B*-quasisymmetric polynomials and a Catalan tetrahedron*

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FPSAC06 – San Diego

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Catalan triangle (1/3)

1						
1	1					
1	2	2				
1	3	5	5			
1	4	9	14	14		
1	5	14	28	42	42	
1	6	20	48	90	132	132

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- $B(1,0) = 1$
- $\forall n > 1, k < n,$

  $B(n,k) = \sum_{l=0}^{k} B(n-1,l)$

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$B(n,k) =$ number of Dyck paths of length $2n$ with $n-k$ DOWN steps in the last sequence of DOWN steps

Catalan triangle (2/3)

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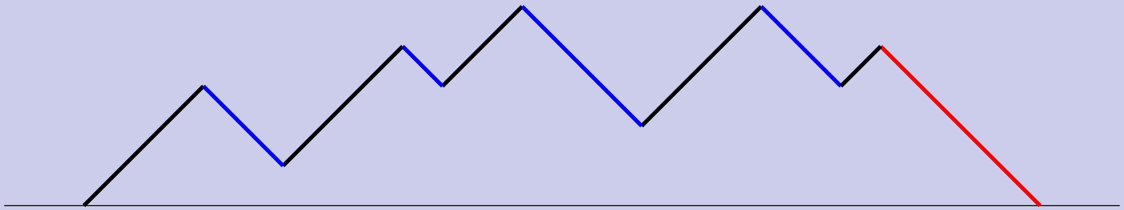

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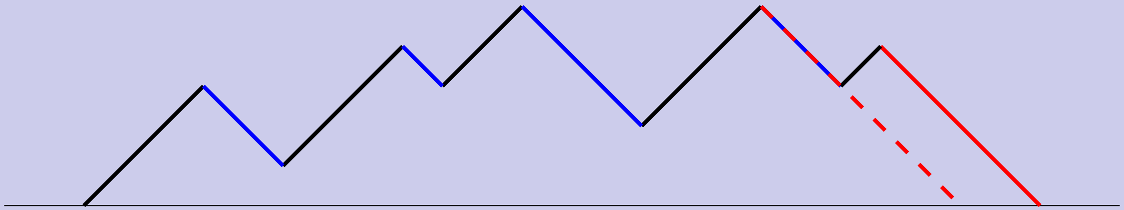

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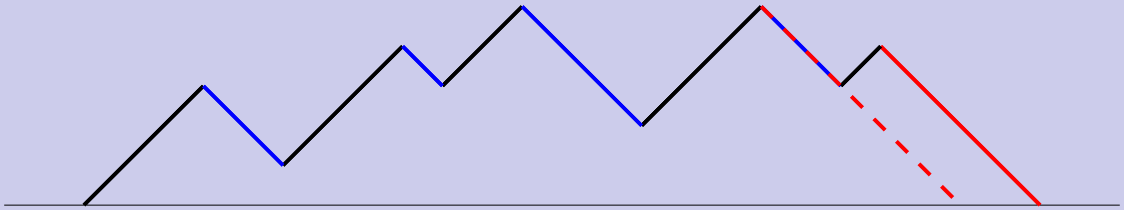
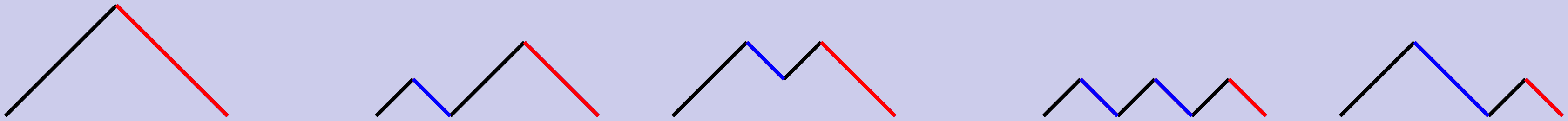


