\,1\,4, 4\,\underline{3}\,\underline{2}\,1)$	$(6,14,15,7)$				
45.	$(\underline{1}\,4\,\underline{3}\,2, \underline{1}\,2\,\underline{4}\,\underline{3})$	$(6,14,15,7)$				
46.	$(\underline{1}\,4\,\underline{2}\,3, 1\,2\,\underline{4}\,\underline{3})$	$(7,15,16,8)$				

Table 2. Isomoprhism types of pairs in $ES_5(\mathcal{F}_B)$.

types	$\widetilde{R}_{x,y}(q)$	$R_{x,y}(q)$
1–2	$q^7+3q^5+3q^3+q$	$(q-1)(q^2-q+1)^3$
3–5	$q^7+3q^5+2q^3$	$(q-1)^3(q^2+1)(q^2-q+1)$
6–11	$q^7+3q^5+q^3$	$(q-1)^3(q^4-q^3+q^2-q+1)$
12–57	$q^7+2q^5+q^3$	$(q-1)^3(q^2-q+1)^2$
58–89	q^7+2q^5	$(q-1)^5(q^2+1)$
90–217	q^7+q^5	$(q-1)^5(q^2-q+1)$

TABLE 3. Isomorphism types of pairs in $ES_7(\mathcal{F}_A)$.

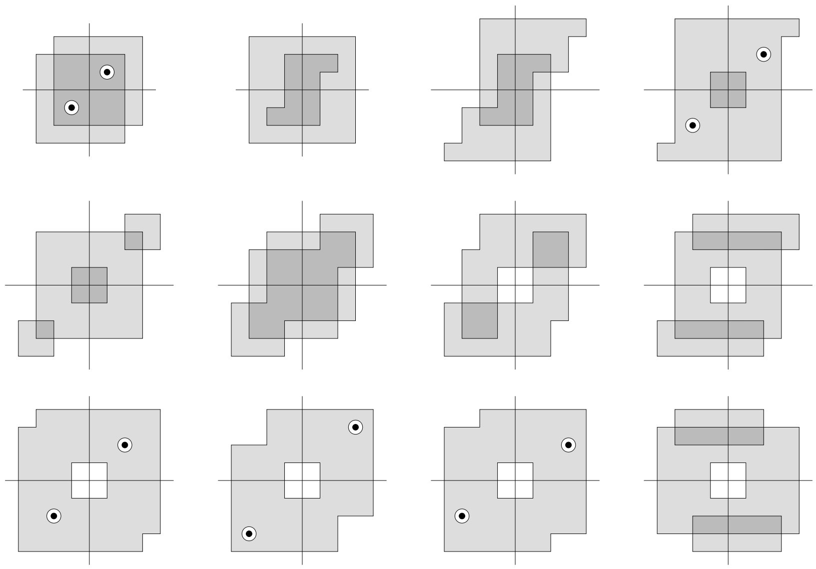


FIGURE 5. Representative diagrams of pairs in $ES_5(\mathcal{F}_D)$.

