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General Augmented Rook Boards & Product Formulas

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Abstract. There are a number of so-called factorization theorems for rook polynomials that have appeared in the literature. For example, Goldman, Joichi, and White [**6**] showed that for any Ferrers board $B = F(b_1, b_2, \ldots, b_n)$,

$$\prod_{i=1}^{n}(x + b_i - (i-1)) = \sum_{k=0}^{n} r_k(B)(x)\downarrow_{(n-k)}$$

where $r_k(B)$ is the k-th rook number of B and $(x)\downarrow_k = x(x-1)\cdots(x-(k-1))$ is the usual falling factorial polynomial. Similar formulas where $r_k(B)$ is replaced by some appropriate generalization of rook numbers and $(x)\downarrow_k$ is replaced by polynomials like $(x)\uparrow_{k,j} = x(x+j)\cdots(x+j(k-1))$ or $(x)\downarrow_{k,j} = x(x-j)\cdots(x-j(k-1))$ can be found in the work of Goldman and Haglund [**5**], Remmel and Wachs [**11**], Haglund and Remmel [**7**], and Briggs and Remmel [**3**]. We shall call such formulas generalized product formulas. The main goal of this paper is to develop a new rook theory setting where we can give a uniform combinatorial proof of a generalized product formula which includes all the cases referred to above. That is, given any two sequences of non-negative integers, $\mathcal{B} = (b_1, \ldots, b_n)$ and $\mathcal{A} = (a_1, \ldots, a_n)$, and two sign functions $sgn, \overline{sgn} : \{1, \ldots, n\} \to \{-1, 1\}$, we shall define a rook theory setting and appropriate generalization of rook numbers $r_k^{\mathcal{A}}(\mathcal{B}, sgn, \overline{sgn})$ such that

$$\prod_{i=1}^{n}(x + sgn(i)b_i) = \sum_{k=0}^{n} r_k^{\mathcal{A}}(\mathcal{B}, sgn, \overline{sgn}) \prod_{j=1}^{n-k}(x + (\sum_{s=1}^{j} \overline{sgn}(s)a_s)).$$

Thus, for example, we obtain a combinatorial interpretations of the connection coefficients between any two bases of the polynomial ring $Q[x]$ of the form $\{(x)\downarrow_{k,j}\}_{k\geq 0}$ or $\{(x)\uparrow_{k,j}\}_{k\geq 0}$. We also find q-analogues and (p,q)-analogues of the above formulas.

Résumé.

Le but principal de cet article est de développer une nouvelle théorie rook dans laquelle nous pouvons fournir des preuves combinatoires uniformes d'une formule de produit généralisée qui inclut toutes les cas cités ci-dessus. C'est-à-dire, se donnant deux suites quelconques de nombres entiers positifs, $\mathcal{B} = (b_1, \ldots, b_n)$ et $\mathcal{A} = (a_1, \ldots, a_n)$, et deux fonctions de signes $sgn, \overline{sgn} : \{1, \ldots, n\} \to \{-1, 1\}$, nous définissons une théorie rook ainsi qu'une généralisation appropriée des nombres rook $r_k^{\mathcal{A}}(\mathcal{B}, sgn, \overline{sgn})$ tel que

$$\prod_{i=1}^{n}(x + sgn(i)b_i) = \sum_{k=0}^{n} r_k^{\mathcal{A}}(\mathcal{B}, sgn, \overline{sgn}) \prod_{j=1}^{n-k}(x + (\sum_{s=1}^{j} \overline{sgn}(s)a_s)).$$

Donc, par exemple, nous obtenons une interprétation combinatoire des coefficients de connexion entre deux bases de l'anneau des polynômes $Q[x]$ de la forme $\{(x)\downarrow_{k,j}\}_{k\geq 0}$ ou $\{(x)\uparrow_{k,j}\}_{k\geq 0}$. Nous trouvons aussi des q-analogues et des (p,q)-analogues de ces formules.

1. Introduction

Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ denote the set of natural numbers. For any positive integer a, we will set $[a] := \{1, 2, \ldots, a\}$. We will say that $\mathcal{B}_n = [n] \times [n]$ is an n by n array of squares (like a chess board), which we

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call *cells*. The cells of $\mathcal{B}_n$ will be numbered from left to right and bottom to top with the numbers from $[n]$, and we will refer to the cell in the i^{th} row and j^{th} column of $\mathcal{B}_n$ as the (i,j) cell of $\mathcal{B}_n$. Any subset of $\mathcal{B}_n$ is called a *rook board*. If B is a board in $\mathcal{B}_n$ with column heights $b_1, b_2, \ldots, b_n$ reading from left to right, with $0 \leq b_i \leq n$ for each i, then we will write $B = F(b_1, b_2, \ldots, b_n) \subseteq \mathcal{B}_n$. In the special case that $0 \leq b_1 \leq b_2 \leq \cdots \leq b_n \leq n$, we will say that $B = F(b_1, b_2, \ldots, b_n)$ is a *Ferrers board*.

Given a board $B = F(b_1, b_2, \ldots, b_n)$, there are three sets of numbers we can associate with B, namely, the rook, file, and hit numbers of B. The rook number, $r_k(B)$, is the number of placements of k rooks in the board B so that no two rooks lie in the same row or column. The file number, $f_k(B)$, is the number of placements of k rooks in the board B so that no two rooks lie in the same column but where we allow any given row to contain more than one rook. Given a permutation $\sigma = \sigma_1\sigma_2\ldots\sigma_n$ in the symmetric group S_n, we shall identify σ with the placement $\mathbb{P}_\sigma = \{(1,\sigma_1),(2,\sigma_2),\ldots,(n,\sigma_n)\}$. Then the hit number, $h_k(B)$, is the number of $\sigma \in S_n$ such that the placement $\mathbb{P}_\sigma$ intersects the board in exactly k cells.

