An analogue of the Robinson-Schensted-Knuth Algorithm and its application to standard bases

FPSAC 2006

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Outline

1. Skyline augmented fillings and nonsymmetric Schur functions

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2. A decomposition of the Schur functions

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2. A decomposition of the Schur functions
3. An analogue of the Robinson-Schensted-Knuth Algorithm
4. Applications of the RSK analogue to keys and standard bases

Compositions and partitions

composition of a positive integer n

$$(\gamma_1, \gamma_2, \dots \gamma_k), \qquad \text{where} \quad \sum_{i=1}^{k} \gamma_i = n$$

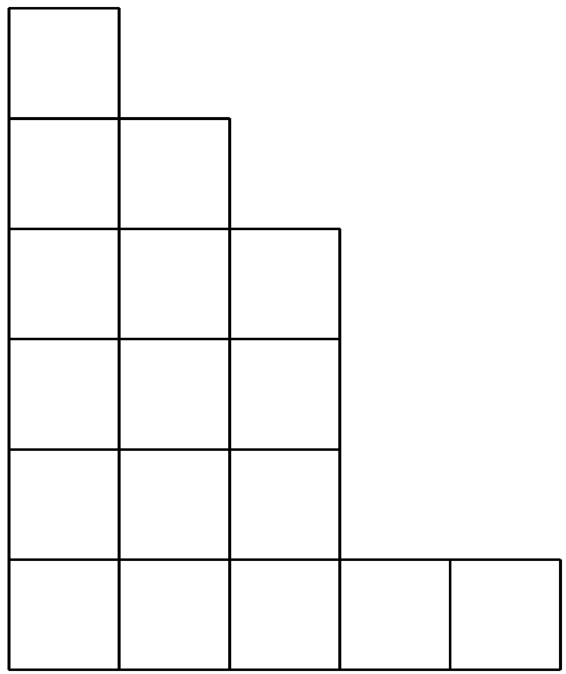
partition of a positive integer n

$$\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_k$$

Ferrers diagram

The **Ferrers diagram** of a partition $\lambda = \lambda_1, ..., \lambda_k$ consists of rows of cells, where the i^{th} row contains λ_i cells.

Example $\lambda = (5, 3, 3, 3, 2, 1)$



SSYT

A **semi-standard young tableau (SSYT)** is a filling of the cells of a Ferrers diagram with positive integers in such a way that the numbers are **weakly increasing across rows** and **strictly increasing up columns**.

Example: $\lambda = (5, 3, 3, 3, 2, 1)$

$$T = \begin{array}{|c|c|c|c|c|}
\cline{1-1}
10 \\
\cline{1-2}
9 & 11 \\
\cline{1-3}
8 & 10 & 10 \\
\cline{1-3}
4 & 7 & 8 \\
\cline{1-3}
2 & 5 & 5 \\
\hline
1 & 2 & 3 & 5 & 10 \\
\hline
\end{array}$$

weight(T) = $x^T = x_1\, x_2^2\, x_3\, x_4\, x_5^3\, x_7\, x_8^2\, x_9\, x_{10}^4\, x_{11}$

The Schur function s_λ

For a partition λ, to get the **Schur function** s_λ, determine all the possible SSYTs of shape λ. Then s_λ equals the sum over all such SSYTs of the weights.

$$s_\lambda = \sum_{T \in SSYT(\lambda)} x^T$$

Example: $\lambda = (2,1)$

$$s_{(2,1)} = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_2 x_3 +$$

$$\begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 1 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 1 & 1 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 2 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 1 & 2 \\ \hline \end{array}$$

$$x_1 x_2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2$$

$$\begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 3 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 1 & 3 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 2 & 2 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 2 & 3 \\ \hline \end{array}$$

Symmetric functions

$$\sigma(f)(x_1, x_2, \ldots) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots) = f(x_1, x_2, \ldots)$$

for every permutation σ of $\mathbb{Z}^+$.

