Formal Power Series and Algebraic Combinatorics
Séries Formelles et Combinatoire Algébrique
San Diego, California 2006

Enumerating Bases of Self-Dual Matroids

Molly Maxwell

Abstract. We define involutively self-dual matroids and prove a relationship between the bases and self-dual bases of these matroids. We use this relationship to prove an enumeration formula for the higher dimensional spanning trees in a class of cell complexes. This gives a new proof of Tutte's theorem that the number of spanning trees of a central reflex is a perfect square and solves a problem posed by Kalai about higher dimensional spanning trees in simplicial complexes. We also give a weighted version of the latter result.

The critical group of a graph is a finite abelian group whose order is the number of spanning trees of the graph. We prove that the critical group of a central reflex is a direct sum of two copies of an abelian group. We conclude with an analogous result in Kalai's setting.

Résumé. Nous définissons la notion de matroide auto-dual par involution et nous démontrons une relation entre les bases et les bases auto-duales de ces matroides. Nous utilisons le relation pour démontrer une formule d'énumération pour les arbres couvrants de dimension supérieure dans une classe de complexes de cellules. Ceci mène à une nouvelle démonstration d'un théorème de Tutte – le nombre d'arbres couvrants d'un *central reflex* est un carré parfait – et résoud un problème posé par Kalai concernant les arbres couvrants de dimension supérieure à 1 de complexes simpliciaux. Nous donnons également une version pondérée de ce dernier résultat.

Le groupe critique d'un graphe est un groupe abélien fini dont l'ordre est le nombre d'arbres couvrants du graphe. Nous prouvons que le groupe critique d'un *central reflex* est la somme directe de deux copies d'un groupe abéliens. Nous concluons avec un résultat analogue dans le cadre posé par Kalai.

1. Introduction

A *matroid* $\mathcal{M}$ is a finite set E along with a collection $\mathcal{I}$ of subsets of E called *independent sets* which satisfy the following conditions:

(1) The empty set $\emptyset$ is in $\mathcal{I}$.
(2) If $I_1 \in \mathcal{I}$ and $I_2 \subseteq I_1$, then $I_2 \in \mathcal{I}$.
(3) If $I_1, I_2 \in \mathcal{I}$ and $|I_2| > |I_1|$, then there exists $e \in I_2 \backslash I_1$ such that $I_1 \cup \{e\} \in \mathcal{I}$.

The *bases* $\mathcal{B}$ of a matroid $\mathcal{M}$ are the maximal independent sets. The bases satisfy the conditions:

(1) $\mathcal{B}$ is non-empty.
(2) If $B_1, B_2 \in \mathcal{B}$ and $e \in B_1 \backslash B_2$, then there exists $e' \in B_2 \backslash B_1$ with
$$(B_1 \backslash \{e\}) \cup \{e'\} \in \mathcal{B}.$$

For a matroid $\mathcal{M}$, its *dual matroid* $\mathcal{M}^\perp$ has bases
$$\mathcal{B}(\mathcal{M}^\perp) := \{E \backslash B : B \in \mathcal{B}(\mathcal{M})\}.$$

Definition 1.1. A matroid $\mathcal{M}$ is said to be *involutively self-dual* if it can be represented by an $n \times 2n$ $\mathbb{Z}$-valued matrix with columns indexed by $E = \{e_1, \ldots, e_n, \tilde{e}_1, \ldots, \tilde{e}_n\}$ of the form

2000 *Mathematics Subject Classification.* Primary 05B35; Secondary 05C05.

Key words and phrases. matroid, Pfaffian, duality, regular cell complex, simplicial complex, simplicial matroid, central reflex, critical group.

This work partially satisfies the requirements for the author's doctoral dissertation at the University of Minnesota, under the supervision of Professor Vic Reiner, and is partially supported by NSF grant DMS-0245379.

$$\begin{array}{c} \begin{array}{cccc|ccc} e_1 & \dots & e_n & & \tilde{e}_1 & \dots & \tilde{e}_n \end{array} \\ M = \left[\begin{array}{c|c} \quad N \quad & \quad I \quad \end{array} \right], \end{array}$$

such that the matrix

$$\begin{array}{c} \begin{array}{cccc|ccc} e_1 & \dots & e_n & & \tilde{e}_1 & \dots & \tilde{e}_n \end{array} \\ M^{\perp} := \left[\begin{array}{c|c} \quad -I \quad & \quad -N \quad \end{array} \right] \end{array}$$

satisfies $\text{Rowspace}(M^{\perp}) = \text{Rowspace}(M)^{\perp}$ (or equivalently $N^T = -N$). In this case, the map $\phi : E \to E$ given by $e_i \mapsto \tilde{e}_i$ is a fixed-point free involution which induces a matroid isomorphism $\mathcal{M} \to \mathcal{M}^{\perp}$.

A basis B is said to be *self-dual* if it contains *exactly* one of e_i and $\tilde{e}_i$ from each pair. Equivalently, B is self-dual if $\phi(E \backslash B) = B$. From the matrix M, we see that $B_0 := \{\tilde{e}_1, \dots, \tilde{e}_n\}$ is a self-dual basis of $\mathcal{M}$.

In this paper, we use the method of Pfaffians to prove the following result.

Theorem 1.2. *If $\mathcal{M}$ is an involutively self-dual matroid, then*

$$\sum_{\text{bases } B \text{ of } M} \det(M|_B)^2 = \left(\sum_{\substack{\text{self-dual} \\ \text{bases } B \text{ of } M}} |\det(M|_B)| \right)^2 .$$

A matrix is *unimodular* if all non-singular square submatrices have determinant ± 1.

Corollary 1.3. If $\mathcal{M}$ is an involutively self-dual matroid and the associated matrix M is unimodular, then the number of bases of $\mathcal{M}$ equals the square of the number of self-dual bases of $\mathcal{M}$.

Theorem 1.4. *Let $\mathcal{M}$ be an involutively self-dual matroid and let A be the concatenated matrix $A := \begin{bmatrix} M \\ M^{\perp} \end{bmatrix}$. Then*

$$\text{coker}(MM^T) \cong \text{coker}(A) \cong H \oplus H,$$

where H is an abelian group of order

$$\sum_{\substack{\text{self-dual} \\ \text{bases } B \text{ of } \mathcal{M}}} |\det(M|_B)|.$$

In Section 3, we show that involutively self-dual matriods arise from cellular $2k$-spheres for k odd that are isomorphic to their duals via the antipodal map. These include the central reflexes studied by Tutte and the boundaries of simplices studied by Kalai. We apply the matroid results above to prove Theorem 1.6, Proposition 1.2 and Theorem 1.9 below.