All of these numbers have been studied extensively by combinatorialists. Here are three fundamental identities involving these numbers. Define $(x)\downarrow_m = x(x-1)\cdots(x-(m-1))$ and $(x)\uparrow_m = x(x+1)\cdots(x+(m-1))$. Then

$$\sum_{k=0}^{n} h_k(B)x^k = \sum_{k=0}^{n} r_k(B)(n-k)!(x-1)^k, \tag{1.1}$$

$$\prod_{i=1}^{n}(x+b_i-(i-1)) = \sum_{k=0}^{n} r_{n-k}(B)(x)\downarrow_k, \text{ and} \tag{1.2}$$

$$\prod_{i=1}^{n}(x+b_i) = \sum_{k=0}^{n} f_{n-k}(B)x^k. \tag{1.3}$$

Identity (1.1) is due to Kaplansky and Riordan [**8**] and holds for any board $B \subseteq \mathcal{B}_n$. Identity (1.2) holds for all Ferrers boards $B = F(b_1, \ldots, b_n)$ and is due to Goldman, Joichi, and White [**6**]. Identity (1.3) is due to Garsia and Remmel [**4**] and holds for all boards of the form $B = F(b_1, \ldots, b_n)$. Formulas (1.2) and (1.3) are examples of what we shall call *product formulas* in rook theory.

We note that in the special case where $B = \mathbf{B}_n := F(0,1,2,\ldots,n-1)$, Equations (1.2) and (1.3) become

$$x^n = \sum_{k=0}^{n} r_{n-k}(\mathbf{B}_n)(x)\downarrow_k \text{ and} \tag{1.4}$$

$$(x)\uparrow_n = \sum_{k=0}^{n} f_{n-k}(\mathbf{B}_n)x^k. \tag{1.5}$$

This shows that $r_{n-k}(\mathbf{B}_n) = S_{n,k}$, where $S_{n,k}$ is the Stirling number of the second kind, and $(-1)^{n-k}f_{n-k}(\mathbf{B}_n) = s_{n,k}$, where $s_{n,k}$ is the Stirling number of the first kind, and thus, we obtain rook theory interpretations for the Stirling numbers of the first and second kind.

There are natural q-analogues of formulas (1.1), (1.2), and (1.3). That is, define $[n]_q = 1+q+\cdots+q^{n-1} = \frac{1-q^n}{1-q}$. We then define q-analogues of the factorials and falling factorials by $[n]_q! = [n]_q[n-1]_q\cdots[2]_q[1]_q$ and $[x]_q\downarrow_m = [x]_q[x-1]_q\cdots[x-(m-1)]_q$, Garsia and Remmel [**4**] defined q-analogues of the hit numbers, $h_k(B,q)$, q-analogues of the rook numbers, $r_k(B,q)$, and q-analogues of file numbers, $f_k(B,q)$, for Ferrers boards B so that the following hold:

$$\sum_{k=0}^{n} h_k(B,q)x^{n-k} = \sum_{k=0}^{n} r_{n-k}(B,q)[k]_q!x^k(1-xq^{k+1})\cdots(1-xq^n), \tag{1.6}$$

$$\prod_{i=1}^{n}[x+b_i-(i-1)]_q = \sum_{k=0}^{n} r_{n-k}(B,q)[x]_q\downarrow_k, \quad \text{and} \tag{1.7}$$

$$\prod_{i=1}^{n}[x+b_i]_q = \sum_{k=0}^{n} f_{n-k}(B,q)([x]_q)^k. \tag{1.8}$$

Finally, we should mention that there are also (p, q)-analogues of such formulas (see Wachs and White [**12**], Briggs and Remmel [**2**], and Briggs [**1**]).

In recent years, a number of researchers have developed new rook theory models which give rise to new classes of product formulas. For example, Haglund and Remmel [**7**] developed a rook theory model where the analogue of the the rook number $m_k(B)$ counts partial matchings in the complete graph $\mathcal{K}_n$. They defined an analogue of a Ferrers board $\tilde{F}(a_1, \ldots a_{2n-1})$ where $2n-1 \geq a_1 \geq \cdots \geq a_{2n-1} \geq 0$ and where the nonzero entries in $(a_1, \ldots, a_{2n-1})$ are strictly decreasing, and, in their setting, they proved the following identity,

$$\prod_{i=1}^{2n-1} (x + a_{2n-i} - 2i + 2) = \sum_{k=0}^{2n-1} m_k(F) x(x-2)(x-4)\cdots(x-2(n-(k-1))). \tag{1.9}$$

Remmel and Wachs [**11**] defined a more restricted class of rook numbers, $\tilde{r}_k^j(B)$, in their *j-attacking* rook model and proved that for Ferrers boards $B = F(b_1, \ldots, b_n)$, where $b_{i+1} - b_i \geq j - 1$ if $b_i \neq 0$,

$$\prod_{i=1}^{n} (x + b_i - j(i-1)) = \sum_{k=0}^{n} \tilde{r}_{n-k}^j(B) x(x-j)(x-2j)\cdots(x-(k-1)j). \tag{1.10}$$

Goldman and Haglund [**5**] developed an *i-creation rook theory model* and proved that for Ferrers boards one has the following identity,

$$\prod_{j=1}^{n} (x + b_i + j(i-1)) = \sum_{k=0}^{n} r_{n-k}^{(i)}(B) x(x+(i-1))\cdots(x+(k-1)(i-1)). \tag{1.11}$$

In all of these new models, the authors proved q-analogues and or (p, q)-analogues of their product formulas.

2. A General Product Formula

Suppose we are given any two sequences of natural numbers: $\mathcal{B} = \{b_i\}_{i=1}^n, \mathcal{A} = \{a_i\}_{i=1}^n \in \mathbb{N}^n$. Define $A_i = a_1 + a_2 + \cdots + a_i$, the ith partial sum of the a_i's, and let $B = F(b_1, b_2, \ldots, b_n)$ be a rook board. We will also define two functions, sgn and $\overline{sgn}$, such that $sgn, \overline{sgn} : [n] \to \{-1, +1\}$. Our goal is to define a rook theory model with an appropriate notion of the rook numbers $r_k^{\mathcal{A}}(\mathcal{B}, sgn, \overline{sgn})$ such that the following product formula holds:

$$\prod_{i=1}^{n} (x + sgn(i)(b_i)) = \sum_{k=0}^{n} r_k^{\mathcal{A}}(\mathcal{B}, sgn, \overline{sgn}) \prod_{j=1}^{n-k} (x + \sum_{s \leq j} \overline{sgn}(s)(a_s)). \tag{2.1}$$

We will refer to Equation (2.1) as the *general product formula* and the number $r_k^{\mathcal{A}}(\mathcal{B}, sgn, \overline{sgn})$ as the k^{th} *augmented rook number of $\mathcal{B}$ with respect to $\mathcal{A}$, sgn, and $\overline{sgn}$.*

