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Classifying ascents and descents with specified equivalences mod k

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ABSTRACT. Given a permutation τ of length j, we say that a permutation σ has a τ-match starting at position i, if the elements in position $i, i+1, \ldots, i+j-1$ in σ have the same relative order as the elements of τ. If Υ is set of permutations of length j, then we say that a permutation σ has an Υ-match starting at position j if it has a τ-match at position j for some $\tau \in \Upsilon$. A number of recent papers have studied the distribution of τ-matches and Υ-matches in permutations. In this paper, we consider a more refined pattern matching condition where we take into account conditions involving the equivalence classes of the elements mod k for some integer $k \geq 2$. In general, when one includes parity conditions or conditions involving equivalence mod k, then the problem of counting the number of pattern matchings becomes more complicated. In this paper, we prove explicit formulas for the number of permutations of n which have s τ-equivalence mod k matches when τ is of length 2. We also show that similar formulas hold for Υ-equivalence mod k matches for certain subsets of permutations of length two.

RÉSUMÉ. Étant donnée une permutation τ de longueur j, on dit qu'une permutation σ a un τ-motif débutant en position i si les éléments en position $i, i+1, \ldots, i+j-1$ de σ ont le même ordre relatif que les éléments de τ. Si Υ est un ensemble de permutations de longueur j, alors on dit que σ a un Υ-motif en position i si σ a un τ-motif en position i pour une permutation τ de Υ. Plusieurs travaux récents ont portés sur la distribution des τ-occurrences et Υ-occurrences dans les permutations. Dans ce travail, nous étudions un raffinement de la notion de motif prenant en compte de conditions basée sur les classes d'équivalences des éléments mod k. De manière générale, lorsque l'on prend en compte la parité ou l'équivalence mod k, le problème de l'énumération du nombre d'occurrences d'un motif devient plus compliqué. Nous démontrons une formule explicite pour le nombre de permutations de n qui ont s τ-motifs équivalents mod k quand τ est de longueur 2. Nous montrons aussi que des formules similaires existent pour les Υ-motifs quand Υ est limité à certains sous-ensembles de permutations de longueur 2.

1. Introduction

Given any sequence $\sigma = \sigma_1 \cdots \sigma_n$ of distinct integers, we let $red(\sigma)$ be the permutation that results by replacing the i-th largest integer that appears in the sequence σ by i. For example, if $\sigma = 2\ 7\ 5\ 4$, then $red(\sigma) = 1\ 4\ 3\ 2$. Given a permutation τ in the symmetric group S_j, we define a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ to have a τ-match at place i provided $red(\sigma_i \cdots \sigma_{i+j-1}) = \tau$. Let τ-$mch(\sigma)$ be the number of τ-matches in the permutation σ. To prevent confusion, we note that a permutation not having a τ-match is different than a permutation being τ-avoiding. A permutation is called τ-avoiding if there are no indices $i_1 < \cdots < i_j$ such that $red[\sigma_{i_1} \cdots \sigma_{i_j}] = \tau$. For example, if $\tau = 2\ 1\ 4\ 3$, then the permutation 3 2 1 4 6 5 does not have a τ-match but it does not avoid τ since $red[2\ 1\ 6\ 5] = \tau$.

In the case where $|\tau| = 2$, then τ-$mch(\sigma)$ reduces to familiar permutation statistics. That is, if $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, let $Des(\sigma) = \{i : \sigma_i > \sigma_{i+1}\}$ and $Rise(\sigma) = \{i : \sigma_i < \sigma_{i+1}\}$. Then it is easy to see that $(2\ 1)$-$mch(\sigma) = des(\sigma) = |Des(\sigma)|$ and $(1\ 2)$-$mch(\sigma) = rise(\sigma) = |Rise(\sigma)|$.

A number of recent publications have analyzed the distribution of τ-matches in permutations. See, for example, [**EN03, Kit03, Kit**]. A number of interesting results have been proved. For example, let τ-nlap(σ)

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be the maximum number of non-overlapping τ-matches in σ where two τ-matches are said to overlap if they contain any of the same integers. Then Kitaev [**Kit03, Kit**] proved the following.

THEOREM 1.1.

$$\sum_{n\geq 0}\frac{t^n}{n!}\sum_{\sigma\in S_n} x^{\tau\text{-nlap}(\sigma)} = \frac{A(t)}{(1-x)+x(1-t)A(t)} \tag{1.1}$$

where $A(t) = \sum_{n\geq 0}\frac{t^n}{n!}|\{\sigma\in S_n : \tau\text{-mch}(\sigma)=0\}|$.

In other words, if the exponential generating function for the number of permutations in S_n without any τ-matches is known, then so is the exponential generating function for the entire distribution of the statistic τ-nlap.

In this paper, we consider a more refined pattern matching condition where we take into account conditions involving equivalence mod k for some integer $k \geq 2$. That is, suppose we fix $k \geq 2$ and we are given some sequence of distinct integers $\tau = \tau_1 \cdots \tau_j$. Then we say that a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ has a τ-k-equivalence match at place i provided $red(\sigma_i \cdots \sigma_{i+j-1}) = red(\tau)$ and for all $s \in \{0, \ldots, j-1\}$, $\sigma_{i+s} = \tau_{1+s}$ mod k. For example, if $\tau = 1\ 2$ and $\sigma = 5\ 1\ 7\ 4\ 3\ 6\ 8\ 2$, then σ has τ-matches starting at positions 2, 5, and 6. However, if $k = 2$, then only the τ-match starting at position 5 is a τ-2-equivalence match. Later, it will be explained that the τ-match starting a position 2 is a (1 3)-2-equivalence match and the τ-match starting a position 6 is a (2 4)-2-equivalence match. Let τ-k-$emch(\sigma)$ be the number of τ-k-equivalence matches in the permutation σ. Let τ-k-$enlap(\sigma)$ be the maximum number of non-overlapping τ-k-equivalence matches in σ where two τ-matches are said to overlap if they contain any of the same integers.

More generally, if Υ is a set of sequences of distinct integers of length j, then we say that a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ has a Υ-k-equivalence match at place i provided there is a $\tau \in \Upsilon$ such that $red(\sigma_i \cdots \sigma_{i+j-1}) = red(\tau)$ and for all $s \in \{0, \ldots, j-1\}$, $\sigma_{i+s} = \tau_{1+s} \mod k$. Let Υ-k-$emch(\sigma)$ be the number of Υ-k-equivalence matches in the permutation σ and Υ-k-$enlap(\sigma)$ be the maximum number of non-overlapping Υ-k-equivalence matches in σ.

In this paper, we shall begin the study of the polynomials

$$T_{\tau,k,n}(x) = \sum_{\sigma\in S_n} x^{\tau\text{-}k\text{-}emch(\sigma)} = \sum_{s=0}^{n} T^s_{\tau,k,n}x^s \text{ and} \tag{1.2}$$

$$U_{\Upsilon,k,n}(x) = \sum_{\sigma\in S_n} x^{\Upsilon\text{-}k\text{-}emch(\sigma)} = \sum_{s=0}^{n} U^s_{\Upsilon,k,n}x^s. \tag{1.3}$$

In particular, we shall focus on certain special cases of these polynomials where we consider only patterns of length 2. That is, fix $k \geq 2$ and let A_k equal the set of all sequences $(a\ b)$ such that $1 \leq a < b \leq 2k$ where there is no lexicographically smaller sequence $x\ y$ having the property that $x \equiv a \mod k$ and $y \equiv b \mod k$. For example,

$$A_4 = \{1\ 2, 1\ 3, 1\ 4, 1\ 5, 2\ 3, 2\ 4, 2\ 5, 2\ 6, 3\ 4, 3\ 5, 3\ 6, 3\ 7, 4\ 5, 4\ 6, 4\ 7, 4\ 8\}.$$

Let $D_k = \{b\ a : a\ b \in A_k\}$ and $E_k = A_k \cup D_k$. Thus E_k consists of all k-equivalence patterns of length 2 that we could possibly consider. Note that if $\Upsilon = A_k$, then Υ-k-$emch(\sigma) = rise(\sigma)$ and if $\Upsilon = D_k$, then Υ-k-$emch(\sigma) = des(\sigma)$.