For a p-dimensional regular cell complex X, the *dual block complex* $D(X)$ of X is a partition of X into disjoint blocks such that every i-cell σ of X is associated to a unique $(p-i)$-block $D(\sigma)$ of $D(X)$. If X is self-dual, then $D(X)$ is a regular cell complex and the blocks $D(\sigma)$ are its cells.

Definition 1.5. Let k be an odd positive integer. An *antipodally self-dual cell complex* X is a regular cell complex such that $|X| = \mathbb{S}^{2k}$ and $a(X) = D(X)$, where $a : \mathbb{S}^{2k} \to \mathbb{S}^{2k}$ is the antipodal map and $D(X)$ is the dual block complex of X.

For each k-cell σ of X, its dual block $D(\sigma)$ is a k-cell in $D(X)$ and its conjugate $\tilde{\sigma}$ is defined by $\tilde{\sigma} := a(D(\sigma))$. The cells σ and $\tilde{\sigma}$ are distinct k-cells of X, and when k is odd, X and $D(X)$ can be oriented in such a way that $\tilde{\tilde{\sigma}} = \sigma$. It follows that the k-cells can be partitioned into n pairs $\{\sigma, \tilde{\sigma}\}$.

Let $\mathcal{T}_k(X)$ be the set of all k-dimensional subcomplexes T of X such that

(1) T contains the $(k-1)$-skeleton of X,
(2) $Z_k(T) = \widetilde{H}_k(T) = 0$,
(3) $\widetilde{H}_{k-1}(T)$ is a finite group.

Complexes in $\mathcal{T}_k(X)$ will be called *k-dimensional spanning trees* of X. A k-dimensional spanning tree T is said to be *self-dual* if it contains *exactly* one of σ_i and $\tilde{\sigma}_i$ from each pair.

PROPOSITION 1.1. *Let k be an odd positive integer. If X is an antipodally self-dual cell complex which contains an acyclic, self-dual spanning tree T_0, then X gives rise to an involutively self-dual matroid.*

We use Proposition 1.1, Theorem 1.2 and Lemma 3.2 to obtain the following result.

THEOREM 1.6. *Let k be an odd positive interger and let X be an antipodally self-dual cell complex which contains an acyclic, self-dual spanning tree T_0. Then*

$$\sum_{T\in\mathcal{T}_k(X)} |\widetilde{H}_{k-1}(T)|^2 = \left(\sum_{\substack{\textit{self-dual}\\ T\in\mathcal{T}_k(X)}} |\widetilde{H}_{k-1}(T)|\right)^2.$$

We next discuss how Theorem 1.6 implies a result of Tutte. A *central reflex* G is an embedding of a connected, directed planar graph on the sphere $\mathbb{S}^2$ with the property that the antipodal map a sends G to an embedding of its planar dual graph G^* on $\mathbb{S}^2$. When $k = 1$, the antipodally self-dual cell complexes are precisely the central reflexes with no loops and no isthmuses. We show that every central reflex G is equivalent to a central reflex G' with no loops and no isthmuses in the sense that G and G' have the same spanning tree numbers and the same critical groups. The dual block complex of a central reflex G is an embedding of the planar dual graph G^* on the sphere $\mathbb{S}^2$. For each edge e, its dual block $D(e)$ is the edge e^* which crosses e in the dual graph and its *conjugate* $\tilde{e}$ is defined by $\tilde{e} := a(e^*)$. A *self-dual spanning tree* is a spanning tree that contains exactly one of e and $\tilde{e}$ from each pair. We let $\mathcal{D}(G)$ denote the number of self-dual spanning trees of G. In [**12**], Tutte uses the theory of electrical networks to prove the following theorem.

THEOREM 1.7. *(Tutte) If G is a central reflex, then the spanning tree number $\kappa(G) = \mathcal{D}(G)^2$.*

In Section 4.1, we show that every central reflex contains a self-dual tree. Theorem 1.6 then gives a new proof of Tutte's theorem.

The critical group of a graph is an abelian group whose order is the number of spanning trees of the graph. We use Theorem 1.4 to prove the following result.

PROPOSITION 1.2. *The critical group of a central reflex G is of the form*

$$K(G) \cong H \oplus H,$$

where H is an abelian group of order $\mathcal{D}(G)$.

Theorem 1.6 also resolves a question posed by Kalai, as we now discuss. Let $\mathcal{T}(n, k)$ be the set of all simplicial complexes T on the vertex set $\{1, 2, \ldots, n\} = [n]$ such that

(1) T has a complete $(k-1)$-skeleton,
(2) T has exactly $\binom{n-1}{k}$ k-faces,
(3) $H_k(T) = 0$.

Complexes in $\mathcal{T}(n, k)$ will be called *k-dimensional spanning trees* on the vertex set $[n]$. To each vertex i we associate a variable x_i. Let $\mathbf{x}^{\deg(T)} := \prod_{i=1}^n x_i^{\deg_T(i)}$, $m_1 := \binom{n-2}{k-1}$, and $m_2 := \binom{n-2}{k}$. Kalai ([**5**, Theorem 1, Theorem 3']) proved the following analogues of Cayley's Theorem and the Cayley-Prüfer Theorem for these k-dimensional trees:

THEOREM 1.8. *(Kalai)*

$$\sum_{T\in\mathcal{T}(n,k)} |H_{k-1}(T, \mathbb{Z})|^2 = n^{\binom{n-2}{k}},$$

and more generally

$$\sum_{T\in\mathcal{T}(n,k)} |H_{k-1}(T, \mathbb{Z})|^2 \, \mathbf{x}^{\deg(T)} = (x_1 + x_2 + \cdots + x_n)^{m_2} \prod_{i=1}^n x_i^{m_1}.$$

The *blocker* or *Alexander dual* of a simplicial complex C is defined by $C^\vee := \{S \subseteq [2k+2] : S^c \notin C\}$. A complex $T \in \mathcal{T}(2k+2,k)$ is said to be *self-dual* if $T^\vee = T$.