2.1. Special Cases of the General Product Formula. We first wish to consider the case where $sgn(i) = +1$ and $\overline{sgn}(i) = -1$ for every $1 \leq i \leq n$. In this case we will set

$$r_k^{\mathcal{A}}(\mathcal{B}, sgn, \overline{sgn}) = r_k^{\mathcal{A}}(\mathcal{B}).$$

Thus, we want to prove Equation (2.2):

$$\prod_{i=1}^{n} (x + b_i) = \sum_{k=o}^{n} r_k^{\mathcal{A}}(\mathcal{B})(x - A_1)(x - A_2)\cdots(x - A_{n-k}). \tag{2.2}$$

To do this, we first construct an *augmented rook board*, $B^{\mathcal{A}} = F(b_1 + A_1, b_2 + A_2, \ldots, b_n + A_n)$. In $B^{\mathcal{A}}$, the cells in the i-th column are $(1, i), \ldots, (b_i + a_1 + \cdots + a_i, i)$ reading from bottom to top. We shall refer to the cells $(1, i), \ldots, (b_i, i)$ as the b_i part of column i, the cells $(b_i + 1, i), \ldots, (b_i + A_i, i)$ as the A_i part of column i, and, for each $s \leq i$, the cells $(b_i + a_1 + \cdots a_{s-1} + 1, i), \ldots, (b_i + a_1 + \cdots + a_s, i)$ as the a_s part of column i where by convention $a_{-1} = 0$. We call the part of the board $B^{\mathcal{A}}$ which corresponds to the A_i's the *augmented part of* $B^{\mathcal{A}}$. We now consider rook placements in $B^{\mathcal{A}}$ with at most one rook in each column. We define the following cancellation rule: a rook r placed in column j of $B^{\mathcal{A}}$ will cancel, in each column to its right, all of the cells which lie in the a_i part of that column where i is the highest subscript j such that the a_j part of that column has not been canceled by a rook to the left of r. For example, in Figure 1,

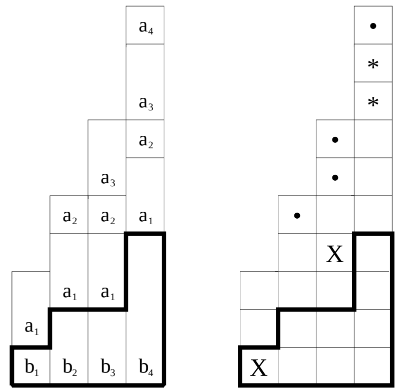

Figure 1. $B^{\mathcal{A}}$, with $B = F(1,2,2,4)$ and $\mathcal{A} = (2,1,2,1)$, and a placement of two rook in $B^{\mathcal{A}}$.

where $B = F(1,2,2,4)$ and $\mathcal{A} = (2,1,2,1)$, the rook in the first column cancels the cells in the a_2 part of the second column, the a_3 part of the third column, and the a_4 part of the fourth column (those cells which contain a "•"). The rook in the third column cancels the cells in the a_3 part the fourth column (those cells which contain a "$*$"). We then define $r_k^{\mathcal{A}}(\mathcal{B})$ to be the number of ways of placing k such rooks in $B^{\mathcal{A}}$ so that no rook lies in a cell which is canceled by a rook to its left.

We can now construct a *general augmented rook board*, $B_x^{\mathcal{A}}$, defined by the sequences $\mathcal{B} = \{b_i\}_{i=1}^n$ and $\mathcal{A} = \{a_i\}_{i=1}^n$ and some nonnegative integer x. The board $B_x^{\mathcal{A}}$ will be the board $B^{\mathcal{A}}$ (the augmented part of $B^{\mathcal{A}}$ will here be referred to as the *upper augmented part of* $B_x^{\mathcal{A}}$), with x rows appended below, called the *x-part* and then a "mirror image" of the augmented part of $B^{\mathcal{A}}$ below that, called the *lower augmented part of* $B_x^{\mathcal{A}}$. In the lower augmented part, we number the cells in i-th column with $(1,i),\ldots,(b_i + A_i, i)$ reading from top to bottom and we define the a_s part of the i-th column of the lower augmented board to consist of the cells $(a_1 + \cdots + a_{s-1} + 1, i),\ldots,(a_1 + \cdots + a_s, i)$. We say that the board $B^{\mathcal{A}}$ is separated from the x-part by the *high bar* and the x-part is separated from the lower augmented part by the *low bar*. An illustration of this type of board with $B = F(1,2,2,4)$, $\mathcal{A} = (2,1,2,1)$, and $x = 4$ can be seen in the left side of Figure 2.

In order to define a proper rook placement in the board $B_x^{\mathcal{A}}$, we make the rule that exactly one rook must be placed in every column of $B_x^{\mathcal{A}}$. When placing rooks in $B_x^{\mathcal{A}}$, we will define the following cancellation rules:

1. (1) A rook placed above the high bar in the j^{th} column of $B_x^{\mathcal{A}}$ will cancel all of the cells in columns $j+1, j+2, \ldots, n$, both in the upper and lower augmented parts, which belong to the a_i part of the column where i is largest j such that cells in the a_j part of the column are not canceled by a rook to their left.
2. (2) Rooks placed below the high bar do not cancel any cells.

An example of a rook placement in these boards can be seen in the right side of Figure 2. In this placement, the rook placed in the first column is placed above the high bar, thus it cancels in the columns to its right those cells contained in the a_i part of highest subscript in both the upper and lower augmented parts (denoted by a "•"). The rook placed in the second column is placed below the the high bar so that it cancels nothing. The rook placed in the third column is again placed above the high bar so that it cancels as does the rook placed in the first column (denoted by a "$*$"), and the last rook may be placed in any available cell.

We will now prove two lemmas in order to prove Equation (2.2).

Lemma 2.1. *If there are $b_j + A_m$ cells to place a rook above the high bar in column j, then there are A_m cells below the low bar to place a rook in column j.*

Proof: By how we define our cancellation, a block of cells from a_i gets canceled above the high bar if and only if a block of cells from a_i gets canceled below the low bar.