Example: $n = 3$ $\quad 1 \quad 2 \quad 2$



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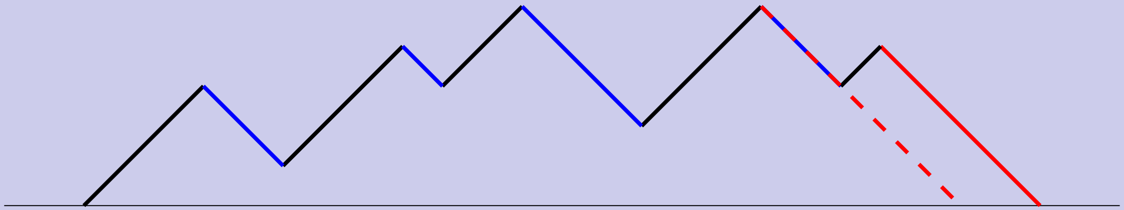

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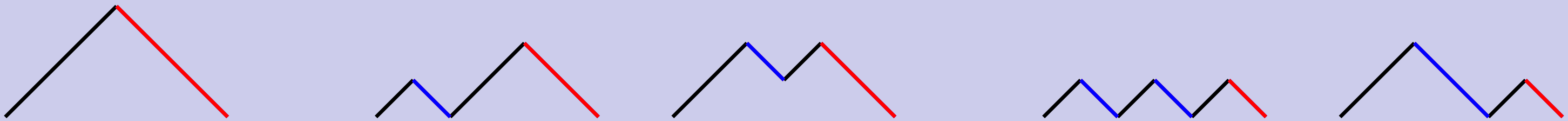

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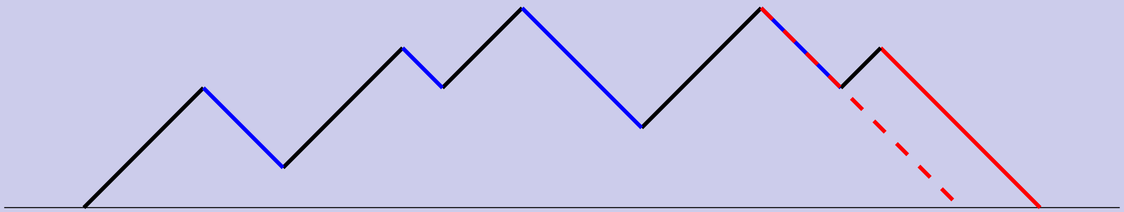

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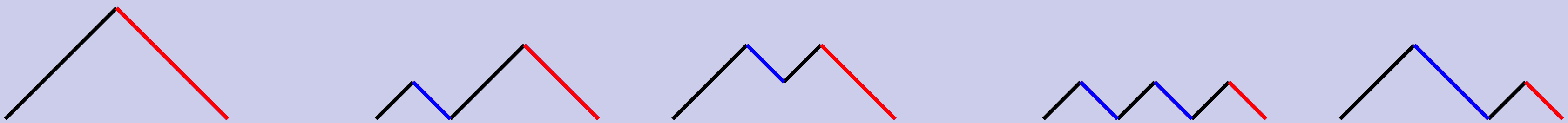

$$B(n,k) = \frac{n-k}{n+k}\binom{n+k}{n}$$

$$\sum_{k=0}^{n-1} B(n,k) = C(n) = \frac{1}{n+1}\binom{2n}{n}$$

Catalan triangle (3/3)

$$\sum_{k=0}^{n-1} B(n,k) = C(n) = \frac{1}{n+1}\binom{2n}{n}$$

1							1
1	1						2
1	2	2					5
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1							1
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What happens if we let the same recurrence grow in dimension 3 ? $\longrightarrow$ Catalan tetrahedron