In Section 4.2 we show that when k is odd the complete $2k$-dimensional simplicial complex on the vertex set $[2k+2]$ can be embedded on the sphere $\mathbb{S}^{2k}$ in such a way that it forms an antipodally self-dual cell complex X. In this case, $\mathcal{T}_k(X) = \mathcal{T}(2k+2,k)$ and the two descriptions of self-dual trees given above agree.

In [**5**, Problem 3], Kalai posed a problem about the relationship between the trees and the self-dual trees in these complexes. The next result gives a solution to this problem when k is odd. We apply Theorem 1.6 to prove the first assertion. In Section 4.2 we use the method of Pfaffians to prove the second assertion.

Theorem 1.9. *If k is an odd positive integer, then*

$$\left(\sum_{\substack{\textit{self-dual} \\ T\in\mathcal{T}(2k+2,k)}} |\widetilde{H}_{k-1}(T,\mathbb{Z})| \right)^2 = \sum_{T\in\mathcal{T}(2k+2,k)} |\widetilde{H}_{k-1}(T,\mathbb{Z})|^2,$$

and more generally

$$\sum_{\substack{\textit{self-dual} \\ T\in\mathcal{T}(2k+2,k)}} |\widetilde{H}_{k-1}(T,\mathbb{Z})| \, \mathbf{x}^{\deg(T)} = (x_1^2 + x_2^2 + \cdots + x_{2k+2}^2)^{\frac{m_2}{2}} \prod_{i=1}^{2k+2} x_i^{m_1};$$

or in other words,

$$\left(\sum_{\substack{\textit{self-dual} \\ T\in\mathcal{T}(2k+2,k)}} |\widetilde{H}_{k-1}(T,\mathbb{Z})| \, \mathbf{x}^{\deg(T)} \right)^2 = \sum_{T\in\mathcal{T}(2k+2,k)} |\widetilde{H}_{k-1}(T,\mathbb{Z})|^2 \, \mathbf{x}^{\deg(T)} \Bigg|_{x_i \to x_i^2}.$$

Corollary 1.10. If k is an odd positive integer, then

$$\sum_{\substack{\text{self-dual} \\ T\in\mathcal{T}(2k+2,k)}} |\widetilde{H}_{k-1}(T,\mathbb{Z})| = (2k+2)^{\binom{2k-1}{k}}.$$

2. Proofs of Theorems 1.2 and 1.4

Before we begin the proof of Theorem 1.2, we recall that for a skew-symmetric matrix A, the *Pfaffian* of A, $\mathrm{Pf}(A)$, is a polynomial in the entries of A defined, up to a sign, by the formula

$$\mathrm{Pf}(A)^2 = \det(A).$$

More information about the general theory of Pfaffians can be found in [**7**].

Sketch Proof of Theorem 1.2. Since $N^T = -N$, the matrix

$$A := \begin{bmatrix} M \\ M^\perp \end{bmatrix} = \begin{bmatrix} N & I \\ -I & -N \end{bmatrix}$$

is skew-symmetric, and hence $\mathrm{Pf}(A)^2 = \det(A)$. We prove that

$$|\mathrm{Pf}(A)| = |\det(N+I)| \tag{2.1}$$

and that

$$\det(N+I) = \sum_{\substack{\text{self-dual} \\ \text{bases} B \text{ of } \mathcal{M}}} |\det(M|_B)|. \tag{2.2}$$

Then the result follows from the fact that

$$\sum_{\text{bases } B \text{ of } M} \det(M|_B)^2 = \det(A),$$

which comes from generalized Laplace expansion along the first n rows of A, and the relation between complementary minors of M and $M^\perp$.

Proof of (2.1): We begin by noting that

$$PAQ = \begin{bmatrix} I & 0 \\ 0 & N^2 - I \end{bmatrix}, \tag{2.3}$$

where the matrices

$$P = \begin{bmatrix} I & 0 \\ -N & I \end{bmatrix}, Q = \begin{bmatrix} 0 & -I \\ -I & -N \end{bmatrix}$$

both have determinant ± 1. Since N is skew-symmetric, this implies that

$$\pm \det(A) = \det(N+I)(N-I) = \pm \det(N+I)^2$$

where the last equality uses the fact that

$$N - I = -(N^T + I) = -(N+I)^T.$$

Since $\det(A) = \mathrm{Pf}(A)^2$, it follows that

$$|\mathrm{Pf}(A)| = |\det(N+I)|.$$

Proof of (2.2): Now we set $X = N$ and $Y = I$ and use the general fact that if X and Y are $n \times n$ matrices, then

$$\det(X+Y) = \sum_{U \subseteq [n]} \det X_U,$$

where X_U denotes the matrix formed by replacing the columns in X indexed by $U \subseteq [n]$ with the corresponding columns in Y. This formula can be proved using the multilinearity of the determinant and induction.

□

In this paper, we'll let $\mathbb{Z}_d$ denote the cyclic group $\mathbb{Z}/d\mathbb{Z}$.

Proof of Theorem 1.4. In [**6**, Theorem 18], Kuperberg proves that for any skew-symmetric $2n \times 2n$ matrix A, there exists a matrix $B \in GL_{2n}(\mathbb{Z})$ such that B^TAB is a direct sum of matrices of this form:

$$B^TAB = \bigoplus_{i=1}^{r} \begin{bmatrix} 0 & a_i \\ -a_i & 0 \end{bmatrix}.$$

Hence

$$\mathrm{coker}(A) \cong \bigoplus_{i=1}^{r} \mathrm{coker} \begin{bmatrix} 0 & a_i \\ -a_i & 0 \end{bmatrix} \cong \bigoplus_{i=1}^{r} \mathbb{Z}_{a_i}^2 \cong H \oplus H,$$

where $H := \bigoplus_{i=1}^{r} \mathbb{Z}_{a_i}$.

We've shown that $|\mathrm{coker}(A)| = \det(A) = |H|^2$. From the proof of Theorem 1.2, we have

$$\det(A) = \left(\sum_{\substack{\text{self-dual} \\ \text{bases } B \text{ of } M}} |\det(M|_B)| \right)^2,$$

and it follows that

$$|H| = \sum_{\substack{\text{self-dual} \\ \text{bases } B \text{ of } M}} |\det(M|_B)|.$$

□

As one might expect, the matrix N controls the behavior of $\mathrm{coker}(A)$. We make this more precise in the next proposition. Let $\mathrm{Syl}_p(G)$ denote the p-primary component of an abelian group G.