Our goal is to give explicit formulas for the coefficients of $T^s_{\tau,k,n}$ and $U^s_{\Upsilon,k,n}$. First we shall show that we can use inclusion-exclusion to find a formula for $U^s_{\Upsilon,k,n}$ for any $\Upsilon \subset E_k$ in terms of certain rook numbers of a sequences of boards associated with Υ. While this approach is straightforward, it is unsatisfactory since it reduces the computation of $U^s_{\Upsilon,k,n}$ to another difficult problem, namely, computing rook numbers for general boards. However, we can give two other more direct formulas for the coefficients $T^s_{\tau,k,n}$ where $\tau \in E_k$. For example, in the case where $\tau = (1\ k)$, our results will imply that for all $0 \leq s \leq n$ and for all $0 \leq j \leq k-1$,

$$
\begin{aligned}
(1.4)\ T^s_{(1\ k),k,kn+j} &= \sum_{r=s}^{n} (-1)^{r-s}(kn+j-r)!\binom{r}{s} S_{n+1,n+1-r} \\
&= ((k-1)n+j)!\sum_{r=0}^{s}(-1)^{s-r}((k-1)n+j+r)^n\binom{(k-1)n+j+r}{r}\binom{kn+j+1}{s-r} \\
&= ((k-1)n+j)!\sum_{r=0}^{n-s}(-1)^{n-s-r}(1+r)^n\binom{(k-1)n+j+r}{r}\binom{kn+j+1}{n-s-r}
\end{aligned}
$$

where $S_{n,k}$ is the Stirling number of the second kind, i.e., $S_{n,k}$ is the number of partitions of an n-set into k parts. These formulas lead to interesting identities in their own right. For example, we see that for all $k \geq 2$, $0 \leq s \leq n$ and $0 \leq j \leq k-1$,

$$
\sum_{r=0}^{s}(-1)^{s-r}((k-1)n+j+r)^n\binom{(k-1)n+j+r}{r}\binom{kn+j+1}{s-r} =
\sum_{r=0}^{n-s}(-1)^{n-s-r}(1+r)^n\binom{(k-1)n+j+r}{r}\binom{kn+j+1}{n-s-r}.
$$

The general problem of finding explicit expressions for the coefficients $U^s_{\Upsilon,k,n}$ for arbitrary Υ is open. However, Kitaev and Remmel [**KR05, KR06**] have developed formulas for $U^s_{\Upsilon,k,n}$ in certain other special cases. In particular, Kitaev and Remmel studied permutation statistics which classified the descents of a permutation according to whether either the first element or the second element of a descent pair is equivalent to $0 \mod k$. In our language, they computed explicit formulas for $U^s_{\Upsilon,k,n}$ where either $\Upsilon = \{b\ a : (b\ a) \in D_k\ \&\ b \equiv 0 \mod k\}$ or $\Upsilon = \{b\ a : (b\ a) \in D_k\ \&\ a \equiv 0 \mod k\}$. In this paper, we shall generalize some of their results by deriving explicit formulas for $U^s_{\Upsilon,k,n}$ in the special cases where Υ is a subset of the form $\{(x_1,y_1),(x_2,y_2),\ldots,(x_n,y_n)\}$ where for all i,j $y_i \equiv y_j \mod k$ and either $\Upsilon \subseteq A_k$ or $\Upsilon \subseteq D_k$.

The outline of this paper is as follows. In section 2, we shall discuss some of the previous results of Kitaev and Remmel [**KR05, KR06**] and give some examples of the polynomials $T_{\tau,k,n}(x)$. In section 3, we will show how to one can use inclusion-exclusion to derive an $U_{\Upsilon,k,n}(x)$ in terms of certain rook numbers. In section 4, we shall prove formulas in the case where Υ consists of a sequences of pairs $\{(x_1,y_1),\ldots,(x_t,y_t)\} \subseteq A_k$ such that for all i and j, $y_i = y_j \mod k$. Using the bijection which sends each permutation $\sigma = \sigma_1 \cdots \sigma_n$ to its reverse, $\sigma^r = \sigma_n \cdots, \sigma_1$, one can show that the same formulas hold for $\Upsilon^r = \{(y_1,x_1),\ldots,(y_t,x_t)\} \subseteq D_k$. We shall also see that the identities that result by equating the different formulas for any given coefficient are interesting in their own right. Then, we shall make a few comments about the problem of finding $U_{\Upsilon,k,n}(x)$ for arbitrary Υ.

2. Previous results and examples

In this section, we shall state some previous results and give some examples of the polynomials $T_{\tau,k,n}(x)$ and $U_{\Upsilon,k,n}(x)$. As mentioned in the introduction, Kitaev and Remmel [**KR05, KR06**], found explicit formulas for the coefficients $U^s_{\Upsilon,k,n}$ in certain special cases. In particular, they studied descents according to the equivalence class $\mod k$ of either the first or second element in a descent pair. That is, for any set $X \subseteq \{0,1,2,\ldots\}$, define

- $\overleftarrow{Des}_X(\sigma) = \{i : \sigma_i > \sigma_{i+1}\ \&\ \sigma_i \in X\}$ and $\overleftarrow{des}_X(\sigma) = |\overleftarrow{Des}_X(\sigma)|$
- $\overrightarrow{Des}_X(\sigma) = \{i : \sigma_i > \sigma_{i+1}\ \&\ \sigma_{i+1} \in X\}$ and $\overrightarrow{des}_X(\sigma) = |\overrightarrow{Des}_X(\sigma)|$

In [**KR05**], Kitaev and Remmel studied the following polynomials.

(1) $R_n(x) = \sum_{\sigma \in S_n} x^{\overleftarrow{des}_E(\sigma)} = \sum_{k=0}^{n} R_{k,n}x^k$,
(2) $P_n(x,z) = \sum_{\sigma \in S_n} x^{\overrightarrow{des}_E(\sigma)} z^{\chi(\sigma_1 \in E)} = \sum_{k=0}^{n}\sum_{j=0}^{1} P_{j,k,n}z^jx^k$
(3) $M_n(x) = \sum_{\sigma \in S_n} x^{\overleftarrow{des}_O(\sigma)} = \sum_{k=0}^{n} M_{k,n}x^k$, and
(4) $Q_n(x,z) = \sum_{\sigma \in S_n} x^{\overrightarrow{des}_O(\sigma)} z^{\chi(\sigma_1 \in O)} = \sum_{k=0}^{n}\sum_{j=0}^{1} Q_{j,k,n}z^jx^k$.

where $E = \{0, 2, 4, \ldots, \}$ is the set of even numbers, $O = \{1, 3, 5, \ldots\}$ is the set of odd numbers, and for any statement A, we let $\chi(A) = 1$ is A is true and $\chi(A) = 0$ if A is false. Thus, for example, in our language, $R_n(x) = U_{\Upsilon,2,n}(x)$ where $\Upsilon = \{2\ 1, 4\ 2\}$ and $P_n(x,1) = U_{\Upsilon,2,n}(x)$ where $\Upsilon = \{3\ 2, 4\ 2\}$. In this case, there are some surprisingly simple formulas for the coefficients of this polynomials. For example, Kitaev and Remmel [**KR05**] proved the following.