Catalan tetrahedron (1/2)

$$
\begin{bmatrix} 1 \end{bmatrix}
\begin{bmatrix} 1 & \\ 1 & 1 \end{bmatrix}
\begin{bmatrix} 2 & & \\ 2 & 3 & \\ 1 & 2 & 2 \end{bmatrix}
\begin{bmatrix} 5 & & & \\ 5 & 10 & & \\ 3 & 8 & 10 & \\ 1 & 3 & 5 & 5 \end{bmatrix}
\begin{bmatrix} 14 & & & & \\ 14 & 35 & & & \\ 9 & 30 & 45 & & \\ 4 & 15 & 30 & 35 & \\ 1 & 4 & 9 & 14 & 14 \end{bmatrix}
$$

$$n = 1 \qquad n = 2 \qquad n = 3 \qquad n = 4 \qquad n = 5$$

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- $\forall k + l \geqslant n,\ B_3(n, k, l) = 0$
- $B_3(1, 0, 0) = 1$
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$$\sum_{k+l<n} B_3(n, k, l) = ?$$

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$$\sum_{k+l<n} B_3(n, k, l) = ?$$

1 $\quad$ 3 $\quad$ 12 $\quad$ 55 $\quad$ 273

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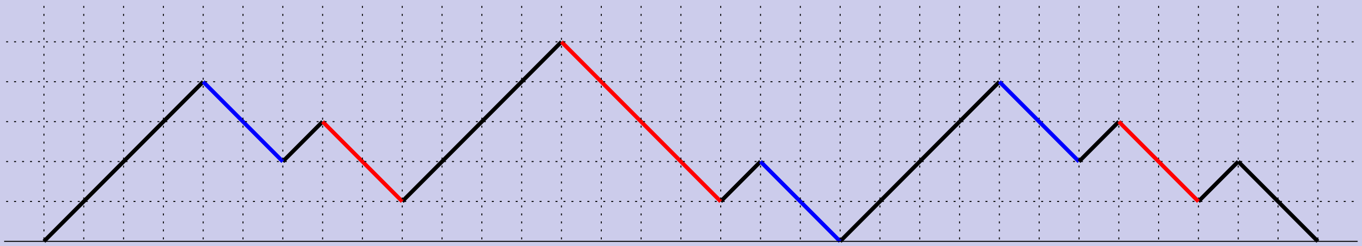

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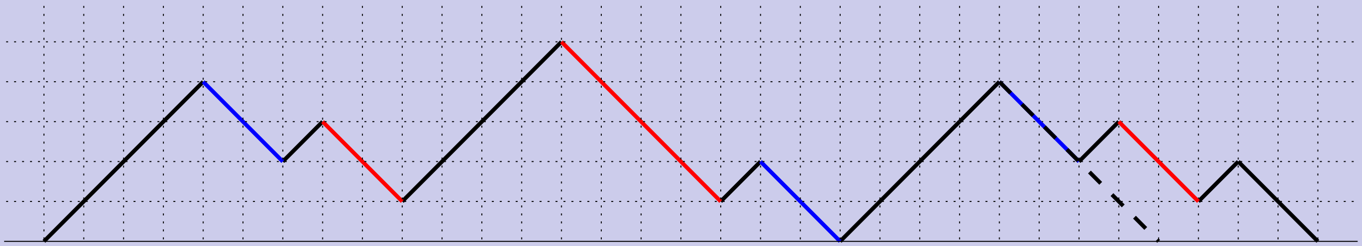

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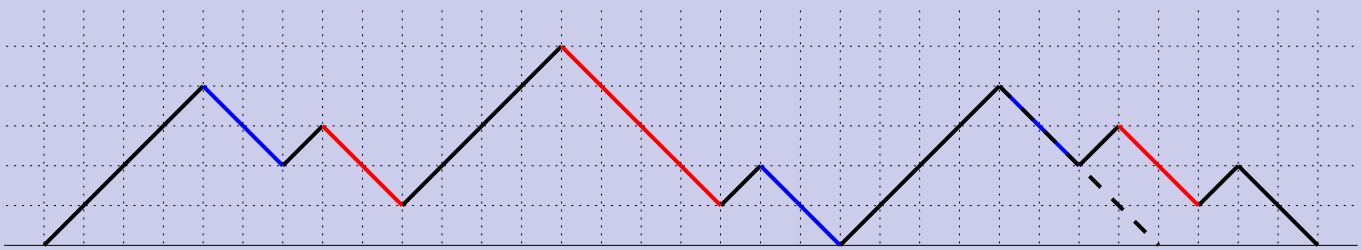