Proposition 2.1. *If a matrix A has the form*

$$A = \begin{bmatrix} N & I \\ -I & -N \end{bmatrix}$$

and is skew-symmetric, then for primes $p \neq 2$,

$$\mathrm{Syl}_p(\mathrm{coker}(A)) \cong \mathrm{Syl}_p(\mathrm{coker}(N+I)) \oplus \mathrm{Syl}_p(\mathrm{coker}(N+I)).$$

Proof. From line (2.3) we have $\mathrm{coker}(A) = \mathrm{coker}(N+I)(N-I)$. We note that $(N+I)-(N-I) = 2I$ and $N - I = -(N^T + I) = -(N+I)^T$. The result then follows from Lemma 2.1 below. □

LEMMA 2.1. *([**4**, Lemma 16], [**1**, Proposition 3.1]) Let G be a finite abelian group, and let α, β be two endomorphisms $G \to G$ satisfying $\beta - \alpha = m \cdot I_G$ for some $m \in \mathbb{Z}$. Then for any prime p that does not divide m, we have*

$$\mathrm{Syl}_p(\mathrm{coker}(\alpha\beta)) \cong \mathrm{Syl}_p(\mathrm{coker}(\alpha)) \oplus \mathrm{Syl}_p(\mathrm{coker}(\beta)).$$

COROLLARY 2.2. The group H in Theorem 1.4 is "almost" $\mathrm{coker}(N+I)$: for primes $p \neq 2$, one has

$$\mathrm{Syl}_p(H) \cong \mathrm{Syl}_p(\mathrm{coker}(N+I)).$$

Example 4.3 below shows that it is necessary to exclude $p = 2$ in the previous corollary.

3. Antipodally Self-Dual Regular Cell Complexes

We begin this section by briefly describing the dual block complex of a regular cell complex X. More information on this topic can be found in [**9**]. The dual block complex $D(X)$ of a p-dimensional regular cell complex X is a partition of X into disjoint blocks. For an i-cell τ in X, its *dual block* $D(\tau)$ is a $(p-i)$-block in $D(X)$. When X is self-dual, $D(X)$ is a regular cell complex and the dual blocks $D(\tau)$ are its cells.

DEFINITION 3.1. Let k be an odd positive integer. An *antipodally self-dual cell complex* X is a regular cell complex such that $|X| = \mathbb{S}^{2k}$ and $a(X) = D(X)$, where $a : \mathbb{S}^{2k} \to \mathbb{S}^{2k}$ is the antipodal map and $D(X)$ is the dual block complex of X.

For each k-cell σ of X, its dual block $D(\sigma)$ is a k-cell in $D(X)$ and its conjugate $\tilde{\sigma}$ is defined by $\tilde{\sigma} := a(D(\sigma))$. The cells σ and $\tilde{\sigma}$ are distinct k-cells of X. When k is odd, we use an inductive argument similar to that in [**9**, Theorem 65.1] to orient X and $D(X)$ in such a way that $\tilde{\tilde{\sigma}} = \sigma$. It follows that the k-cells can be partitioned into n pairs $\{\sigma,\tilde{\sigma}\}$.

Let $\mathcal{T}_k(X)$ be the set of all k-dimensional subcomplexes T of X such that

(1) T contains the $(k-1)$-skeleton of X,
(2) $Z_k(T) = \widetilde{H}_k(T) = 0$,
(3) $\widetilde{H}_{k-1}(T)$ is a finite group.

Complexes in $\mathcal{T}_k(X)$ will be called *k-dimensional spanning trees* of X. A k-dimensional spanning tree T is said to be *self-dual* if it contains *exactly* one of σ_i and $\tilde{\sigma}_i$ from each pair. Equivalently, T is self-dual if $\widetilde{X \backslash T} = \{\tilde{\tau} : \tau \nsubseteq T\} = T$.

For a collection C of k-cells of X, the closure of C is the cell complex defined by $\overline{C} := C \cup X^{(k-1)}$, where $X^{(k-1)}$ denotes the $(k-1)$-skeleton of X. X gives rise to a matroid $\mathcal{M}$ by setting

- $E =$ the set of all k-cells of X,
- $\mathcal{I} =$ collections C of k-cells of X with $Z_k(\overline{C}) = \widetilde{H}_k(\overline{C}) = 0$,
- $\mathcal{B} =$ collections C of k-cells of X with $\overline{C} \in \mathcal{T}_k(X)$.

In the proof of the next proposition, we see that the boundary of each k-cell can be represented as a vector. Then the elements of $\mathcal{I}$ correspond to collections of vectors that are independent over $\mathbb{Z}$ (and hence over $\mathbb{Q}$) and the elements of $\mathcal{B}$ correspond to $\mathbb{Q}$-bases for the span of the vectors.

PROPOSITION 3.1. *Let k be an odd positive integer. If X is an antipodally self-dual cell complex which contains an acyclic, self-dual spanning tree T_0, then X gives rise to an involutively self-dual matroid.*

Then we use Proposition 3.1, Theorem 1.2 and Lemma 3.2 to obtain Theorem 1.6.

SKETCH PROOF OF PROPOSITION 3.1: The k^{th} incidence matrix $I^k(X)$ is the matrix whose rows are labeled by the $(k-1)$-faces of X, whose columns are labeled by the k-faces of X, and whose entries are the incidence numbers

$$\epsilon(\sigma,\tau) = \begin{cases} 0 & \text{if } \sigma \nsubseteq \tau \\ 1 & \text{if } \sigma \subseteq \tau \text{ and } \sigma \text{ is oriented coherently with } \tau \\ -1 & \text{if } \sigma \subseteq \tau \text{ and } \sigma \text{ has the opposite orientation of } \tau. \end{cases}$$

The columnns of $I^k(X)$ represent the boundaries of the k-faces in X. We can order the columns of $I^k(X)$ so it has the form

$$I^k(X) = \begin{array}{cc} & \begin{array}{cc} \overbrace{\tilde{\sigma}_1 \; \dots \; \tilde{\sigma}_n}^{\substack{k\text{-faces} \\ \text{not in } T_0}} & \overbrace{\sigma_1 \; \dots \; \sigma_n}^{\substack{k\text{-faces} \\ \text{in } T_0}} \end{array} \\ \text{(k-1)-faces} \left\{ \begin{array}{c} \tau_1 \\ \vdots \\ \tau_m \end{array} \right. & \left[\begin{array}{c|c} \epsilon(\tau_i, \tilde{\sigma_j}) & \epsilon(\tau_i, \sigma_j) \end{array} \right] \end{array}.$$