THEOREM 2.1.

$$R_{k,2n} = \binom{n}{k}^2 (n!)^2, \tag{2.1}$$

$$R_{k,2n+1} = (k+1)\binom{n}{k+1}^2 (n!)^2 + (2n+1-k)\binom{n}{k}^2 (n!)^2 = \frac{1}{k+1}\binom{n}{k}^2((n+1)!)^2, \tag{2.2}$$

$$P_{1,k,2n} = \binom{n-1}{k}\binom{n}{k+1}(n!)^2, \tag{2.3}$$

$$P_{0,k,2n} = \binom{n-1}{k}\binom{n}{k}(n!)^2, \tag{2.4}$$

$$P_{0,k,2n+1} = (k+1)\binom{n}{k}\binom{n+1}{k+1}(n!)^2 = (n+1)\binom{n}{k}^2 (n!)^2, \text{ and} \tag{2.5}$$

$$P_{0,k,2n+1} = \binom{n}{k}(n!)^2\left(n\binom{n-1}{k} + (k+1)\binom{n}{k}\right). \tag{2.6}$$

In [**KR06**], Kitaev and Remmel studied the polynomials

(1) $A_n^{(k)}(x) = \sum_{\sigma \in S_n} x^{\overleftarrow{des}_{kN}(\sigma)} = \sum_{j=0}^{\lfloor \frac{n}{k} \rfloor} A_{j,n}^{(k)} x^j$ and

(2) $B_n^{(k)}(x,z) = \sum_{\sigma \in S_n} x^{\overrightarrow{des}_{kN}(\sigma)} z^{\chi(\sigma_1 \in kN)} = \sum_{j=0}^{\lfloor \frac{n}{k} \rfloor} \sum_{i=0}^{1} B_{i,j,n}^{(k)} z^i x^j$.

where $kN = \{0, k, 2k, \ldots\}$. Again both $A_n^{(k)}(x)$ and $B_n^{(k)}(x,z)$ are special cases of $U_{\Upsilon,k,n}(x)$. When $k \geq 2$, the formulas for $A_n^{(k)}(x)$ and $B_n^{(k)}(x,z)$ become more complicated. Nevertheless, certain nice formulas arise. For example, Kitaev and Remmel [**KR06**] proved the following.

THEOREM 2.2. *For all $0 \leq j \leq k-1$ and all $n \geq 0$, we have*

$$A_{s,kn+j}^{(k)} = ((k-1)n+j)! \sum_{r=0}^{s} (-1)^{s-r} \binom{(k-1)n+j+r}{r}\binom{kn+j+1}{s-r} \prod_{i=0}^{n-1}(r+1+j+(k-1)i) \tag{2.7}$$

$$= ((k-1)n+j)! \sum_{r=0}^{n-s} (-1)^{n-s-r} \binom{(k-1)n+j+r}{r}\binom{kn+j+1}{n-s-r} \prod_{i=1}^{n}(r+(k-1)i) \tag{2.8}$$

In general, when one includes parity conditions or conditions involving equivalence mod k, then the problem of counting the number of pattern matchings become more complicated. For example, if $\tau = 2\ 1$, then the number of permutations of S_n with no τ-matches is 1 since the only permutation of S_n with no (2 1)-matches is the identity permutation $1\ 2\ \cdots\ n-1\ n$. However, according to Theorem 2.1, the number of permutations of S_m with no $\{(2\ 1), (4\ 2)\}$-2-equivalences matches is $(n!)^2$ if $m = 2n$ and is $((n+1)!)^2$ if $m = 2n+1$. Similarly, the analogue of the Kitaev's result (1.1) fails to hold in general. For example, in the case where $k = 2$ and $\tau = 1\ 2$, then (1.4) implies that for $n \geq 1$, $T^0_{(1\ 2),2,2n} = n^n(n!)$ and $T^0_{(1\ 2),2,2n+1} = (n+1)^n((n+1)!)$,

$$A(t) = \sum_{n\geq 0} \frac{t^n}{n!} |\{\sigma \in S_n : (1\ 2)\text{-}2\text{-}emch(\sigma) = 0\}| = 1 + \sum_{n\geq 1} \frac{t^{2n}}{(2n)!} n^n(n!) + \sum_{n\geq 0} \frac{t^{2n+1}}{(2n+1)!}(n+1)^n(n+1)!.$$

Moreover for any $\sigma \in S_n$, $(1\ 2)$-2-$emch(\sigma) = (1\ 2)$-2-$enlap(\sigma)$. But is easy to check that

$$\sum_{n\geq 0} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{(1\ 2)\text{-}2\text{-}emch(\sigma)} \neq \frac{A(t)}{(1-x) + x(1-t)A(t)}.$$

Next we give some examples of our polynomials. Here is a table that lists $T_{(a\ b),k,n}(x)$ for all possible values of a and b where $k=3$ and $2 \le n \le 8$.

$T_{(12),3,2}(x) = 1+x$	$T_{(13),3,2}(x) = 2$	$T_{(14),3,2}(x) = 2$
$T_{(12),3,3}(x) = 4+2x$	$T_{(13),3,3}(x) = 4+2x$	$T_{(14),3,3}(x) = 6$
$T_{(12),3,4}(x) = 18+6x$	$T_{(13),3,4}(x) = 18+6x$	$T_{(14),3,4}(x) = 18+6x$
$T_{(12),3,5}(x) = 54+60x+6x^2$	$T_{(13),3,5}(x) = 96+24x$	$T_{(14),3,5}(x) = 96+24x$
$T_{(12),3,6}(x) = 384+312x+24x^2$	$T_{(13),3,6}(x) = 384+312x+24x^2$	$T_{(14),3,6}(x) = 600+120x$
$T_{(12),3,7}(x) = 3000+1920x+120x^2$	$T_{(13),3,7}(x) = 3000+1920x+120x^2$	$T_{(14),3,7}(x) = 3000+1920x+120x^2$
$T_{(12),3,8}(x) = 15000+20520x+4680x^2+120x^3$	$T_{(13),3,8}(x) = 25920+13680x+720x^2$	$T_{(14),3,8}(x) = 25920+13680x+720x^2$

Glancing at these values, certain things become apparent. First, observe that for each of these polynomials all the coefficients are divisible by the coefficient of the highest power of x appearing in the polynomial. Second, one can observe that polynomials $T_{(ab),3,n}(x)$ depend only on b. Finally, one can also observe that for any given n, the function $T_{(ab),k,n}(x)$ takes at most three distinct values. For example when $n=5$, one can see that all the polynomials $T_{(ab),3,5}(x)$ are equal to one of $T_{(12),3,5}(x)$, $T_{(13),3,5}(x)$, or $T_{(36),3,5}(x)$ and that these three polynomials are distinct. All of these facts are true in general for any k and n since they follow from our closed forms for $T_{(ab),k,n}(x)$.

$T_{(23),3,2}(x) = 2$	$T_{(24),3,2}(x) = 2$	$T_{(25),3,2}(x) = 2$
$T_{(23),3,3}(x) = 4+2x$	$T_{(24),3,3}(x) = 6$	$T_{(25),3,3}(x) = 6$
$T_{(23),3,4}(x) = 18+6x$	$T_{(24),3,4}(x) = 18+6x$	$T_{(25),3,4}(x) = 24$
$T_{(23),3,5}(x) = 96+24x$	$T_{(24),3,5}(x) = 96+24x$	$T_{(25),3,5}(x) = 96+24x$
$T_{(23),3,6}(x) = 384+312x+24x^2$	$T_{(24),3,6}(x) = 600+120x$	$T_{(25),3,6}(x) = 600+120x$
$T_{(23),3,7}(x) = 3000+1920x+120x^2$	$T_{(24),3,7}(x) = 3000+1920x+120x^2$	$T_{(25),3,7}(x) = 4320+720x$
$T_{(23),3,8}(x) = 25920+13680x+720x^2$	$T_{(24),3,8}(x) = 25920+13680x+720x^2$	$T_{(25),3,8}(x) = 25920+13680x+720x^2$

$T_{(34),3,2}(x) = 2$	$T_{(35),3,2}(x) = 2$	$T_{(36),3,2}(x) = 2$
$T_{(34),3,3}(x) = 6$	$T_{(35),3,3}(x) = 6$	$T_{(36),3,3}(x) = 6$
$T_{(34),3,4}(x) = 18+6x$	$T_{(35),3,4}(x) = 24$	$T_{(36),3,4}(x) = 24$
$T_{(34),3,5}(x) = 96+24x$	$T_{(35),3,5}(x) = 96+24x$	$T_{(36),3,5}(x) = 120$
$T_{(34),3,6}(x) = 600+120x$	$T_{(35),3,6}(x) = 600+120x$	$T_{(36),3,6}(x) = 600+120x$
$T_{(34),3,7}(x) = 3000+1920x+120x^2$	$T_{(35),3,7}(x) = 4320+720x$	$T_{(36),3,7}(x) = 4320+720x$
$T_{(34),3,8}(x) = 25920+13680x+720x^2$	$T_{(35),3,8}(x) = 25920+13680x+720x^2$	$T_{(36),3,8}(x) = 25920+13680x+720x^2$

3. Finding the coefficients for $U_{\Upsilon,k,n}(x)$ by inclusion-exclusion

In this section, we shall show how we can use inclusion-exclusion to obtain an expression for $U_{\Upsilon,k,n}(x)$ for any $\Upsilon \subset E_k$. The idea is as follows. Suppose that we fix k and $\Upsilon \subseteq E_k$. Given any two element sequence $ab \in E_k$, we shall write $ab \approxeq xy \mod k$ if (i) $x \equiv a \mod k$, (ii) $y \equiv b \mod k$, (iii) $a<b$ implies $x<y$, and (iv) $a>b$ implies $x>y$. Then for each $n \ge 1$, we let $\Upsilon_n = \{xy : 1 \le x,y \le n \ \&\ xy \approxeq ab \mod k \text{ where } (ab) \in \Upsilon\}$. For each $xy \in \Upsilon_n$, we let $C_{xy,n}$ equal the set of all $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ such that there exist an $1 \le i < n$ such that $\sigma_i = x$ and $\sigma_{i+1} = y$. Given $\sigma \in S_n$, we define $Pr_{\Upsilon,n}(\sigma)$, the property set of σ relative to Υ, to be the set of all $xy \in \Upsilon_n$ such that $\sigma \in C_{xy,n}$. Then we define the following.