Lemma 2.2. *If k rooks are placed above the high bar in $B_x^{\mathcal{A}}$, then the column heights of the uncanceled cells in the lower augmented part of $B_x^{\mathcal{A}}$, when read from left to right, are $A_1, A_2, \ldots, A_{n-k}$.*

Proof: Suppose the first rook above the high bar is placed in the j^{th} column. The columns below the low bar which lie to the left of column j have heights $A_1, A_2, \ldots, A_{j-1}$. Now, the rook that was placed in the

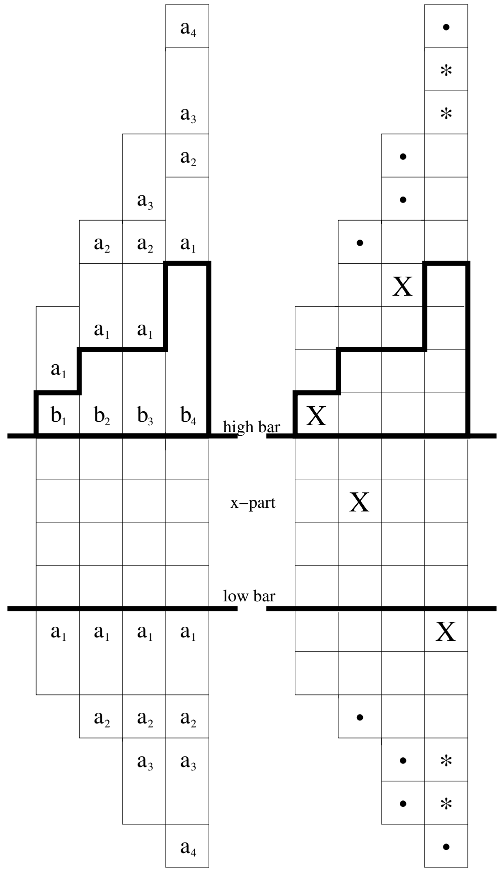

FIGURE 2. $B_x^{\mathcal{A}}$, with $B = F(1,2,2,4), \mathcal{A} = (2,1,2,1)$, and $x = 4$, and a placement of rooks in $B_x^{\mathcal{A}}$.

the j^{th} column will cancel all the cells in the a_{j+1} part of the $(j+1)^{st}$ column, all the cells in the a_{j+2} part of the $(j+2)^{nd}$ column, etc.. Thus after this cancellation, the heights of the columns below the low bar into which a rook may be placed are $A_1, A_2, \ldots, A_{j-1}, A_j, \ldots A_{n-1}$. Now suppose that the leftmost rook to the right to column j is in column k. Then the rook in column k will cancel all the cells in a_k part of the $(k+1)^{st}$ column, all the cells in the a_{k+1} part of the $(k+2)^{nd}$ column, etc.. Thus after this second cancellation, the heights of the columns below the low bar into which a rook may be placed are $A_1, A_2, \ldots, A_{j-1}, A_j, \ldots A_{k-1}, A_k, \ldots, A_{n-2}$. We can continue this type of reasoning to show that if there are k rooks are placed above the high bar in $B_x^{\mathcal{A}}$, then the column heights of the uncanceled cells in the lower augmented part of $B_x^{\mathcal{A}}$, when read from left to right, are $A_1, A_2, \ldots, A_{n-k}$.

We are now in position to prove (2.2). We shall show that (2.2) is the result of computing the sum S of the weights of all placements of n rooks in $B_x^{\mathcal{A}}$ in two different ways, where we define the weight of the rooks placed above the low bar to be "+1", the weight of the rooks placed below the low bar to be "−1", and the weight of any placement to be the product of the weights of the rooks in the placement.

If we first place the rooks starting with the leftmost column and working to the right, then we can see that in the first column there are exactly $x + b_1 + 2a_1$ cells in which to place the first rook, where the "$2a_1$" corresponds to placing the rook in either the upper or lower augmented part of the 1^{st} column. Since all of the rooks above the high bar have weight "+1" and all the rooks placed below the low bar have weight "−1", it is easy to see that the possible placements of rooks in the first column contributes a factor of $x + b_1 + a_1 + (-a_1) = x + b_1$ to S. When we consider the possible placements of a rook in the second column, we have two cases.

Case I: Suppose the rook that the 1^{st} column was placed below the high bar. Then nothing was canceled in the second column so we can place a rook in any cell of the second column. Thus there are a total

$x+b_2+2(a_1+a_2)$ ways to place the rook in the second column in this case. However, given our weighting of the rooks, we see that the possible placements of rooks in the second column contributes a factor of $x+b_2+(a_1+a_2)+(-a_1-a_2)=x+b_2$ to S.

Case II: If the rook in the first column was placed above the high bar, then the cells corresponding to a_2 part in both the upper and lower augmented parts of the 2^{nd} column are canceled. Thus in this case, there are $x+b_2+2a_1$ cells left to place the rook in the second column. However, given our weighting of rooks, the possible placements of rooks in the second column contributes a factor of $x+b_2+(a_1)+-a_1=x+b_2$ to S in this case.

In general, suppose we are placing a rook in the j^{th} column that does not have a rook above the high bar reading from left to right. Assume that we have placed s rooks above the high bar and t rooks below the high bar in the first $j-1$ columns. Then by Lemma 2.1, we have, $x+b_j+2(A_{j-s})$ choices as to where to place the rook in that column. Again, due to our weighting, it is easy to see that possible placement of rooks contributes a factor of $x+b_j+A_{j-s}+(-A_{j-s})=x+b_j$ to S. It follows that $S=\prod_{i=1}^{n}(x+b_i)$.

The second way of counting this sum of the weights of all the rook placements in $B_x^{\mathcal{A}}$ is to organize the placements by how many rooks lie above the high bar. Suppose that we place k rooks above the high bar and then wish to extend that to a placement in the entire board. The number of ways of placing the k rooks above the high bar is given by $r_k^{\mathcal{A}}(\mathcal{B})$. For any such placement of k rooks above the high bar, we are left with $n-k$ columns in which to place rooks below the high bar. We consider the placement of the remaining rooks in these available columns starting with the leftmost one and working right. By Lemma 2.2, the number of ways we can do this will be $(x+A_1)(x+A_2)\cdots(x+A_{n-k}))$. However, these placements come with a weighting of $(x+(-A_1)(x+(-A_2)\cdots(x+(-A_{n-k}))$ since the cells below the low bar have weight $''-1''$. Thus the sum of the weights of the set of placements in $B_x^{\mathcal{A}}$ with k rooks above the high bar is $r_k^{\mathcal{A}}(\mathcal{B})(x-A_1)(x-A_2)\cdots(x-A_{n-k}))$. Summing over all possible k gives us the RHS of (2.2).