There is a one-to-one correspondence between the $(k-1)$-cells and $(k+1)$-cells of X given by $\tau \mapsto \tilde{\tau}$. Thus the transpose of the $(k+1)^{st}$ incidence matrix can be written as

$$I^{k+1}(X)^T = \begin{array}{cc} & \begin{array}{cc} \overbrace{\tilde{\sigma}_1 \; \dots \; \tilde{\sigma}_n}^{\substack{k\text{-faces} \\ \text{not in } T_0}} & \overbrace{\sigma_1 \; \dots \; \sigma_n}^{\substack{k\text{-faces} \\ \text{in } T_0}} \end{array} \\ \text{(k+1)-faces} \left\{ \begin{array}{c} \tilde{\tau}_1 \\ \vdots \\ \tilde{\tau}_m \end{array} \right. & \left[\begin{array}{c|c} \epsilon(\tilde{\sigma_j}, \tilde{\tau_i}) & \epsilon(\sigma_j, \tilde{\tau_i}) \end{array} \right] \end{array}.$$

Again, using an inductive argument as in [**9**, Theorem 65.1], we orient X and $D(X)$ in such a way that $\epsilon(\tau_i, \sigma_j) = \epsilon(\tilde{\sigma_j}, \tilde{\tau_i})$ and $\epsilon(\tau_i, \tilde{\sigma_j}) = \epsilon(\sigma_j, \tilde{\tau_i})$. Thus the matrices $I^k(X)$ and $I^{k+1}(X)^T$ are of the forms

$$\begin{array}{rcl} I^k(X) & = & [\ P \mid Q\ ], \\ I^{k+1}(X)^T & = & [\ Q \mid P\ ]. \end{array} \tag{3.1}$$

We show that there exists a matrix $R \in \mathbb{Z}^{n \times m}$ such that $RI^{k+1}(X)^T = [\ I \mid N\ ]$. We define the *reduced incidence matrices* $I_r^k(X) := RI^k(X)$ and $I_r^{k+1}(X)^T := RI^{k+1}(X)^T$. These matrices are of the forms

$$\begin{array}{rcl} I_r^k(X) & = & [\ N \mid I\ ] \ =: \ M \\ I_r^{k+1}(X)^T & = & [\ I \mid N\ ] \ =: \ M^{\perp}. \end{array}$$

Since $\partial_k\ \partial_{k+1} = 0$, we have $\mathrm{Rowspace}(M)^{\perp} = \mathrm{Rowspace}(M^{\perp})$.

□

When k is even, we can form the matrices $I^k(X)$ and $I^{k+1}(X)^T$ as above. However, our method of orienting X and $D(X)$ now yields $\epsilon(\tau_i, \sigma_j) = \epsilon(\tilde{\sigma_j}, \tilde{\tau_i})$ and $\epsilon(\tau_i, \tilde{\sigma_j}) = -\epsilon(\sigma_j, \tilde{\tau_i})$. Thus the matrices $I^k(X)$ and $I^{k+1}(X)^T$ have the forms

$$\begin{array}{rcl} I^k(X) & = & [\ P \mid Q\ ] \\ I^{k+1}(X)^T & = & [\ Q \mid -P\ ], \end{array}$$

and the reduced incidence matrices $I_r^k(X)$ and $I_r^{k+1}(X)^T$ have the forms

$$\begin{array}{rcl} I_r^k(X) & = & [\ N \mid I\ ] \ =: \ M \\ I_r^{k+1}(X)^T & = & [\ I \mid -N\ ] \ =: \ M^{\perp}. \end{array}$$

With this orientation, X does not give rise to an involutively self-dual matroid and the concatenated matrix $A = \left[\frac{M}{M^{\perp}}\right]$ is symmetric rather than skew-symmetric, so the matroid results do not apply. Of course this does not preclude the possibility that a different method of orienting X and $D(X)$ could yield a version of Theorem 1.6 for even k. However, the fact that certain trees had to be excluded to give a similar formula for simplicial complexes when $k = 2$ makes it seem less promising (see [**5**, page 350]).

We conclude this section with the following analogue of Kalai's Lemma 2 [**5**]. The ideas of this proof are almost exactly the same as those in Kalai's proof.

LEMMA 3.2. *Let $k \geq 1$ and let X be an antipodally self-dual cell complex, which contains an acyclic, self-dual tree T_0. Then for each collection C of k-cells of X we have*

(1) $\det I_r^k(\overline{C}) = 0$ *if and only if* $\widetilde{H}_k(\overline{C}) \neq 0$,
(2) *If* $\widetilde{H}_k(\overline{C}) = 0$, *then* $|\det I_r^k(\overline{C})| = |\widetilde{H}_{k-1}(\overline{C})|$.

PROOF. The proof of (1) is exactly the same as Kalai's proof of [**5**, Lemma 2]. For (2), we first consider the case when $k > 1$. Since $\widetilde{H}_{k-1}(X) = \widetilde{H}_{k-1}(\mathbb{S}^{2k}) = 0$ and $X^{(k-1)} \subseteq \overline{C}$, we have $B_{k-1}(X) = Z_{k-1}(X) = Z_{k-1}(\overline{C})$. The columns of $I^k(X)$ represent $B_{k-1}(X)$, while the columns of $I^k(\overline{C})$ represent $B_{k-1}(\overline{C})$. Hence,

$$\widetilde{H}_{k-1}(\overline{C}) = I^k(X)\mathbb{Z}^{2n}/I^k(\overline{C})\mathbb{Z}^n \cong RI^k(X)\mathbb{Z}^{2n}/RI^k(\overline{C})\mathbb{Z}^n = I_r^k(X)\mathbb{Z}^{2n}/I_r^k(\overline{C})\mathbb{Z}^n \cong \mathbb{Z}^n/I_r^k(\overline{C})\mathbb{Z}^n,$$

where the last congruence uses the fact that $I_r^k(X) = [\ N \mid I\ ]$ contains an $n \times n$ identity matrix.