(1) For each $T \subseteq \Upsilon_n$, let $E_{=T,\Upsilon,n} = \{\sigma \in S_n : Pr_{\Upsilon,n}(\sigma) = T\}$ and $\beta_{T,\Upsilon,n} = |E_{=T,\Upsilon,n}|$.
(2) For each $T \subseteq \Upsilon_n$, let $E_{\supseteq T,\Upsilon,n} = \{\sigma \in S_n : Pr_{\Upsilon,n}(\sigma) \supseteq T\}$ and $\alpha_{T,\Upsilon,n} = |E_{\supseteq T,\Upsilon,n}|$.
(3) For each $r \ge 0$, let $\beta_{r,\Upsilon,n} = \sum_{S \subseteq \Upsilon_n, |S|=r} \beta_{S,\Upsilon,n}$ and $\alpha_{r,\Upsilon,n} = \sum_{S \subseteq \Upsilon_n, |S|=r} \alpha_{S,\Upsilon,n}$.

It is an easy consequence of the inclusion-exclusion principle that

$$\sum_{t \ge 0} \beta_{t,\Upsilon,n} x^t = \sum_{t \ge 0} \alpha_{t,\Upsilon,n} (x-1)^t. \tag{3.1}$$

It is also easy to see from our definitions that

$$\sum_{t \ge 0} \beta_{t,\Upsilon,n} x^t = U_{\Upsilon,k,n}(x). \tag{3.2}$$

Thus we get an expression for $U_{\Upsilon,k,n}(x)$ by calculating the RHS of (3.1).

Next we observe that it is easy to compute $\alpha_{T,\Upsilon,n}$. We say that $T \subseteq \Upsilon_n$ is consistent if there does not exist distinct ab and cd in T such that either $a = c$ or $b = d$. For example, if $k = 4$ and $\Upsilon = \{12, 34, 32, 46\}$, then $\Upsilon_7 = \{12, 16, 56, 34, 32, 72, 76, 46\}$. Then $T_1 = \{12, 16, 34\}$ and $T_2 = \{12, 32, 76\}$ are not consistent while $T_3 = \{12, 34, 46\}$ is consistent. First we claim that if T is consistent, then $\alpha_{T,\Upsilon,n} = (n - |T|)!$. That is, we need to construct $E_{\supseteq T,\Upsilon,n}$ which consists of all permutations $\sigma \in S_n$ such that each pattern in T occurs consecutively in σ. We do this by first constructing the maximal blocks of elements of $\{1, \ldots, n\}$ where xy occurs consecutively in a block if and only if $xy \in T$. For example, if $n = 7$ and $T = T_3$ as given above, then the maximal blocks constructed from T are 12, 346, 5 and 7. Then it is easy to see that any permutation of the maximal blocks constructed from T corresponds to a permutation $\sigma \in E_{\supseteq T,\Upsilon,n}$. For example, the permutation of the maximal blocks 346 5 12 7 corresponds to the permutation $3\ 4\ 6\ 5\ 1\ 2\ 7 \in E_{\supseteq T_3,\Upsilon,7}$. Now it is easy to see that the number of maximal blocks of $\{1, \ldots, n\}$ constructed from T is $n - |T|$. Thus $\alpha_{T,\Upsilon,n} = |E_{\supseteq T,\Upsilon,n}| = (n - |T|)!$. Of course, if T is inconsistent, there there is no permutation $\sigma \in S_n$ such that all the sequences in T occur consecutively in σ. In this situation, $\alpha_{T,\Upsilon,n} = 0$.

Thus to compute $\alpha_{t,\Upsilon,n}$, we need only count the number of consistent subsets of size t in Υ_n. We can think of this problems as counting the number of rook placements of size t in a certain board associated with Υ_n. That is, given Υ_n, let $B_{\Upsilon,n}$ be the set of all (x, y) such that $xy \in \Upsilon_n$. For example, if $k = 4$ and $\Upsilon = \{12, 34, 32, 46\}$ so that $\Upsilon_7 = \{12, 16, 56, 34, 32, 72, 76, 46\}$, then $B_{\Upsilon,7}$ consists of the shaded squares on the board pictured in Figure 1.

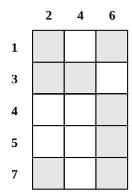

FIGURE 1. The board $B_{\Upsilon,7}$.

Given any board $B \subseteq \{1, \ldots, n\} \times \{1, \ldots, n\}$, we let $r_k(B)$ denote the number of placements of k rooks in B such that no two rooks lie in the same row or the same column. It is then easy to see that number of consistent subsets of size t in Υ_n equals $r_t(B_{\Upsilon,n})$ and thus, $\alpha_{t,\Upsilon,n} = (n - t)! r_t((B_{\Upsilon,n})$. It follows that

$$\begin{aligned} U_{\Upsilon,k,n}(x) &= \sum_{t\geq 0} \beta_{t,\Upsilon,n} x^t = \sum_{t\geq 0} \alpha_{t,\Upsilon,n}(x-1)^t \\ &= \sum_{t\geq 0} (n-t)! r_t(B_{\Upsilon,n}) \sum_{s=0}^{t} (-1)^{t-s} \binom{t}{s} x^s = \sum_{s\geq 0} x^s \sum_{t=s}^{n} (n-t)!(-1)^{t-s} \binom{t}{s} r_t(B_{\Upsilon,n}). \end{aligned}$$

The problem with formula (3.3) is that we obtain an expression for the coefficients of $U_{\Upsilon,k,n}(x)$ in terms of the numbers $r_t(B_{\Upsilon,n})$ which are not easy to compute in general. There are however some special cases of (3.3) where the numbers $r_t(B_{\Upsilon,n})$ are familiar. That is, suppose $\Upsilon = \{(1k)\}$. Then it is easy to see that $B_{\Upsilon,kn+j}$ consists of the set of squares $\{(1 + ik, jk) : 0 \leq i < j \leq n\}$. For example, if $k = 3$ and $\Upsilon = \{(13)\}$, then $B_{\Upsilon,12}$ consists of the shaded squares on the board pictured in Figure 2.

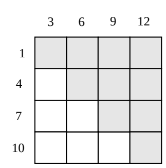

FIGURE 2. The board $B_{\{13\},12}$.

It is well known that the Stirling number of the second kind, $S_{n+1,k}$, is the number of placements of $n + 1 - k$ rooks on the staircase board, consisting of columns of heights $0, 1, \ldots, n$ reading from right to left, so that no two rooks lie in the same row or column. It then easily follows that

$$T^s_{(1k),k,kn+j} = U^s_{\{(1k)\},k,kn+j} = \sum_{r=s}^{n} (-1)^{r-s} \binom{r}{s} (kn+j-r)! S_{n+1,n+1-r}. \tag{3.3}$$

Another case that involves the Stirling numbers is when $\Upsilon = D_k$. As pointed out in the introduction, in that case, Υ-k-$emch(\sigma) = des(\sigma)$. In this case the board the $B_{\Upsilon,n}$ equals $\{(j,i) : 0 \leq i < j \leq n\}$ which is equivalent to a staircase board with column heights $0, 1, \ldots, n-1$.