□

Now suppose we change the weights which are assigned to rooks in $B_x^{\mathcal{A}}$ by declaring that the weight of a rook placed in the upper augmented part is "-1" and all other rooks have weight "$+1$". Again the weight of the placement is the product of the weights of the rooks in the placement. This weighting corresponds to the case where $sgn(i)=+1$ and $\overline{sgn}(i)=+1$ for every $1\le i\le n$. We will define $\tilde{r}_k^{\mathcal{A}}(\mathcal{B})$ to be the weighting of all placements of k rooks in $B^{\mathcal{A}}$ with this newly assigned weight, and this yields an equation which is analogous to Equation (2.2), namely,

$$\prod_{i=1}^{n}(x+b_i)=\sum_{k=o}^{n}\tilde{r}_k^{\mathcal{A}}(\mathcal{B})(x+A_1)(x+A_2)\cdots(x+A_{n-k}). \tag{2.3}$$

Proof of Equation (2.3): This proof follows exactly the proof of Equation (2.2) with the weights from the upper and lower augmented parts switched.

□

We can see that these two special cases encapsulate all of the product formulas stated in the Introduction. Next we sketch a proof for the general product formula (2.1).

2.2. The General Product Formula. We have now shown how to generate our general product formula in the special cases where the functions sgn and $\overline{sgn}$ are certain constant functions; however, the proofs of Equations (2.2) and (2.3) do not depend on sgn and $\overline{sgn}$ being constant. Rather, the proofs depend only on the condition that, for each column j, if the cells corresponding to the a_i part of the upper augmented part in column j are weighted with $\omega(a_i)$, then the cells corresponding to the a_i part in the lower augmented part in column j must be weighted with -$\omega(a_i)$. Moreover, the proofs do not depend on the weighting of rooks placed in the cells of the cells in the b_i part of column i in $B^{\mathcal{A}}$. Thus, if we define $r_k^{\mathcal{A}}(\mathcal{B},sgn,\overline{sgn})$ to be the weight of all placements of k rooks in the board $B^{\mathcal{A}}$, with each rook in the b_i part of column i having weight $sgn(i)$ and each rook in the a_i part of any column below the low bar having weight $\overline{sgn}(i)$, and the weight of any placement to be the product of the rooks in that placement, then we can show that Equation (2.1) is the result of computing the sum S of the weights of all placements of n rooks in $B_x^{\mathcal{A}}$ exactly as in the proofs of Equations (2.2) and (2.3).

3. Q-Analogues of General Product Formulas

In this section, we shall describe how one can derive q-analogues of some of the general product formulas described in Section 2. We do this by q-counting rook placements considered in Section 2. To simplify our notation, we shall use the convention that for any negative integer x, $[x]_q := -[|x|]_q$. If we set $\overline{A}_k = \sum_{i=1}^k \overline{sgn}(i)a_i$, then we can prove the following q-analogue of Equation 2.1:

$$\prod_{i=1}^{n}([x]_q + sgn(i)[b_i]_q) = \sum_{k=0}^{n} r_k^{\mathcal{A}}(\mathcal{B}, sgn, \overline{sgn}, q) \prod_{s=1}^{n-k}([x]_q + [\overline{A}_s]_q). \tag{3.1}$$

For each cell c in the board $B^{\mathcal{A}}$, we let $below_{B^{\mathcal{A}}}(c)$ denote the number of cells that lie directly below c in its part. That is, if c is a cell in the augmented part of $B^{\mathcal{A}}$, then $below_{B^{\mathcal{A}}}(c)$ is the number of cells below c in the augmented part of $B^{\mathcal{A}}$ and if c is not in the augmented part of $B^{\mathcal{A}}$, then $below_{B^{\mathcal{A}}}(c)$ is just the number of cells below c in B. We may then extend this definition to the board $B_x^{\mathcal{A}}$ by defining $below_{B_x^{\mathcal{A}}}(c)$ to be the number of cells below a given cell c in $B_x^{\mathcal{A}}$ in its part. To each cell c in the board $B_x^{\mathcal{A}}$ we will assign a q-weight, $\omega_q(c)$. Given a placement P in $B_x^{\mathcal{A}}$, we will define the q-weight of that placement to be $\omega_q(P) = \prod_{r \in P} \omega_q(r)$, where $\omega_q(r) = \omega_q(c)$ if the rook r is in cell c. First, we define $\omega_q(c) = q^{below_{B_x^{\mathcal{A}}}(c)}$ if c is in the x-part of the board. Next we set $\omega_q(c) = sgn(i)q^{below_{B_x^{\mathcal{A}}}(c)}$ if c is in the i^{th} column of the board B. For the lower augmented part of the board, the definition of $\omega_q(c)$ is slightly more involved. Suppose we are at the k^{th} column of the lower augmented part of $B_x^{\mathcal{A}}$, which has column height $a_1 + a_2 + \cdots + a_k$. Recall that we labeled the cells in k-th column of the lower augmented board from top to bottom with the pairs $(1,k), (2,k), \ldots, (a_1 + \cdots + a_k, k)$. Then, for $i \le a_1$, we set $\omega_q((i,k)) = \overline{sgn}(1)q^{i-1}$. Now, assume by induction that we have assigned weights to the cells $(1,k), (2,k), \ldots, (a_1+\cdots+a_i, k)$ so that $\sum_{j=1}^{a_1+\cdots+a_i} \omega_q((j,k)) = [\overline{A}_i]_q$. Then we will label the cells $(a_1 + \cdots + a_i + 1, k), \ldots, (a_1 + \cdots + a_i + a_{i+1}, k)$ in the following manner:

1. Case I: $\overline{A}_i \ge 0$
   - (a) If $\overline{A}_i \le \overline{A}_{i+1}$, then we assign the q-weight of the cells $(a_1 + \cdots + a_i + 1, k), \ldots, (a_1 + \cdots + a_i + a_{i+1}, k)$ to be $q^{\overline{A}_i}, q^{\overline{A}_i+1}, \ldots, q^{\overline{A}_{i+1}-1}$, respectively.
   - (b) If $0 \le \overline{A}_{i+1} \le \overline{A}_i$, then we assign the q-weight of the cells $(a_1 + \cdots + a_i + 1, k), \ldots, (a_1 + \cdots + a_i + a_{i+1}, k)$ to be $-q^{\overline{A}_i-1}, -q^{\overline{A}_i-2}, \ldots, -q^{\overline{A}_{i+1}}$, respectively.
   - (c) If $\overline{A}_{i+1} < 0$, then we assign the q-weight of the cells $(a_1+\cdots+a_i+1, k), \ldots, (a_1+\cdots+a_i+a_{i+1}, k)$ to be $-q^{\overline{A}_i-1}, -q^{\overline{A}_i-2}, \ldots, -1, -1, -q, -q^2, \ldots, -q^{|\overline{A}_{i+1}|-1}$, respectively.
2. Case II: $\overline{A}_i < 0$
   - (a) If $\overline{A}_i \ge \overline{A}_{i+1}$, then we assign the q-weight of the cells $(a_1 + \cdots + a_i + 1, k), \ldots, (a_1 + \cdots + a_i + a_{i+1}, k)$ to be $-q^{|\overline{A}_i|}, -q^{|\overline{A}_i|+1}, \ldots, -q^{|\overline{A}_{i+1}|-1}$, respectively.
   - (b) If $0 \ge \overline{A}_{i+1} \ge \overline{A}_i$, then we assign the q-weight of the cells $(a_1 + \cdots + a_i + 1, k), \ldots, (a_1 + \cdots + a_i + a_{i+1}, k)$ to be $q^{|\overline{A}_i|-1}, q^{|\overline{A}_i|-2}, \ldots, -q^{|\overline{A}_{i+1}|}$, respectively .
   - (c) If $\overline{A}_{i+1} > 0$, then we assign the q-weight of the cells $(a_1+\cdots+a_i+1, k), \ldots, (a_1+\cdots+a_i+a_{i+1}, k)$ to be $q^{|\overline{A}_i|-1}, q^{|\overline{A}_i|-2}, \ldots, 1, 1, q, q^2, \ldots, q^{\overline{A}_{i+1}-1}$, respectively.

Finally, in order to assign the q-weights to the k^{th} column of the upper augmented part of $B_x^{\mathcal{A}}$, we will simply take the weights that we assigned to the lower augmented part of the k^{th} column, flip them upside down and multiply them all by "-1". An example of this weighting can be seen in Figure 3, where the q-number displayed in each cell of the diagram corresponds to the q-weight a rook placed in that cell would be given. For example, we can see that the q-weights assigned to the lower augmented part of the fourth column, read from top to bottom are: $1, q, -q, -1, -1, 1$. The weights in the upper augmented part of the same column are, when read from bottom to top: $-1, -q, q, 1, 1, -1$, which is the previous sequence with every element multiplied by "-1".

Now we can prove Equation 3.1 similar to the way we proved Equation 2.1 in the previous section. That is, Equation 3.1 results by computing the sum S_q of the q-weights of all placements of n rooks in $B_x^{\mathcal{A}}$ in two different ways.

3.1. Special Cases of the General Q-Analogue Formula. Now consider the special cases where sgn and $\overline{sgn}$ are the constant functions -1 or $+1$. In this case, it is easy to see that

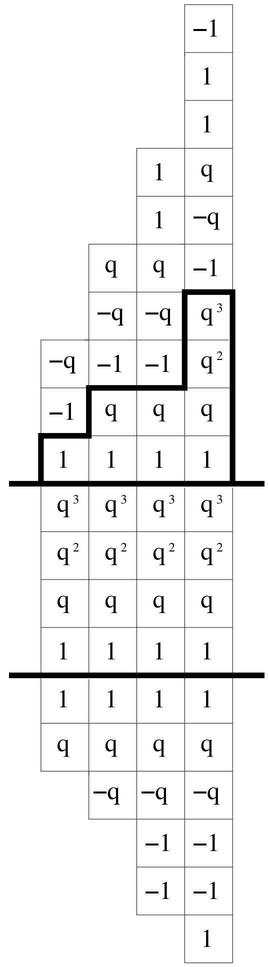

FIGURE 3. A q-analogue of the rook placement in Figure 2 with $sgn(i) = +1$ for $i = 1, 2, 3, 4$ and $\overline{sgn}(i) = +1$ for $i = 1, 3, 4$ $\overline{sgn}(i) = -1$ for $i = 2$. Here each cell is labeled with the q-weight that a rook placed in that cell would be given.

(i) if a rook in is the b_i part of column i of $B^{\mathcal{A}}$, then its q-weight will be $sgn(i)q^{below_{B_x^{\mathcal{A}}}(c)}$,
(ii) if a rook in is the A_i part of column i of $B^{\mathcal{A}}$, then its q-weight will be $-\overline{sgn}(i)q^{below_{B_x^{\mathcal{A}}}(c)}$,
(iii) if rook in is the x part of column i of $B^{\mathcal{A}}$, then its q-weight will be $q^{below_{B_x^{\mathcal{A}}}(c)}$, and
(iv) the q-weights of a cell in the lower augmented part of the board is just the q-weight of its mirror image in the upper augmented part of the board multiplied by $''-1''$.
In this case (3.1) becomes

$$\prod_{i=1}^{n}([x]_q + sgn(i)[b_i]_q) = \sum_{k=0}^{n} r_k^{\mathcal{A}}(\mathcal{B}, sgn, \overline{sgn}, q) \prod_{s=1}^{n-k}([x]_q + \overline{sgn}(s)[A_s]_q). \tag{3.2}$$

It turns out that by slightly modifying our q-counting of rook placements, we can prove analogues of (3.2) where we replace $[x] - [c]$ by $[x - c]$ or $[x] + [c]$ by $[x + c]$.