When $k = 1$, we use part (1) along with the standard facts from graph theory and topology that for a collection C of edges of a graph G

$$\det I_r^1(C) = \begin{cases} \pm 1 & \text{if } C \text{ is a tree} \\ 0 & \text{otherwise,} \end{cases}$$

$$|\widetilde{H}_0(\overline{C})| = \begin{cases} 1 & \text{if } \overline{C} \text{ is connected} \\ \infty & \text{otherwise.} \end{cases}$$

□

4. Applications and Further Results

In this section we first discuss a class of graphs called central reflexes. We apply the results from the previous sections to show that their spanning tree numbers are perfect squares and that their critical groups have a special form. Then we discuss a class of simplicial complexes and apply the previous results to solve a problem that was posed by Kalai (see [**5**, problem 3]).

4.1. Spanning Trees and Critical Groups of Central Reflexes. Central reflexes are a special class of directed, connected self-dual graphs on $\mathbb{S}^2$ for which the graph isomorphism sending G to G^* is the antipodal map $a : \mathbb{S}^2 \to \mathbb{S}^2$. Some examples of central reflexes include odd wheels embedded on $\mathbb{S}^2$. Figure 1 shows a 5-wheel on $\mathbb{S}^2$ and a planar representation of a 5-wheel. Another interesting class of central reflexes arises from squared rectangles and is described in [**11**].

When $k = 1$, the antipodally self-dual cell complexes are precisely the central reflexes with no loops and no isthmuses. The dual block complex $D(G)$ of a central reflex is just an embedding of the planar dual graph G^* on the sphere $\mathbb{S}^{2k}$. For each edge e, its dual block $D(e)$ is the edge e^* in G^* which crosses e and its conjugate $\tilde{e}$ is defined by $\tilde{e} := a(e^*)$. See Figure 1 for some examples of conjugate edges. Central reflexes can be oriented in such a way that the property $\tilde{\tilde{e}} = e$ holds. For each conjugate pair $\{e, \tilde{e}\}$, we arbitrarily orient one edge e. Its dual edge e^* is oriented so that it crosses e from right to left. Then, since $\tilde{e} = a(e^*)$, the orientation of $\tilde{e}$ is determined. Tutte [**12**, (3.4)] proves that the property $\tilde{\tilde{e}} = e$ holds.

A *self-dual spanning tree* of a central reflex G is a spanning tree that contains exactly one edge from each conjugate pair $\{e, \tilde{e}\}$. Equivalently, a spanning tree T is self-dual if $a((E(G)\backslash T)^*) = \{\tilde{e} : e \notin T\} = T$. An example of a self-dual spanning tree is given in Figure 1. We let $\mathcal{D}(G)$ denote the number of self-dual spanning trees of G.

An edge e is a loop in G if and only if e^* is an isthmus in G^*. Since the antipodal map a is a homeomorphism, it follows that e is a loop in G if and only if $\tilde{e}$ is an isthmus in G.

In this paper, we'll let $G\backslash e$ denote deletion of e from G and G/e denote contraction of G on e. Deleting a non-isthmus edge e in G corresponds to contracting its dual edge e^* in G^*. Likewise, contracting a non-loop edge e in G corresponds to deleting its dual edge e^* in G^*. Also, the self-dual spanning trees in $G\backslash\tilde{e}/e$ correspond to the self-dual spanning trees in G that contain e, while the self-dual spanning trees in $G\backslash e/\tilde{e}$ correspond to the self-dual spanning trees in G that contain $\tilde{e}$. Tutte uses these facts to prove the following proposition [**12**, (4.4) and (4.5)].

PROPOSITION 4.1. *If G is a central reflex and e is an edge of G that is neither a loop nor an isthmus, then $G\backslash\tilde{e}/e$ and $G\backslash e/\tilde{e}$ are central reflexes and*

$$\mathcal{D}(G) = \mathcal{D}(G\backslash\tilde{e}/e) + \mathcal{D}(G\backslash e/\tilde{e}).$$

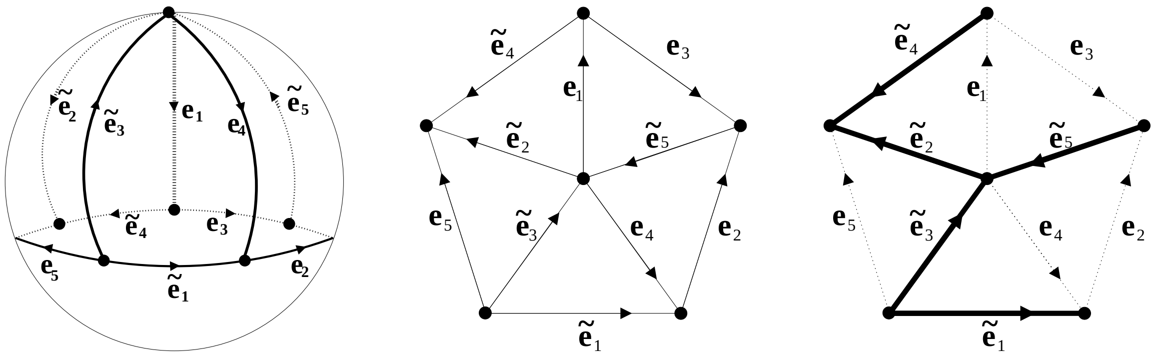

FIGURE 1. An example of a central reflex G on $\mathbb{S}^2$, a planar representation of G, and a self-dual spanning tree T.

We use this proposition and induction on the number of conjugate pairs that are not loop-isthmus pairs to prove the next lemma.

LEMMA 4.1. *If G is a central reflex, then G has at least one self-dual spanning tree.*

In [**12**], Tutte allows loops and isthmuses in central reflexes. The antipodally self-dual cell complexes are regular and hence cannot contain loops and isthmuses. However, every central reflex is equivalent to a regular central reflex in the following sense. Given a central reflex G, let G' be the graph that results from deleting all of the loops and contracting all of the isthmuses. By [**12**, (4.3)], G' is a central reflex. A spanning tree of G contains no loops and contains every isthmus, hence $\kappa(G) = \kappa(G')$.

Since $\widetilde{H}_0(T) = 0$ for any spanning tree T, Theorem 1.6 gives a new proof of Theorem 1.7.