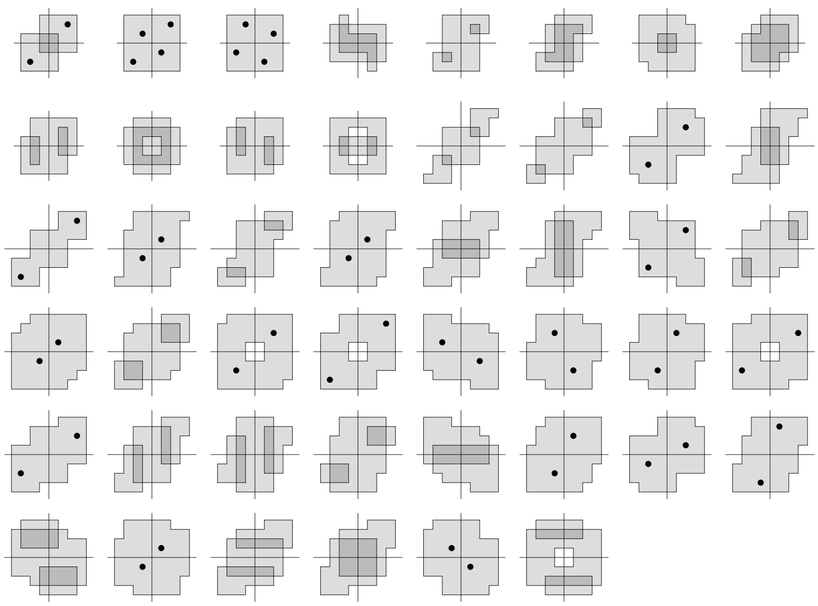


FIGURE 6. Representative diagrams of pairs in $ES_5(\mathcal{F}_B)$.

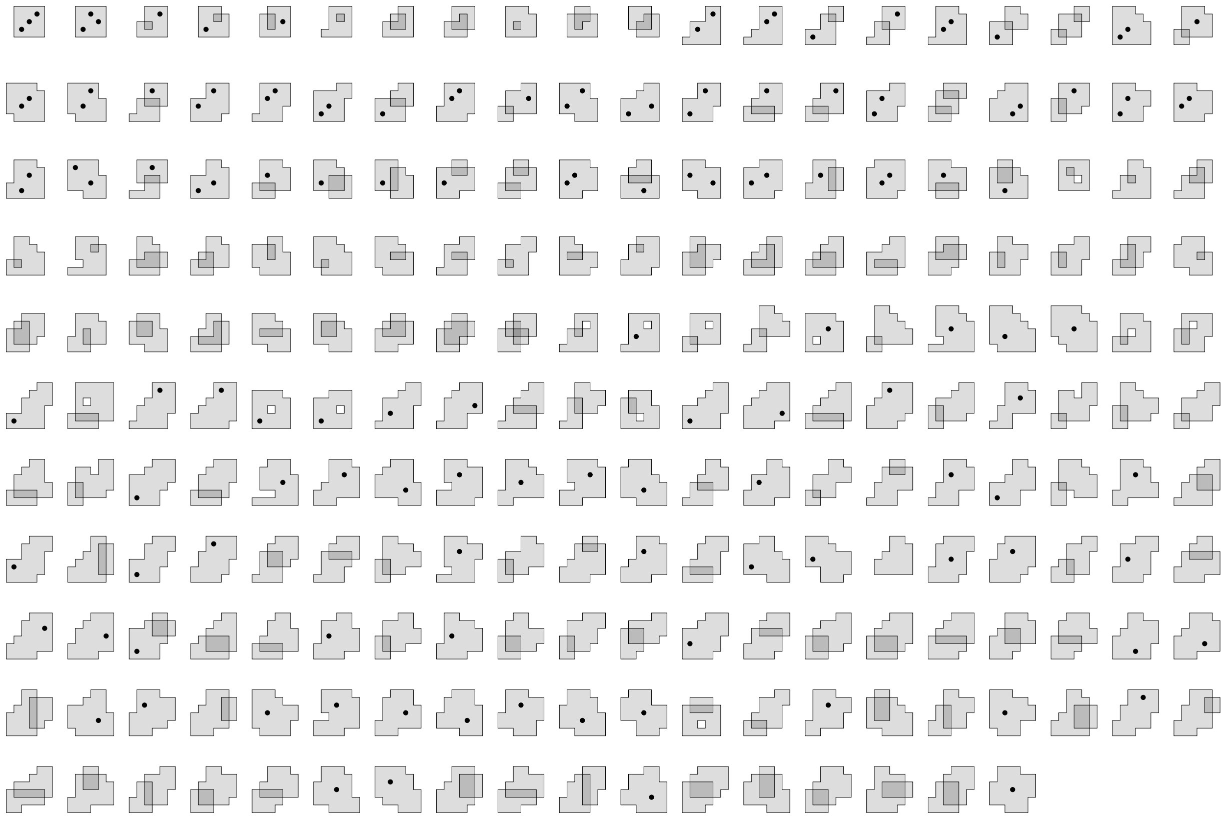


FIGURE 7. Representative diagrams of pairs in $ES_7(\mathcal{F}_A)$.

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