3.1.1. *Case I: $sgn(i) = \overline{sgn}(i) = -1$.* For $x, c \in \mathbb{N}$ with $x > c$, we have that $[x]_q - [c]_q = q^c[x - c]_q$. Thus (3.2) becomes

$$\prod_{i=1}^{n} q^{b_i}[x - b_i]_q = \sum_{k=0}^{n} r_k^{\mathcal{A}}(\mathcal{B}, sgn, \overline{sgn}, q) \prod_{s=1}^{n-k} q^{A_s}[x - A_s]. \tag{3.3}$$

It is then easy to see that if we replace $r_k^{\mathcal{A}}(\mathcal{B},q)$ with $\hat{r}_k^{\mathcal{A}}(\mathcal{B},q)$ by

$$\hat{r}_k^{\mathcal{A}}(\mathcal{B},q) := q^{(A_1+A_2+\cdots+A_{n-k})-(b_1+\cdots+b_n)} r_k^{\mathcal{A}}(\mathcal{B},q),$$

we obtain the following form of Formula 3.2:

$$\prod_{i=1}^{n}([x-b_i]_q) = \sum_{k=o}^{n} \hat{r}_k^{\mathcal{A}}(\mathcal{B},q)([x-A_1]_q)([x-A_2]_q)\cdots([x-A_{n-k}]_q). \tag{3.4}$$

3.1.2. *Case II:* $sgn(i) = -1, \overline{sgn}(i) = +1$. For $x, c \in \mathbb{N}$ we have that $[x]_q + q^x[c]_q = [x+c]_q$. Thus if we want to replace $[x]_q + [A_i]_q$ by $[x+A_i]_q = [x]_q + q^x[A_i]$, then we should weight each rook that lies in upper augmented part of $B^{\mathcal{A}}$ by an extra factor of q^x. This means that when we consider placements in $B_x^{\mathcal{A}}$, then we must also weight each rook that lies in the lower augmented part of $B_x^{\mathcal{A}}$ with an extra factor of q^x so that for any given column the weights of possible placements in the lower and upper augmented parts cancel each other as in the proofs in Section 2. Thus we define $\hat{\hat{r}}_k^{\mathcal{A}}(\mathcal{B},q)$ to be the sum of the q-weight over all placements of k rooks in $B^{\mathcal{A}}$ where each rook placed in the augmented part receiving an extra factor of q^x. Then it is easy to see that (3.2) becomes

$$\prod_{i=1}^{n}([x]_q - [b_i]_q) = \sum_{k=0}^{n} \hat{\hat{r}}_k^{\mathcal{A}}(\mathcal{B},q)[x+A_1]\cdots[x+A_{n-k}]. \tag{3.5}$$

Finally we replace $\hat{\hat{r}}_k^{\mathcal{A}}(\mathcal{B},q)$ by a new q-rook number, $\tilde{\tilde{r}}_k^{\mathcal{A}}(\mathcal{B},q)$ where $\tilde{\tilde{r}}_k^{\mathcal{A}}(\mathcal{B},q) := q^{-(b_1+\cdots+b_n)} r_k^{\mathcal{A}}(\mathcal{B},q)$. In doing this, we obtain the following formula:

$$\prod_{i=1}^{n}([x-b_i]_q) = \sum_{k=0}^{n} \tilde{\tilde{r}}_k^{\mathcal{A}}(\mathcal{B},q)([x+A_1]_q)([x+A_2]_q)\cdots([x+A_{n-k}]_q). \tag{3.6}$$

We can also use methods similar to the ones used in Cases I and II, to prove the following product formulas for appropriate choices of $\overline{r}_k^{\mathcal{A}}(\mathcal{B},q)$ and $\overline{\overline{r}}_k^{\mathcal{A}}(\mathcal{B},q)$.

3.1.3. *Case III:* $sgn(i) = +1, \overline{sgn}(i) = -1$.

$$\prod_{i=1}^{n}([x+b_i]_q) = \sum_{k=o}^{n} \overline{r}_k^{\mathcal{A}}(\mathcal{B},q)([x-A_1]_q)([x-A_2]_q)\cdots([x-A_{n-k}]_q) \tag{3.7}$$

3.1.4. *Case IV:* $sgn(i) = \overline{sgn}(i) = +1$.

$$\prod_{i=1}^{n}([x+b_i]_q) = \sum_{k=o}^{n} \overline{\overline{r}}_k^{\mathcal{A}}(\mathcal{B},q)([x+A_1]_q)([x+A_2]_q)\cdots([x+A_{n-k}]_q) \tag{3.8}$$

4. (P,Q)-Analogues of General Product Formulas

For any $n \in \mathbb{N}$ we define $[n]_{p,q} = p^{n-1} + qp^{n-2} + \cdots + q^{n-2}p + q^{n-1}$, and we again use the convention that for a negative integer x, $[x]_{p,q} := -[|x|]_{p,q}$. Then we can give a combinatorial interpretation of the following (p,q)-analogue formula:

$$\prod_{i=1}^{n}([x]_{p,q} + sgn(i)[b_i]_{p,q}) = \sum_{k=0}^{n} r_k^{\mathcal{A}}(\mathcal{B}, sgn, \overline{sgn}, p, q) \prod_{s=1}^{n-k}([x]_{p,q} + [\overline{A}_s]_{p,q}). \tag{4.1}$$

Again, we will assign a weight to each cell c of the board $B_x^{\mathcal{A}}$, which we will call the (p,q)-weight of c, and this will be denoted by $\omega_{p,q}(c)$. We will also define the statistic $above_{B_x^{\mathcal{A}}}(c)$, for any cell c in the board $B_x^{\mathcal{A}}$, to be the number of cells that lie above c in its part, that is, if c is in the x-part of the board, then $above_{B_x^{\mathcal{A}}}(c)$ is the number of cells that lie above c in the x-part. For a placement P of rooks in $B_x^{\mathcal{A}}$, we will let the (p,q)-weight of P be $\omega_{p,q}(P) = \prod_{r \in P} \omega_{p,q}(r)$, where $\omega_{p,q}(r) = \omega_{p,q}(c)$ if the rook r is placed in cell c. Now, we can (p,q)-weight the cells of $B_x^{\mathcal{A}}$ in the following manner:

1. If c is in the x-part of the board, then $\omega_{p,q}(c) = p^{above_{B_x^{\mathcal{A}}}(c)} q^{below_{B_x^{\mathcal{A}}}(c)}$.
2. If c is in the i^{th} column of the board B, then $\omega_{p,q}(c) = sgn(i) p^{above_{B_x^{\mathcal{A}}}(c)} q^{below_{B_x^{\mathcal{A}}}(c)}$.

(3) If c is in the k^{th} column of the lower augmented part of the board, then we will set $\omega(1,k)=[\overline{A}_1]_{p,q}$. We will then set $\omega(a_1+\cdots+a_i+1,k)=[\overline{A}_{i+1}]_{p,q}-[\overline{A}_i]_{p,q}$ and $\omega(j,k)=0$ otherwise.
(4) If c is in the k^{th} column of the upper augmented part of the board, then weights will be assigned, from bottom to top, as they were in the lower augmented part, with all of the weights multiplied by "-1".