Explicit formula (proved using the cycle lemma):

$$B_3(n,k,l) = \binom{n+k-1}{k}\binom{n+l-1}{l}\frac{n-k-l}{n}$$

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Symmetric polynomials

Alphabet $X_n = x_1, \ldots, x_n$

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$$
\begin{aligned}
h_1(X_3) &= x_1 + x_2 + x_3 \\
h_2(X_3) &= x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2 \\
h_3(X_3) &= x_1^3 + x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_1x_2x_3 + x_1x_3^2 \\
&\quad + x_2^3 + x_2^2x_3 + x_2x_3^2 + x_3^3
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$$h_3(X_3) = x_1^3 + x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_1x_2x_3 + x_1x_3^2$$
$$+x_2^3 + x_2^2x_3 + x_2x_3^2 + x_3^3$$

Space $Sym_n = \mathbb{Q}[h_k(X_n),\ k \geqslant 0]$
Ideal $\langle Sym_n^+ \rangle = \langle h_k(X_n),\ k > 0 \rangle$

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Theorem (Artin – 1944)

$$\dim \mathbb{Q}[X_n]/\langle Sym_n^+ \rangle = n!$$

Quasisymmetric polynomials

Basis F_c indexed by compositions:

Definition-example:

$$F_{(2,1,3)}(X_n) = \sum_{1 \leqslant i_1 \leqslant i_2 < i_3 < i_4 \leqslant i_5 \leqslant i_6 \leqslant n} x_{i_1}\, x_{i_2}\, x_{i_3}\, x_{i_4}\, x_{i_5}\, x_{i_6}$$

$$F_{(1,2)}(X_3) = \sum_{1 \leqslant i < j \leqslant k \leqslant 3} x_i\, x_j\, x_k = x_1 x_2^2 + x_1 x_2 x_3 + x_1 x_3^2 + x_2 x_3^2$$

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Theorem (Aval, Bergeron, Bergeron – 2002)

$$\dim \mathbb{Q}[X_n]/\langle QSym_n^+ \rangle = C(n) = \frac{1}{n+1}\binom{2n}{n}$$

B-quasisymmetric polynomials (1/2)

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Bi-composition $42 \; 02 \; 10 \; 31$ (patterns 00 and 00 forbidden)

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$$F_{12\,,\,01\,,\,20}(X_n, Y_n) = \sum_{i_1 \leqslant i_2 \leqslant i_3 < i_4 < i_5 \leqslant i_6} x_{i_1}\, y_{i_2}\, y_{i_3}\, y_{i_4} x_{i_5}\, x_{i_6}$$

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[defined by Poirier, studied by Baumann-Hohlweg]

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Space $BQSym_n = Span(F_c(X_n, Y_n),\ |c| \geqslant 0)$
Ideal $\langle BQSym_n^+ \rangle = \langle F_c(X_n, Y_n),\ |c| > 0 \rangle$

B-quasi-symmetric polynomials (2/2)

Theorem (Aval)
The quotient space $\mathbb{Q}[X_n, Y_n]/\langle BQSym_n^+ \rangle$

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- has dimension $\frac{1}{2n+1}\binom{3n}{n}$ (number of ternary trees)
- is bi-graded, and its Hilbert series is

$$H_n(q,t) = \sum_{0\leqslant k+l\leqslant n} \binom{n+k-1}{k}\binom{n+l-1}{l}\frac{n-k-l}{n}\, q^k\, t^l$$