Example

$$f(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_1 + x_2^2 x_3 + x_3^2 x_1 + x_3^2 x_2$$

Symmetric functions

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Example

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The Schur functions form a basis for the algebra of symmetric functions.

Schur functions

- other symmetric function bases

Schur functions

- other symmetric function bases
- characters of the irreducible representations of GL_n.

Schur functions

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- characters of the irreducible representations of GL_n.
- cohomology ring of the Grassmannian.

Schur functions

- other symmetric function bases
- characters of the irreducible representations of GL_n.
- cohomology ring of the Grassmannian.
- RSK algorithm and other combinatorial properties

Rearrangements ($\gamma^+ = \lambda$)

Each partition λ can be considered as a collection of compositions which **rearrange** to λ. For this talk, we want compositions of n into n parts, so we allow zeros as well.

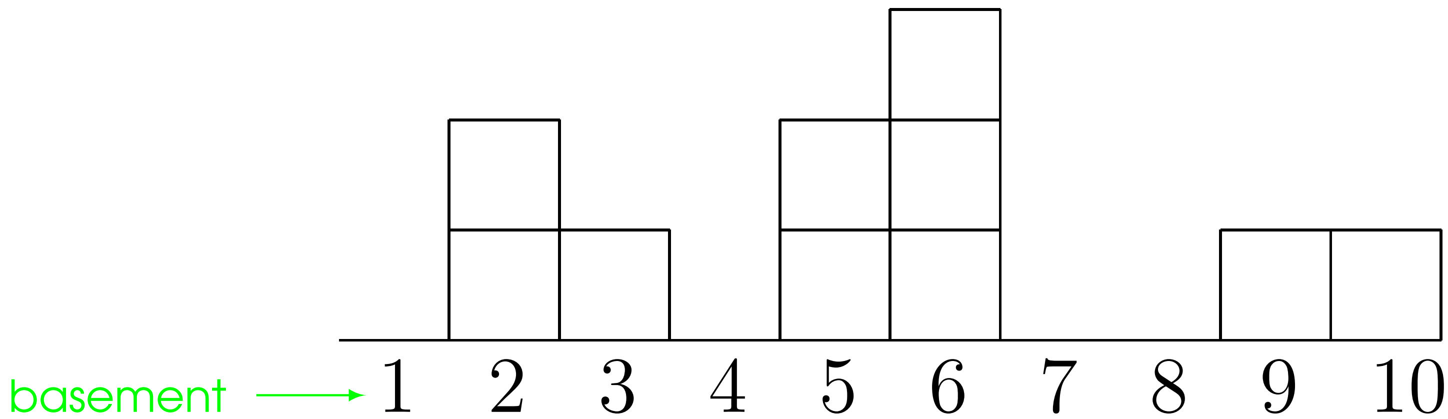
Example *There are six compositions of* 3 *into* 3 *parts which rearrange to* $(2,1)$*:*

$\{(2,1,0),(2,0,1),(1,0,2),(1,2,0),(0,1,2),(0,2,1)\}$

Composition Diagram

To each composition $\gamma = \gamma_1 + \gamma_2 + \ldots + \gamma_n$, we associate a **composition diagram**, which consists of n columns of cells, where the i^{th} column contains γ_i cells.

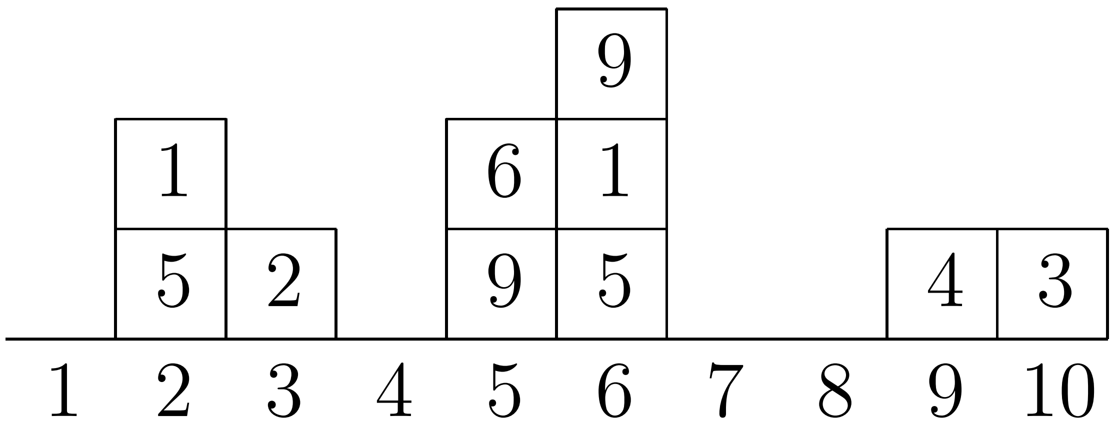
$$\gamma = (0, 2, 1, 0, 2, 3, 0, 0, 1, 1)$$