We note that this type of weighting is more complicated than our q-weighting since now a cell can receive a (p,q)-weight which is a polynomial in p and q rather than just a plus or minus a power of q. Moreover, there are many other choices we could make for the weights, but none of them reduce to the q-weight when $p=1$. However, in certain special cases, we can assign a more natural (p,q)-weight which is consistent with some of the (p,q)-analogues of product formulas that have appeared in the literature, but we shall not consider these types of results in this paper.
We can now prove Equation 4.1 in the exact same way that we proved Equation 3.1.

5. Conclusion and Perspectives

We have given a rook theory interpretation of the product formula

$$\prod_{i=1}^{n}(x+sgn(i)b_i)=\sum_{k=0}^{n}r_k^{\mathcal{A}}(\mathcal{B},sgn,\overline{sgn})\prod_{j=1}^{n-k}(x+(\sum_{s=1}^{j}\overline{sgn}(s)a_s)),$$

and this interpretation can be used to obtain identities studied by Goldman and Haglund [**5**], Remmel and Wachs [**11**], Haglund and Remmel [**7**], and Briggs and Remmel [**3**]. We also have q- and (p,q)- analogues of this general product formula.

One application of this new theory is in finding the inverses of connection coefficients for different bases of $\mathbb{Q}[x]$ [**9**]. If we define the functions $(x)\uparrow_{k,a}=x(x+a)(x+2a)\cdots(x+(k-1)a)$ and $(x)\downarrow_{k,b}=x(x-b)(x-2b)\cdots(x-(k-1)b)$, then for any $a\in\mathbb{N}$, the sets $\{(x)\uparrow_{n,a}\}_{n\geq 0}$ and $\{(x)\downarrow_{n,a}\}_{n\geq 0}$ will both form bases of $\mathbb{Q}[x]$. Thus, there exist numbers $C_{n,k}(b\downarrow,a\uparrow)$ and $C_{n,k}(a\uparrow,b\downarrow)$ such that

$$(x)\downarrow_{n,b}=\sum_{k=0}^{n}C_{n,k}(b\downarrow,a\uparrow)(x)\uparrow_{k,a} \tag{5.1}$$

and

$$(x)\uparrow_{n,a}=\sum_{k=0}^{n}C_{n,k}(a\uparrow,b\downarrow)(x)\downarrow_{k,b}\ . \tag{5.2}$$

From linear algebra it is known that $||C_{n,k}(b\downarrow,a\uparrow)||^{-1}=||C_{n,k}(a\uparrow,b\downarrow)||$, that is to say,

$$\sum_{j=k}^{n}C_{n,j}(a\uparrow,b\downarrow)C_{j,k}(b\downarrow,a\uparrow)=\chi(n=k). \tag{5.3}$$

However, this result may be obtained from our rook theory model. Given the numbers $a,b\in\mathbb{N}$ we will define $\mathcal{B}=(0,b,2b,\ldots,(n-1)b)$ and $\mathcal{A}=(0,a,a,\ldots,a)$. By now defining $sgn(i)=-1$ and $\overline{sgn}(i)=+1$ we see that $C_{n,k}(b\downarrow,a\uparrow)=r_k^{\mathcal{A}}(\mathcal{B},sgn,\overline{sgn})$ and $C_{n,k}(a\uparrow,b\downarrow)=r_k^{\mathcal{B}}(\mathcal{A},\overline{sgn},sgn)$. We can now write equations (5.1) and (5.2) as

$$(x)\downarrow_{n,b}=\sum_{k=0}^{n}r_k^{\mathcal{A}}(\mathcal{B},sgn,\overline{sgn})(x)\uparrow_{k,a} \tag{5.4}$$

and

$$(x)\uparrow_{n,a}=\sum_{k=0}^{n}r_k^{\mathcal{B}}(\mathcal{A},\overline{sgn},sgn)(x)\downarrow_{k,b}\ . \tag{5.5}$$

In particular, we now have that

$$\sum_{j=k}^{n} r_{n-k}^{\mathcal{A}}(\mathcal{B}, sgn, \overline{sgn}) r_{j-k}^{\mathcal{B}}(\mathcal{A}, \overline{sgn}, sgn) = \chi(n = k), \tag{5.6}$$

and we have a completely combinatorial proof of this fact based solely on involutions on rook placements in an augmented rook board setting. We can give similar combinatorial proofs for all the possible choice of $\uparrow$ and $\downarrow$ in the coefficient $C_{n,k}(b\downarrow, a\uparrow)$. For example, we can find combinatorial interpretations of the inverses of the numbers $C_{n,k}(a\uparrow, b\uparrow)$ and $C_{n,k}(a\downarrow, b\downarrow)$ which satisfy the equations

$$(x)\uparrow_{n,a} = \sum_{k=0}^{n} C_{n,k}(a\uparrow, b\uparrow)(x)\uparrow_{k,b} \tag{5.7}$$

and

$$(x)\downarrow_{n,a} = \sum_{k=0}^{n} C_{n,k}(a\downarrow, b\downarrow)(x)\downarrow_{k,b} \ . \tag{5.8}$$

Another application of our rook theory model relates to the numbers $S_{n,k}^{p(x)}$ defined in [**10**] by

$$S_{n+1,k}^{p(x)} = S_{n,k-1}^{p(x)} + p(k) S_{n,k}^{p(x)}, \tag{5.9}$$

where $p(x)$ is any polynomial with nonnegative integer coefficients and with initial conditions $S_{0,0}^{p(x)} = 1$ and $S_{n,k}^{p(x)} = 0$ whenever $n < 0, k < 0$, or $n < k$. We call such numbers *poly-Stirling numbers of the second kind* [**10**]. Then, for example, in the special case where $p(x) = x^m$, we can use an extension of the theory of general augmented rook boards to give a combinatorial proof of the formula

$$(x^n)^m = \sum_{k=0}^{n} S_{n,k}^{x^m} \prod_{j-1}^{k} (x^m - (j-1)^m). \tag{5.10}$$

Finally, we should note that a theory of hit numbers corresponding to the rook theory for our generalized product formulas has yet to be developed.

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