It is also well known that the Eulerian numbers, $E_{m,n}$ counts the number of permutations in S_m that have exactly n descents. Thus we can derive the following formula for the Eulerian numbers in terms of the Stirling numbers.

$$E_{n,s} = U^s_{A_k,k,n}(x) = \sum_{r=s}^{n} (-1)^{r-s} \binom{r}{s} (n-r)! S_{n,n-r}. \tag{3.4}$$

In some other cases, we have been able to derive formulas that involve sums over products of Stirling numbers. In such cases, the board $B_{\Upsilon,n}$ naturally breaks up as a disjoint union of staircase boards. However, because of lack of space, we shall not give such examples in this paper.

4. Finding the coefficients of $U_{\Upsilon,k,n}$ by iterating recursions

In this section, we shall give an alternative approach to finding the $U_{\Upsilon,k,n}$ that exploits the fact that we can find simple recursion for the polynomials $U_{\Upsilon,k,n}$.

Given any permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, we label with the integers from 0 to n (from left to right) the possible positions of where we can insert $n+1$ to get a permutation in S_{n+1}. In other words, inserting $n+1$ in position 0 means that we insert $n+1$ at the beginning of σ and for $i \geq 1$, inserting $n+1$ in position i means we insert $n+1$ immediately after σ_i. In such a situation, we let $\sigma^{(i)}$ denote the permutation of S_{n+1} that results by inserting $n+1$ in position i.

Throughout the rest of this section, we shall assume that $k \geq 2$ and $\Upsilon \subseteq A_k$ is a subset of the form $\{(x_1,y_1),(x_2,y_2),\ldots,(x_t,y_t)\}$ where for all i,j $y_i \equiv y_j \mod k$. Now, define $y = min(\{y_1,\ldots,y_t\})$ and $\alpha = |\{x_i : x_i < y\}|$. We then let $Asc_{\Upsilon,k}(\sigma) = \{i : \sigma_i < \sigma_{i+1}\ \&\ \sigma_i \equiv x_j \mod k\ \&\ \sigma_{i+1} \equiv y_j \mod k$ for some $(x_j,y_j) \in \Upsilon\}$. We shall call the elements of $Asc_{\Upsilon,k}(\sigma)$ the Υ-ascents of σ.

For $j = y-k+1,\ldots,y-1$, let Δ_{kn+j} be the operator which sends x^s to $sx^{s-1} + (kn+j-s)x^s$ and Γ_{kn+y} be the operator that sends x^s to $((k-|\Upsilon|)n+y+s-\alpha)x^s + (|\Upsilon|n+\alpha-s)x^{s+1}$. Then we have the following.

Theorem 4.1. *Given Υ, y, and α as described above, the polynomials $\{U_{\Upsilon,k,n}(x)\}_{n\geq 1}$ satisfy the following recursions.*

1. $U_{\Upsilon,k,1}(x) = 1$,
2. *For* $j = y-k+1,\ldots,y-1$, $U_{\Upsilon,k,kn+j}(x) = \Delta_{kn+j}(U_{\Upsilon,k,kn+j-1}(x))$, *and*
3. $U_{\Upsilon,k,kn+y}(x) = \Gamma_{kn+y}(U_{\Upsilon,k,kn+y-1}(x))$.

Proof. Part (1) is trivial.

For part (2), fix j such that $y-k+1 \leq j \leq y-1$. Now suppose $\sigma = \sigma_1 \cdots \sigma_{kn+j-1} \in S_{kn+j-1}$ and $asc_{\Upsilon,k}(\sigma) = s$. It is then easy to see that if we insert $kn+j$ in position i where $i \in Asc_{\Upsilon,k}(\sigma)$, then $asc_{\Upsilon,k}(\sigma^{(i)}) = s-1$. However, if we insert $kn+j$ in position i where $i \notin Asc_{\Upsilon,k}(\sigma)$, then $asc_{\Upsilon,k}(\sigma^{(i)}) = s$. Thus $\{\sigma^{(i)} : i = 0,\ldots,kn+j-1\}$ gives a contribution of $sx^{s-1} + (kn+j-s)x^s$ to $U_{\Upsilon,k,kn+j}$.

For part (3), suppose $\sigma = \sigma_1 \cdots \sigma_{kn+y-1} \in S_{kn+y-1}$ and $asc_{\Upsilon,k}(\sigma) = s$. In this situation we can create a Υ-ascent, but we can't lose one. That is, if we place $kn+y$ after any element equivalent to $x_i \mod k$ for some $(x_i,y_i) \in \Upsilon$ which isn't already part of a Υ-ascent, we would create an additional Υ-ascent. There are $|\Upsilon|n+\alpha-s$ such locations. This means that the number of locations that keep the number of ascents the same must be $(k-|\Upsilon|)n+y+s-\alpha$ as the two must sum to $kn+y$. Thus $\{\sigma^{(i)} : i = 0,\ldots,kn+y-1\}$ gives a contribution of $((k-|\Upsilon|)n+y+s-\alpha)x^s + (|\Upsilon|n+\alpha-s)x^{s+1}$ to $U_{\Upsilon,k,kn+y}$. □

We can give combinatorial proofs of two simple formulas for the extreme coefficients of $U_{\Upsilon,k,n}(x)$.

Theorem 4.2. *Let Υ, y, and α be as described above. Then for all $k \geq 2$, for all $j = y-k, \ldots, y-1$ and n such that $kn+j > 0$,*

$$U^0_{\Upsilon,k,kn+j} = ((k-1)n+j)! \prod_{i=0}^{n-1} (k-1)n+j+1-\alpha-i(|\Upsilon|-1) \tag{4.1}$$

$$U^n_{\Upsilon,k,kn+j} = ((k-1)n+j)! \prod_{i=0}^{n-1} \alpha+i(|\Upsilon|-1) \tag{4.2}$$

Proof. Clearly when $n=0$, the only $j \in \{y-k, \ldots, y-1\}$ such that $kn+j>0$ are $j = 1, \ldots y-1$. In these cases, no permutation σ of S_j can have an Υ-k- equivalence match so that $U_{\Upsilon,k,j}(x) = j!$. By convention, we assume the empty product is equal to 1 so that our formulas holds when $n=0$.

Next assume that $n \geq 1$ and $\Upsilon = \{(x_i, y_i) : i = 1, \ldots t\}$ where $x_1, \ldots x_\alpha$ consist of those x_i's such that $(x_i, y) \in \Upsilon$. Suppose that $j \in \{y-k, \ldots, y-1\}$.

First we consider those permutations $\sigma \in S_{kn+j}$ such that Υ-k-*emch*$(\sigma) = 0$. We claim that we can construct all such σ as follows. By our definition, there are $(k-1)n+j$ elements in $\{1, \ldots, kn+j\}$ which are not equivalent to $y \mod k$. We can arrange these elements in $((k-1)n+j)!$ ways. Given an arrangement τ of the elements in $\{1, \ldots, kn+j\}$ which are not equivalent to $y \mod k$, we can extend τ to a permutation $\sigma \in S_{kn+j}$ such that Υ-k-*emch*$(\sigma) = 0$ as follows. First we can insert y into τ so that we do not create any Υ-k-equivalence matches. Clearly this can be done in $(k-1)n+j+1-\alpha$ ways since all we have to do is to ensure that we do not insert y immediately after any of $x_1, \ldots x_\alpha$. Now suppose τ_1 is a sequence that results from inserting y into τ so that we do not create any Υ-k-equivalence matches. Then, the number of ways to insert $y+k$ into τ_1 so that we do not create any Υ-k-equivalence matches is $(k-1)n+j+1-\alpha-(|\Upsilon|-1)$. That is there are $(k-1)n+j+2$ possible ways to insert $y+k$ into τ_1 but that are $\alpha+|\Upsilon|$ elements z such that if we insert $y+k$ after z, then we would form an Υ-k-equivalence match. Now suppose τ_2 is a sequence that results from inserting $y+k$ into τ_1 so that we do not create any Υ-k-equivalence matches. Then, the number of ways to insert $y+2k$ into τ_2 so that we do not create any Υ-k-equivalence matches is $(k-1)n+j+1-\alpha-2(|\Upsilon|-1)$. That is there are $(k-1)n+j+3$ possible ways to insert $y+2k$ into τ_2 but that are $\alpha+2|\Upsilon|$ elements z such that if we insert $y+2k$ after z, then we would form an Υ-k-equivalence match. Continuing on in this way, we see that $U^0_{\Upsilon,k,kn+j} = ((k-1)n+j)! \prod_{i=0}^{n-1} (k-1)n+r+j+1-\alpha-i(|\Upsilon|-1)$.