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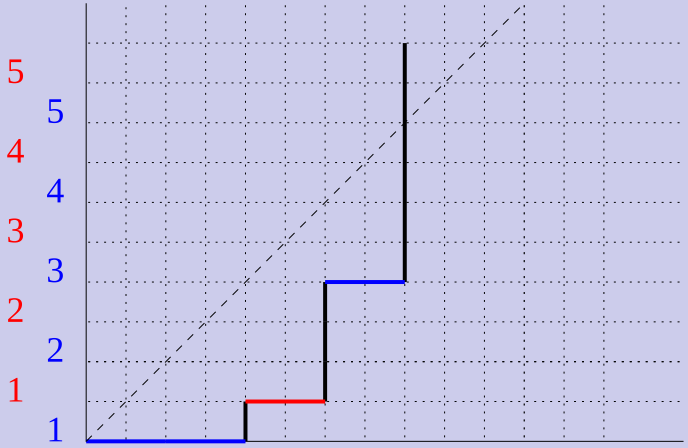
Moreover, there is an effective explicit description of a Gröbner basis for the ideal.

Proof (1/3)

Bijection monomials $\longleftrightarrow$ paths:

$n = 5$

$x_1^2 \, y_1 \, x_3 \quad \longleftrightarrow$



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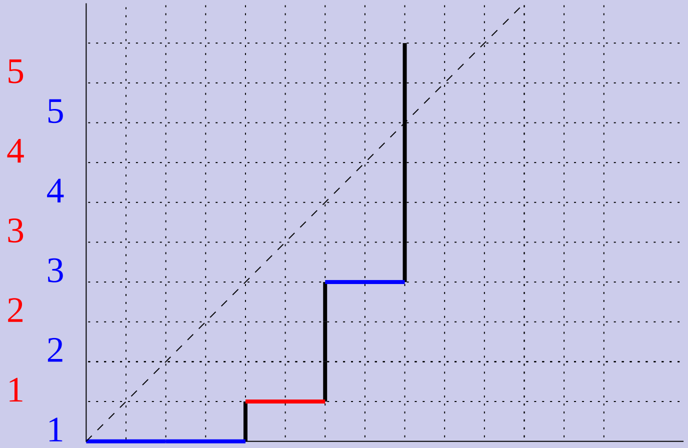

transdiagonal path

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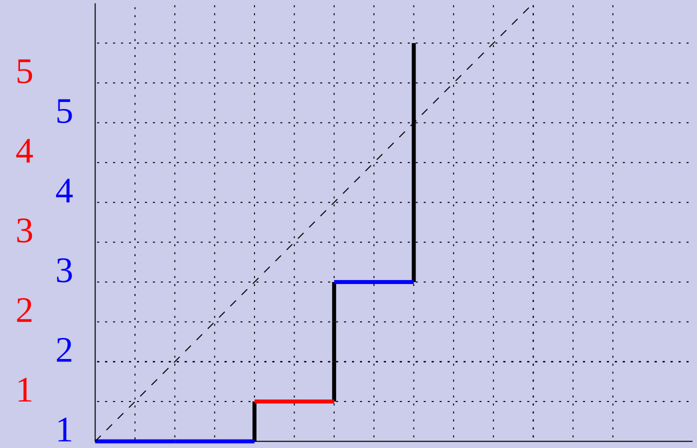
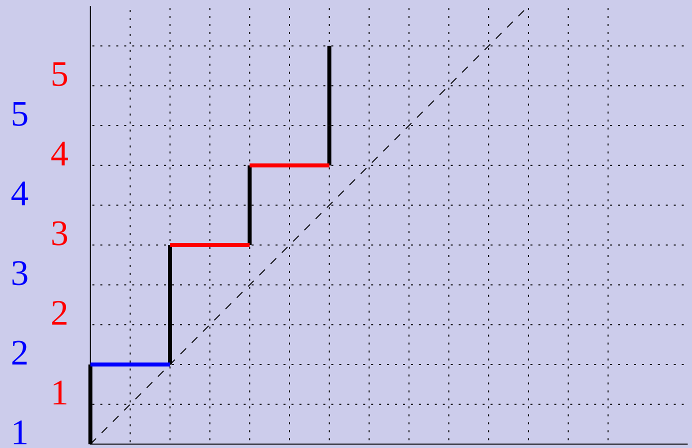