The critical group $K(G)$ of a connected graph G is an abelian group of order $\kappa(G)$. The critical group has several equivalent interpretations. In this paper, we use the form

$$K(G) = \mathbb{Z}^{|E(G)|}/Z_1(G) \oplus B_0(G). \tag{4.1}$$

The formula $\kappa(G) = \mathcal{D}(G)^2$ suggests that the critical group of a central reflex[1] can be written as a direct sum of two copies of a group of order $\mathcal{D}(G)$. Using line (4.1) and Theorem 1.4, we obtain Proposition 1.2.

EXAMPLE 4.2. As noted above, n-wheels are central reflexes when n is odd. For an n-wheel G (with n odd), Biggs [**3**, Theorem 9.2] uses a variation of the chip-firing game to prove that

$$K(G) = \mathbb{Z}_{\ell_n} \oplus \mathbb{Z}_{\ell_n},$$

where ℓ_n is the n^{th} Lucas number.

As we discussed in Section 2, the matrix N controls the behavior of the critical group $K(G) = \operatorname{coker}(A)$. More specifically, Corollary 2.2 states that for $p \neq 2$,

$$\operatorname{Syl}_p(H) \cong \operatorname{Syl}_p(\operatorname{coker}(N + I)),$$

where $\operatorname{Syl}_p(G)$ denote the p-primary component of an abelian group G. The next example demonstrates that it is necessary to exclude $p = 2$ in this corollary.

EXAMPLE 4.3. The double 5-wheel is a central reflex formed by attaching another pentagon to the outside rim of the 5-wheel (see Figure 2). Computing the Smith normal forms of A and $N + I$ gives

$$\operatorname{coker}(A) = (\mathbb{Z}_4)^4 \oplus (\mathbb{Z}_{11})^2 \text{ and thus } H = (\mathbb{Z}_4)^2 \oplus \mathbb{Z}_{11},$$

while

$$\operatorname{coker}(N + I) = (\mathbb{Z}_2)^4 \oplus \mathbb{Z}_{11}.$$

[1]Using the presentation of the critical group $K(G) = \operatorname{coker}\overline{L(G)}$, where $\overline{L(G)}$ denotes the reduced Laplacian matrix, we see that $K(G) = K(G')$. This follows from the fact that deleting a loop has no effect on the Laplacian $L(G)$, while contracting an isthmus corresponds to performing elementary row and column operations on $L(G)$.

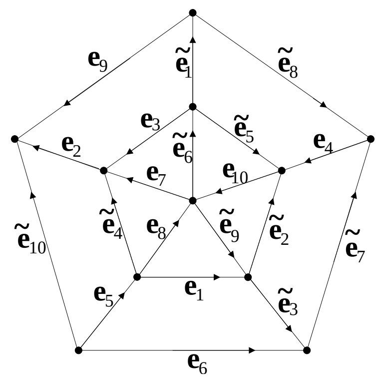

Figure 2. A double 5-wheel.

4.2. Simplicial Complexes. Let $\triangle$ denote the $(2k+1)$-dimensional simplex on the vertex set $V = \{v_0, \dots, v_{2k+2}\}$. We identify the boundary of $\triangle$ with the sphere $\mathbb{S}^{2k}$ in the following way. We first identify $\triangle$ with the $(2k+1)$-simplex in $\mathbb{R}^{2k+1}$ which has vertices

$$v_0 = (0,0,\dots,0), \quad v_1 = (1,0,\dots,0), \quad v_2 = (0,1,\dots,0), \quad \dots, \quad v_{2k+2} = (0,0,\dots,1).$$

We translate $\triangle$ so that its barycenter $\hat{\triangle}$ is at the origin, remove the interior of $\triangle$ and divide the points in the boundary of $\triangle$ by their lengths. Then, for each face F of X, the antipodal map sends $D(F)$ to F^c, i.e. $\tilde{F} = F^c$ when viewed as unoriented cells. When k is odd, X and $D(X)$ can be oriented in such a way that $\tilde{\tilde{F}} = F$ and X is an antipodally self-dual cell complex.

Let $\mathcal{T}(n,k)$ be the set of all simplicial complexes T on the vertex set $\{1,2,\dots,n\} = [n]$ such that

(1) T contains the complete $(k-1)$-skeleton,
(2) T has exactly $\binom{n-1}{k}$ k-faces,
(3) $H_k(T) = 0$.

Let X be the complete $2k$-dimensional complex on the vertex set $[2k+2]$ embedded on $\mathbb{S}^{2k}$. By [**5**, Proposition 2], we see that the definition of $\mathcal{T}(2k+2,k)$ agrees with the definition of $\mathcal{T}_k(X)$. The *blocker* or *Alexander dual* of a simplicial complex C is defined by $C^\vee := \{S \subseteq V : S^c \notin C\}$. A complex $T \in \mathcal{T}(2k+2,k)$ is said to be *self-dual* if $T^\vee = T$. Since

$$\widetilde{X \setminus T} = \{\widetilde{F} : F \notin T\} = \{F^c : F \notin T\} = \{F : F^c \notin T\} = T^\vee,$$

we see that this definition of self-dual complexes agrees with the definition of self-dual trees in Section 3.

Let C be the collection of all k-faces of X that contain vertex 1. We use the fact that vertex 1 is a cone point of $\overline{C}$ to prove the next lemma.

Lemma 4.4. *Let $r := \binom{2k+1}{k} = \binom{2k+1}{k+1}$ and let $C := \{F_1, \dots, F_r\}$ be all of the k-faces of X that contain vertex 1. Then $\overline{C}$ is an acyclic, self-dual spanning tree in $\mathcal{T}_k(X)$.*

Sketch Proof of Theorem 1.9. Combining Lemma 4.4 and Theorem 1.6 gives the proof of the first assertion in Theorem 1.9. We now sketch a proof of the second assertion. Let $\overline{C}$ be the self-dual, acyclic spanning tree from Lemma 4.4. Kalai [**5**, page 342] shows that the reduced incidence matrix $I_r^k(X)$ can be formed from $I^k(X)$ by deleting the rows that correspond to $(k-1)$-faces containing vertex 1. Then $I_r^k(X)$ has rows indexed by the $(k-1)$-faces that don't contain vertex 1 and columns indexed by the k-faces not in $\overline{C}$ followed by the k-faces in $\overline{C}$ and is of the form $[\ N \mid I\ ]$. Also, $I_r^{k+1}(X)^T$ has rows indexed by the $(k+1)$-faces that do contain 1 and columns indexed by the k-faces not in $\overline{C}$ followed by the k-faces in $\overline{C}$ and is of the form $[\ I \mid N\ ]$.