Next we consider those permutations $\sigma \in S_{kn+j}$ such that Υ-k-*emch*$(\sigma) = n$. We claim that we can construct all such σ as follows. By our definition, there are $(k-1)n+j$ elements in $\{1, \ldots, kn+j\}$ which are not equivalent to $y \mod k$. We can arrange these elements in $((k-1)n+j)!$ ways. Given an arrangement τ of the elements in $\{1, \ldots, kn+j\}$ which are not equivalent to $y \mod k$, we can extend τ to a permutation $\sigma \in S_{kn+j}$ such that Υ-k-*emch*$(\sigma) = n$ as follows. Clearly, we must insert $y, y+k, \ldots, y+(n-1)k$ in such a way that each of these elements create an Υ-k-equivalence match. Thus we must insert y into τ so that it immediately follows one of $x_1, \ldots, x_\alpha$. Hence we have α ways to insert y. Now suppose τ_1 is a sequence that results from inserting y into τ so that we did create a Υ-k-equivalence match. Then the number of ways to insert $y+k$ into τ_1 so that we create another Υ-k-equivalence match is $\alpha+(|\Upsilon|-1)$ since there $\alpha+|\Upsilon|$ elements $x < y+k$ such that $(x\ (y+k))$ would be an Υ-k-equivalence match and we can not insert $y+k$ immediately before y. Now suppose τ_2 is a sequence that results from inserting $y+k$ into τ_1 so that we have created a second Υ-k-equivalence match. Then the number of ways to insert $y+2k$ into τ_2 so that we create an additional Υ-k-equivalence matches is $\alpha = 2(|\Upsilon|-1)$ since there $\alpha+2|\Upsilon|$ elements $x < y+k$ such that $(x\ (y+2k))$ would be an Υ-k-equivalence match and we can not insert $y+2k$ immediately before y or $y+2k$. Continuing on in this way, we see that $U^n_{\Upsilon,k,kn+j} = ((k-1)n+j)! \prod_{i=0}^{n-1} \alpha+i(|\Upsilon|-1)$. $\square$

This given, we can derive a general formula $U^s_{\Upsilon,k,n}$ using the recursions implicit in Theorem 4.1. It is easy to see from Theorem 4.1 that we have two following recursions for the coefficients $U^s_{\Upsilon,k,n}$.

For $y-k+1 \leq j \leq y-1$,

$$U^s_{\Upsilon,k,kn+j} = (kn+j-s)U^s_{\Upsilon,k,kn+j-1} + (s+1)U^{s+1}_{\Upsilon,k,kn+j-1} \tag{4.3}$$

Similarly, we have

$$U^s_{\Upsilon,k,kn+y} = ((k-|\Upsilon|)n+y-\alpha+s)U^s_{\Upsilon,k,kn+y-1} + (|\Upsilon|n+\alpha-s+1)U^{s-1}_{\Upsilon,k,kn+y-1} \tag{4.4}$$

We will now turn to a closed form for $U^s_{\Upsilon,k,kn+j}$. This formula was obtained by using (4.3) and iterating these recursions from the bottom up.

THEOREM 4.3. *For all $y-k \le j \le y-1$ and all $s \le n$ such that $kn+j>0$, we have*

$$U^s_{\Upsilon,k,kn+j} = ((k-1)n+j)!\left[\sum_{r=0}^{s}(-1)^{s-r}\binom{(k-1)n+r+j}{r}\binom{kn+j+1}{s-r}\Gamma(r,j,n)\right]$$

$$\text{where } \Gamma(r,j,n) = \prod_{i=0}^{n-1}((k-1)n+r+j+1-\alpha-i(|\Upsilon|-1))$$

PROOF. We shall prove by induction, first on s, and then n that our formulas hold. That is, by Theorem 4.2, our formulas hold when $s=0$ for all $n \ge 0$ and and $y-k \le j \le y-1$ if $kn+j>0$. Next assume that our formulas satisfy the recursions (4.3) and (4.4), which we will verify later in the proof. Then, we can complete the induction as follows. First assume that that our formulas hold at some s for all $n \ge s$ and $y-k \le j \le y-1$ if $kn+j>0$. Note that the recursions (4.3) and (4.4) can be rewritten as

$$U^{s+1}_{\Upsilon,k,kn+j-1} = \frac{1}{s+1}(U^s_{\Upsilon,k,kn+j} - (kn+j-s)U^s_{\Upsilon,k,kn+j-1}), \tag{4.5}$$

for $y-k+1 \le j \le y-1$, and

$$U^{s+1}_{\Upsilon,k,kn+y-1} = \frac{1}{((k-|\Upsilon|)n+y-\alpha+s+1)}(U^{s+1}_{\Upsilon,k,kn+y} - (|\Upsilon|+\alpha-s)U^s_{\Upsilon,k,kn+y-1}) \tag{4.6}$$

Thus in particular, (4.5) implies our formulas hold at $s+1$ when $n \ge s+1$ and $j = y-k, \ldots, y-2$. We are then able to use (4.6) to establish that our formula holds at $s+1$ when $n \ge s+1$ and $j = y-1$.

Thus to complete our proof, we need only verify that our formulas satisfy the recursions (4.3) and (4.4). In order to simplify the algebra, we will convert the form from (4.5) to the following

$$U^s_{\Upsilon,k,kn+j} = \sum_{r=0}^{s}\frac{(-1)^{s-r}((k-1)n+r+j)!(kn+j+1)!\Gamma(r,j,n)}{(kn+j-s+r+1)!r!(s-r)!}. \tag{4.7}$$

So, for $y-k+1 \le j \le y-1$ plugging in the above form into the RHS of (4.3) gives

$$\begin{aligned}&(kn+j-s)\left[\sum_{r=0}^{s}\frac{(-1)^{s-r}((k-1)n+r+j-1)!(kn+j)!\Gamma(r,j-1,n)}{(kn+j-s+r)!r!(s-r)!}\right]\\ &+(s+1)\left[\sum_{r=0}^{s+1}\frac{(-1)^{s+1-r}((k-1)n+r+j-1)!(kn+j)!\Gamma(r,j-1,n)}{(kn+j-s+r-1)!r!(s+1-r)!}\right]\end{aligned}$$

Removing the $s+1$ term from the second summand, recognizing that $\Gamma(r,j-1,n) = \Gamma(r-1,j,n)$ and combining the rest of the terms yields

$$\sum_{r=0}^{s}\frac{(-1)^{s-r}((k-1)n+r+j-1)!(kn+j)!\Gamma(r-1,j,n)\left[-r(kn+j+1)\right]}{(kn+j-s+r)!r!(s+1-r)!} + \frac{((k-1)n+s+j)!\Gamma(s+1,j-1,n)}{s!}$$

Since there is a factor of r in the numerator, we may omit the $r=0$ term from the summand, shift indices and recognize that $\Gamma(s+1,j-1,n) = \Gamma(s,j,n)$ to get

$$\begin{aligned}&\sum_{r=0}^{s-1}\frac{(-1)^{s-r}((k-1)n+r+j)!(kn+j+1)!\Gamma(r,j,n)}{(kn+j-s+r+1)!r!(s-r)!} + \frac{((k-1)n+s+j)!\Gamma(s,j,n)}{s!}\\ &= \sum_{r=0}^{s}\frac{(-1)^{s-r}((k-1)n+r+j)!(kn+j+1)!\Gamma(r,j,n)}{(kn+j-s+r+1)!r!(s-r)!} = U^s_{\Upsilon,k,kn+j}\end{aligned}$$