transdiagonal path

$x_2\, y_3\, y_4 \quad \longleftrightarrow$



2-Dyck path

Proof (2/3)

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Proof
Explicit construction of a Gröbner basis for the ideal $\langle BQSym_n^+ \rangle$ whose leading monomials (for the lexicographic order) are the monomials associated to transdiagonal paths.

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Consequence

A monomial basis for the quotient

$$\dim \mathbb{Q}[X_n, Y_n]/\langle BQSym_n^+ \rangle$$

is given by the monomials associated to 2-Dyck paths.

Proof (3/3)

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A monomial basis for the quotient $\mathbb{Q}[X_n, Y_n]/\langle BQSym_n^+\rangle$ is given by the monomials associated to 2-Dyck paths.

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A monomial basis for the quotient $\mathbb{Q}[X_n, Y_n]/\langle BQSym_n^+\rangle$ is given by the monomials associated to 2-Dyck paths.

$Q_{n,k,l} = \mathbb{Q}[X_n, Y_n]/\langle BQSym_n^+\rangle \cap \mathbb{Q}[X_n, Y_n]_{k,l}$ (polynomials of degree k in $(x_1, \ldots, x_n)$ and l in $(y_1, \ldots, y_n)$)

$$\dim Q_{n,k,l} = \binom{n+k-1}{k}\binom{n+l-1}{l}\frac{n-k-l}{n} = B_3(n,k,l)$$

Proof (3/3)

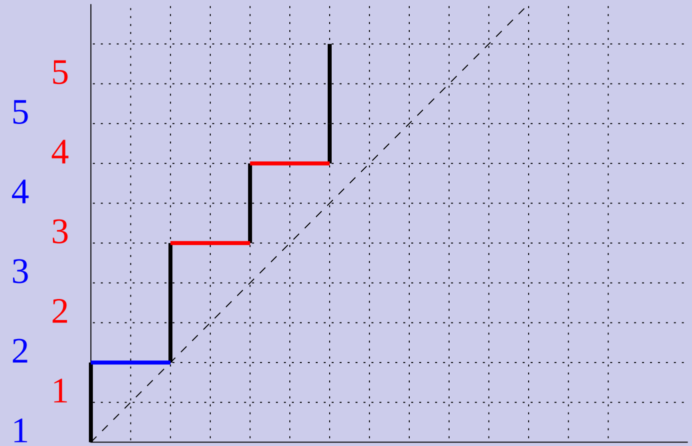
Consequence

A monomial basis for the quotient $\mathbb{Q}[X_n, Y_n]/\langle BQSym_n^+\rangle$ is given by the monomials associated to 2-Dyck paths.

$Q_{n,k,l} = \mathbb{Q}[X_n, Y_n]/\langle BQSym_n^+\rangle \cap \mathbb{Q}[X_n, Y_n]_{k,l}$ (polynomials of degree k in $(x_1, \ldots, x_n)$ and l in $(y_1, \ldots, y_n)$)

$$\dim Q_{n,k,l} = \binom{n+k-1}{k}\binom{n+l-1}{l}\frac{n-k-l}{n} = B_3(n,k,l)$$

Proof



$x_2\, y_3\, y_4 \longleftrightarrow$ 2-Dyck path

$(k,l) = (1, 2) \longleftrightarrow$ East steps: 1 at even height, 2 at odd height

Extensions and open questions

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- open question: find a description of B-quasisymmetric polynomials as invariants