Let A be the concatenated matrix

$$A = \left[\frac{I_r^k(X)}{-I_r^{k+1}(X)^T} \right] = \begin{bmatrix} N & I \\ -I & -N \end{bmatrix}.$$

Since $\partial_k\, \partial_{k+1} = 0$, $\text{Rowspace}(I_r^{k+1}(X)^T) = \text{Rowspace}(I_r^k(X))^\perp$. Thus N and hence A is skew-symmetric.

EXAMPLE 4.5. For $k = 1$, we have

$$A = \begin{array}{c} \\ \begin{matrix} 2 \\ 3 \\ 4 \\ \hline +134 \\ -124 \\ +123 \end{matrix} \end{array} \begin{array}{c} \begin{matrix} -34 & +24 & -23 & +12 & +13 & +14 \end{matrix} \\ \left[\begin{array}{ccc|ccc} & -1 & +1 & +1 & & \\ +1 & & -1 & & +1 & \\ -1 & +1 & & & & +1 \\ \hline -1 & & & & +1 & -1 \\ & -1 & & -1 & & +1 \\ & & -1 & +1 & -1 & \end{array}\right] \end{array}.$$

We associate to each vertex i a weight x_i, and we form a weighted version $A(x)$ of our matrix A by setting

$$A(x)(\tau, \sigma) = A(\tau, \sigma) \cdot x_{\tau \triangle \sigma}.$$

Since $F_j \triangle (F_j \backslash \{1\}) = (\{1\} \cup F_j^c) \triangle F_j^c = 1$, the top right and bottom left blocks of $A(x)$ are $x_1 I$ and $-x_1 I$ respectively. If $n_{ij} \neq 0$, then $F_i \backslash \{1\} \subseteq F_j^c$ and

$$(F_i \backslash \{1\}) \triangle F_j^c = (F_j \backslash \{1\}) \triangle F_i^c.$$

Thus

$$n_{ij} \cdot x_{(F_i \backslash \{1\}) \triangle F_j^c} = -n_{ji} \cdot x_{(F_j \backslash \{1\}) \triangle F_i^c}$$

and it follows that $A(x)$ is skew-symmetric.

EXAMPLE 4.6. For $k = 1$, we have

$$A(x) = \begin{array}{c} \\ \begin{matrix} 2 \\ 3 \\ 4 \\ \hline +134 \\ -124 \\ +123 \end{matrix} \end{array} \begin{array}{c} \begin{matrix} -34 & +24 & -23 & +12 & +13 & +14 \end{matrix} \\ \left[\begin{array}{ccc|ccc} & -x_4 & +x_3 & +x_1 & & \\ +x_4 & & -x_2 & & +x_1 & \\ -x_3 & +x_2 & & & & +x_1 \\ \hline -x_1 & & & & +x_4 & -x_3 \\ & -x_1 & & -x_4 & & +x_2 \\ & & -x_1 & +x_3 & -x_2 & \end{array}\right] \end{array}$$

The ideas of the rest of the proof are very similar to those in Theorem 1.2.

$\square$

In Section 4.1 we discussed the critical groups of graphs. For the complete graph K_n, the critical group has the structure

$$K(K_n) \cong (\mathbb{Z}_n)^{n-2}$$

(see [**3**, Section 8]). The next proposition gives an analogous result for simplicial complexes.

PROPOSITION 4.2. *Let K be the complete k-dimensional simplicial complex on $[n]$ and let $A = \left[\frac{I_r^k(K)}{-I_r^{k+1}(K)^T}\right]$. Then*

$$\operatorname{coker}(A) \cong (\mathbb{Z}_n)^{\binom{n-2}{k}}.$$

The proof of this proposition is divided into three steps:

1. (1) Prove that $\operatorname{coker}(A)$ is all n-torsion.
2. (2) Prove that $\operatorname{coker}(A)$ has a generating set of cardinality $\binom{n-2}{k}$.
3. (3) Finish the proof by using Kalai's result that $\det(A) = \det\left(I_r^k(K) I_r^k(K)^T\right) = n^{\binom{n-2}{k}}$.

We note that in the special case when $n = 2k+2$, Theorem 1.4 takes on the form

$$\operatorname{coker}(A) \cong H \oplus H,$$

where $H = (\mathbb{Z}_{2k+2})^{\binom{2k-1}{k}}$.

References

[1] H. Bai, On the Critical Group of the n-cube, 2003. Linear Algebra Appl., **369**, 251-261.
[2] N. Biggs, Algebraic potential theory on graphs, 1997. Bull. London Math. Soc., **29**, no. 6, 641-682.
[3] N. Biggs, Chip-Firing and the Critical Group of a Graph, 1999. Journal of Algebraic Combinatorics, **9**, 25-45.
[4] B. Jacobson, A. Neidermaier, V. Reiner, Critical groups for complete multipartite graphs and Cartesian products of complete graphs, 2003. J. Graph Theory, **44**, no. 3, 231-250.
[5] G. Kalai, Enumeration of $\mathbb{Q}$-Acyclic Simplicial Complexes, 1983. Israel Journal of Mathematics, **45**, 337-351.
[6] G. Kuperberg, Kasteleyn Cokernels, 2002. Electronic Journal of Combinatorics, **9**, no. 1, R29.
[7] S. Lang, Algebra, Springer-Verlag, 3^{rd} Edition, 2002.
[8] A. Lundell and S. Weingram, The Topology of CW Complexes, Litton Educational Publishing, Inc., 1969.
[9] J. Munkres, Elements of Algebraic Topology, Perseus Books Publishing, L.L.C., 1984.
[10] V. Reiner, Lectures on Matroids and Oriented Matroids, 2005. Available at `http://www.math.umn.edu/~reiner/Talks/Vienna05/index.html`.
[11] C.A.B. Smith and W.T. Tutte, A Class of Self-Dual Maps, 1950. Canadian Journal of Mathematics, **2**, 179-196.
[12] W.T. Tutte, On the Spanning Trees of Self-Dual Maps, 1979. Annals New York Academy of Sciences, 540-548.

SCHOOL OF MATHEMATICS, 206 CHURCH ST. S.E., UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MN 55455
E-mail address: `maxwell@math.umn.edu`