Thus we have shown that our formula for $U^s_{\Upsilon,k,kn+j}$ satisfies (4.3) for $y-k+1 \le j \le y-1$. We will now show that our formula satisfies (4.4). The RHS of (4.4) becomes

$$\begin{aligned}&((k-|\Upsilon|)n+s+y-\alpha)\left[\sum_{r=0}^{s}\frac{(-1)^{s-r}((k-1)n+r+y-1)!(kn+y)!\Gamma(r,y-1,n)}{(kn+y-s+r)!r!(s-r)!}\right]\\ &+(|\Upsilon|n+\alpha-s+1)\left[\sum_{r=0}^{s-1}\frac{(-1)^{s-r-1}((k-1)n+r+y-1)!(kn+y)\Gamma(r,y-1,n)!}{(kn+y-s+r+1)!r!(s-r-1)!}\right]\end{aligned}$$

Removing the s term from the first summand, and combining the rest of the terms yields

$$
\begin{aligned}
& \sum_{r=0}^{s-1} \frac{(-1)^{s-r}((k-1)n+r+y-1)!(kn+y)!\Gamma(r,y-1,n)\left[(kn+y+1)(kn-n|\Upsilon|+r+y-\alpha)\right]}{(kn+y-s+r+1)!r!(s-r)!} \\
& + \frac{((k-|\Upsilon|)n+s+y-\alpha)((k-1)n+y+s-1)!\Gamma(s,y-1,n)}{s!} \\
= & \sum_{r=0}^{s-1} \frac{(-1)^{s-r}((k-1)n+r+y-1)!(kn+y+1)!\Gamma(r,y-1,n)}{(kn+y-s+r+1)!r!(s-r)!} \\
& + \frac{((k-|\Upsilon|)n+s+y-\alpha)((k-1)n+y+s-1)!\Gamma(s,y-1,n)}{s!} \\
= & \sum_{r=0}^{s} \frac{(-1)^{s-r}((k-1)n+r+y-1)!(kn+y+1)!\Gamma(r,y-1,n)((k-|\Upsilon|)n+r+y-\alpha)}{(kn+y-s+r+1)!r!(s-r)!} \\
= & \sum_{r=0}^{s} \frac{(-1)^{s-r}((k-1)n+r+y-1)!(kn+y+1)!\Gamma(r,y-k,n+1)}{(kn+y-s+r+1)!r!(s-r)!} = U^s_{\Upsilon,k,kn+y}
\end{aligned}
$$

Thus we have shown that our formula for $U^s_{\Upsilon,k,kn+j}$ satisfies (4.4) as desired. □

Here is another formula for $U^s_{\Upsilon,k,kn+j}$. This one was obtained by iterating the recursions (4.3) and (4.4) from the top down.

THEOREM 4.4. *For all $y-k \le j \le y-1$ and all $s \le n$ such that $kn+j>0$, we have*

$$
U^{n-s}_{\Upsilon,k,kn+j} = ((k-1)n+j)!\left[\sum_{r=0}^{s}(-1)^{s-r}\binom{(k-1)n+r+j}{r}\binom{kn+j+1}{s-r}\Omega(r,n)\right] \tag{4.8}
$$

$$
\text{where } \Omega(r,n) = \prod_{i=0}^{n-1}(r+\alpha+i(|\Upsilon|-1)).
$$

PROOF. We shall prove by induction, first on s and then on $kn+j$ that our formulas hold. Theorem 4.2 proves our formulas hold when $s=0$ for all $n \ge 0$ and $y-k \le j \le y-1$ such that $kn+j>0$. Now assume that our formulas for $U^{n-s}_{\Upsilon,k,kn+j}$ satisfy the the recursions, (4.3) and (4.4), which we will verify later in the proof. Then, we can complete our induction as follows. Assume that our formulas for $U^{n-s}_{\Upsilon,k,kn+j}$ hold at s for all $n \ge s$ and and $y-k \le j \le y-1$ such that $kn+j>0$. Then, the recursions can be rewritten as

$$
U^{n-(s+1)}_{\Upsilon,k,kn+j} = (kn+j-n+s+1)U^{n-(s+1)}_{\Upsilon,k,kn+j-1} + (n-s)U^{n-s}_{\Upsilon,k,kn+j-1} \tag{4.9}
$$

for $y-k+1 \le j \le y-1$, and

$$
U^{(n+1)-(s+1)}_{\Upsilon,k,k(n+1)+y-k} = ((k-|\Upsilon|)n+y-\alpha+n-s)U^{n-s}_{\Upsilon,k,kn+y-1}) + (|\Upsilon|+\alpha-n+s+1)U^{n-(s+1)}_{\Upsilon,k,kn+y-1} \tag{4.10}
$$

It is easy to see that the recursions (4.10) and (4.10) will allow us to prove our formulas hold for $U^{n-(s+1)}_{\Upsilon,k,kn+j}$, for all $n \ge s+1$ and $y-k \le j \le y-1$ such that $kn+j>0$, by induction on $kn+j$ so long as we can prove a base case. In the base case, we can prove the recursion

$$
U^{(s+1)-(s+1)}_{\Upsilon,k,k(n+1)+y-k} = (k-|\Upsilon|)n+y-\alpha+s-s)U^{s-s}_{\Upsilon,k,kn+y-1} + (|\Upsilon|+\alpha-s+s+1)U^{s-(s+1)}_{\Upsilon,k,kn+y-1} \tag{4.11}
$$

if we interpret each term in the sense of the RHS of (4.8). The problem is that our formulas make sense even in the case

$$
U^{s-(s+1)}_{\Upsilon,k,kn+y-1} = ((k-1)n+y-1)!\left[\sum_{r=0}^{s+1}(-1)^{s+1-r}\binom{(k-1)n+r+y-1}{r}\binom{kn+y}{s+1-r}\Omega(r,s)\right]. \tag{4.12}
$$

However, by our definitions, it must be the case that $U^{s-(s+1)}_{\Upsilon,k,kn+y-1} = U^{-1}_{\Upsilon,k,kn+y-1} = 0$. Thus in order to establish the base case, we need an independent proof that the RHS of (4.12) is 0. In fact, we can prove much more. That is, we can give a direct combinatorial proof that

$$
U^{n+1}_{\Upsilon,k,kn+j} = ((k-1)n+j)!\left[\sum_{r=0}^{n+1}(-1)^{n+1-r}\binom{(k-1)n+r+j}{r}\binom{kn+j+1}{n+1-r}\Omega(r,n)\right] = 0
$$

for any $y-k \leq j \leq y-1$. We will not give this combinatorial proof here due to lack of space.

Thus to complete our induction, we need only show that our formulas satisfy the recursions (4.3) and (4.4). In order to simplify the algebra, we will again convert the form from (4.8) to the following

$$U^s_{\Upsilon,k,kn+j} = \sum_{r=0}^{n-s} \frac{(-1)^{n-s-r}((k-1)n+r+j)!(kn+j+1)!\Omega(r,n)}{((k-1)n+j+s+r+1)!r!(n-s-r)!}$$

So, for $y-k+1 \leq j \leq y-1$ plugging in the above form into the RHS of (4.3) gives

$$\begin{aligned}
&(kn+j-s)\left[\sum_{r=0}^{n-s} \frac{(-1)^{n-s-r}((k-1)n+r+j-1)!(kn+j)!\Omega(r,n)}{((k-1)n+j+s+r)!r!(n-s-r)!}\right] \\
&+(s+1)\left[\sum_{r=0}^{n-s-1} \frac{(-1)^{n-s-r-1}((k-1)n+r+j-1)!(kn+j)!\Omega(r,n)}{((k-1)n+j+s+r+1)!r!(n-s-r-1)!}\right]
\end{aligned}$$

Removing the $n-s$ term from the first summand, and combining the rest of the terms yields

$$\begin{aligned}
&\sum_{r=0}^{n-s-1} \frac{(-1)^{n-s-r}((k-1)n+r+j-1)!(kn+j)!\Omega(r,n)\left[(kn+j+1)((k-1)n+j+r)\right]}{((k-1)n+j+s+r+1)!r!(n-s-r)!} \\
&+\frac{(kn+j-s)!\Omega(n-s,n)}{(n-s)!} \\
= &\sum_{r=0}^{n-s-1} \frac{(-1)^{n-s-r}((k-1)n+r+j)!(kn+j+1)!\Omega(r,n)}{((k-1)n+j+s+r+1)!r!(n-s-r)!} + \frac{(kn+j-s)!\Omega(n-s,n)}{(n-s)!} \\
= &\sum_{r=0}^{n-s} \frac{(-1)^{n-s-r}((k-1)n+r+j)!(kn+j+1)!\Omega(r,n)}{((k-1)n+j+s+r+1)!r!(n-s-r)!} = U^s_{\Upsilon,k,kn+j}
\end{aligned}$$

Thus we have shown that our formula for $U^s_{\Upsilon,k,kn+j}$ satisfies (4.3) for $y-k+1 \leq j \leq y-1$. We will now show that our formula satisfies (4.4). The RHS of (4.4) becomes

$$\begin{aligned}
&((k-|\Upsilon|)n+s+y-\alpha)\left[\sum_{r=0}^{n-s} \frac{(-1)^{n-s-r}((k-1)n+r+y-1)!(kn+y)!\Omega(r,n)}{((k-1)n+y+s+r)!r!(n-s-r)!}\right] \\
&+(|\Upsilon|n+\alpha-s+1)\left[\sum_{r=0}^{n-s+1} \frac{(-1)^{n-s-r+1}((k-1)n+r+y-1)!(kn+y)!\Omega(r,n)}{((k-1)n+y+s+r-1)!r!(n-s-r+1)!}\right]
\end{aligned}$$

Removing the $n-s+1$ term from the second summand, and combining the rest of the terms yields

$$\begin{aligned}
&\sum_{r=0}^{n-s} \frac{(-1)^{n-s-r}((k-1)n+r+y-1)!(kn+y)!\Omega(r,n)\left[(-1)(\alpha+r+n(|\Upsilon|-1))(kn+y+1)\right]}{((k-1)n+y+s+r)!r!(n-s-r+1)!} \\
&+\frac{(|\Upsilon|n+\alpha-s+1)(kn+y-s)!\Omega(n-s+1,n)}{(n-s+1)!} \\
&\sum_{r=0}^{n-s+1} \frac{(-1)^{n-s-r+1}((k-1)n+r+y-1)!(kn+y+1)!\Omega(r,n)(\alpha+r+n(|\Upsilon|-1))}{((k-1)n+y+s+r)!r!(n-s-r+1)!} \\
&\sum_{r=0}^{n-s+1} \frac{(-1)^{n-s-r+1}((k-1)n+r+y-1)!(kn+y+1)!\Omega(r,n+1)}{((k-1)n+y+s+r)!r!(n-s-r+1)!} = U^s_{\Upsilon,k,kn+y}
\end{aligned}$$

Thus we have shown that our formula for $U^s_{\Upsilon,k,kn+j}$ satisfies (4.4) as desired. □

5. Conclusion and perspectives

This paper can be regarded as some initial results on the study of pattern matching in permutations that include conditions on the equivalence class modulo k of the elements of the pattern. In particular, we studied the polynomials

$$T_{\tau,k,n}(x) = \sum_{\sigma \in S_n} x^{\tau\text{-}k\text{-}emch(\sigma)} = \sum_{s=0}^{n} T^s_{\tau,k,n} x^s \text{ and } U_{\Upsilon,k,n}(x) = \sum_{\sigma \in S_n} x^{\Upsilon\text{-}k\text{-}emch(\sigma)} = \sum_{s=0}^{n} U^s_{\Upsilon,k,n} x^s.$$

We developed a number of explicit formulas for these polynomials in the case where τ is a two-element sequence or when Υ is a set of ascents of the form $\{(x_1, y_1), \ldots, (x_t, y_t)\}$ where for all i and j, $y_i \equiv y_j \mod k$ or a set of descents of the form $\{(y_1, x_1), \ldots, (y_t, x_t)\}$ where for all i and j, $y_i \equiv y_j \mod k$. Our formulas for the coefficients of these polynomials lead to a number of interesting identities. For example, it follows from Theorems 4.3 and 4.4 that we have Υ is set of ascents of the form $\{(x_1, y_1), \ldots, (x_t, y_t)\}$ where for all i and j, $y_i \equiv y_j \mod k$, $y = min(\{y_1, \ldots, y_k\})$, and $\alpha = |\{x_i : x_i < y\}|$, then

$$\begin{aligned}&\left[\sum_{r=0}^{s} (-1)^{s-r} \binom{(k-1)n + r + j}{r} \binom{kn + j + 1}{s - r} \Gamma(r, j, n)\right] \\ = &\left[\sum_{r=0}^{n-s} (-1)^{n-s-r} \binom{(k-1)n + r + j}{r} \binom{kn + j + 1}{n - s - r} \Omega(r, n)\right]\end{aligned}$$

where $\Gamma(r, j, n) = \prod_{i=0}^{n-1}((k-1)n + r + j + 1 - \alpha - i(|\Upsilon| - 1))$ and $\Omega(r, n) = \prod_{i=0}^{n-1}(r + \alpha + i(|\Upsilon| - 1))$. It would be nice to have a more general explanation as to how these types of identities arise.

Also the results of this paper give rise to a number of interesting bijective questions. For example, our formulas show that many of the polynomials $T_{(ab),n,kn+j}(x)$ are identical for certain values of a, b, n and j. One can ask to give a bijective proof of such facts. We have not been able to do this in all cases, but we can give can a bijective proof that $T_{(ab),k,kn+j}(x) = T_{(cd),k,kn+j}(x)$ where for all n and $1 \leq j \leq k$ whenever $n - \chi(b > k) + \chi(j \geq b \mod k) = n - \chi(d > k) + \chi(j \geq d \mod k)$.

There is still much work to be done on the structure of the polynomials $T_{\tau,k,n}(x)$ and $U_{\Upsilon,k,n}(x)$. First one can consider generalized Wilf equivalence questions, i.e., given k, for which patterns α and β do we have $T_{\alpha,k,n}(x) = T_{\beta,k,n}(x)$ for all n. We can also consider more complicated sets of patterns. We should note that when we consider more complicated patterns, the problems get considerably harder. For example, consider $U_{\Upsilon,k,kn+j}(x)$ where $k = 3$ and $\Upsilon = \{12, 23\}$. We can no longer get simple recursions for the coefficients $U^s_{\Upsilon,k,kn+j}$ since we need to keep track of more information than just the number of Υ-k-equivalence matches. That is, let

$$A_n(x, y) = \sum_{\sigma \in S_n} x^{(12)\text{-}3\text{-}emch(\sigma)} y^{(23)\text{-}3\text{-}emch(\sigma)} = \sum_{r,s \geq 0} A^{s,t}_n x^s y^t.$$

Using the methods of this paper, we can derive simple recursions for the coefficients of $A^{r,s}_n$

$$\begin{aligned}A^{s,t}_{3n+1} &= (s+1)A^{s+1,t}_{3n} + (t+1)A^{s,t+1}_{3n} + (3n+1-s-t)A^{s,t}_{3n} \\ A^{s,t}_{3n+2} &= (2+n-s)A^{s-1,t}_{3n+1} + (t+1)A^{s,t+1}_{3n+1} + (2n+1+s-t)A^{s,t}_{3n+1} \\ A^{s,t}_{3n+3} &= (s+1)A^{s+1,t}_{3n+2} + (2+n-t)A^{s,t-1}_{3n+2} + (2n+2+t-s)A^{s,t}_{3n+2}.\end{aligned}$$

These recursions are more difficult to iterate, but we have found explicit formulas similar to the ones described in this paper for the coefficients $A^{r,s}_n$ when either r is the maximum power of x that appears in $A_n(x, y)$ or s is the maximum power of y that appears in $A_n(x, y)$. Similarly, we can use extend the inclusion-exclusion approach of section 3 to show that $A_n(x, y) = \sum_{k,l}(n - k - l)! r_k(B_{(12),n}) r_l(B_{(23),n})$.

Similar problems arise when we consider patterns of length ≥ 3. For example, if one is going to study the number of (123)-k-equivalence matches, then to develop simple recursive formulas, one needs to also keep track of the number of (12)-k-equivalence matches so that one ends up studying polynomials like

$$B_n(x, y) = \sum_{\sigma \in S_n} x^{(12)\text{-}3\text{-}emch(\sigma)} y^{(123)\text{-}3\text{-}emch(\sigma)} = \sum_{r,s \geq 0} B^{r,s}_n x^s y^t.$$

Finally, we should note we have derived q-analogues of the results of this paper.

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