Fillings

A filling of a composition diagram is a filling of the cells with positive integers.

Example

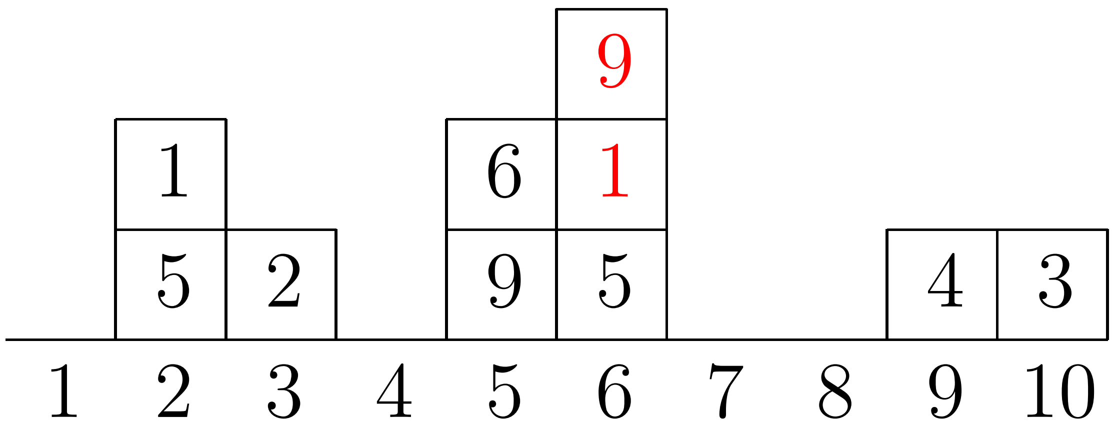


Haglund, Haiman, Loehr

- generalized descent and inversion statistics on fillings of diagrams
- combinatorial description of Macdonald polynomials and nonsymmetric Macdonald polynomials
- nonsymmetric Schur functions obtained as a special case of the nonsymmetric Macdonald polynomials.

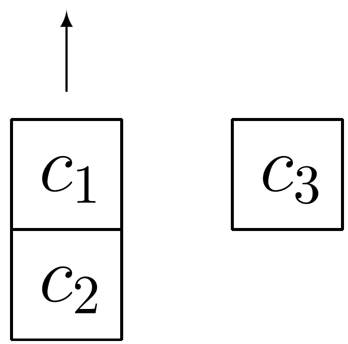
Descent

A descent is a pair of cells, c_1 and c_2, such that c_1 is directly on top of c_2 and $c_1 > c_2$.



Type A triple:

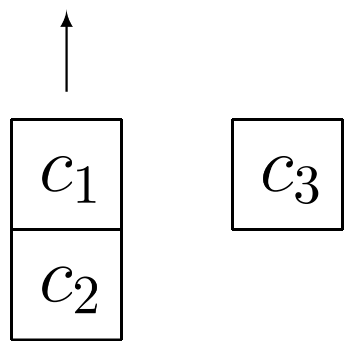
a triple $\{c_1, c_2, c_3\}$ of cells such that:



Type A triple:

a triple $\{c_1, c_2, c_3\}$ of cells such that:

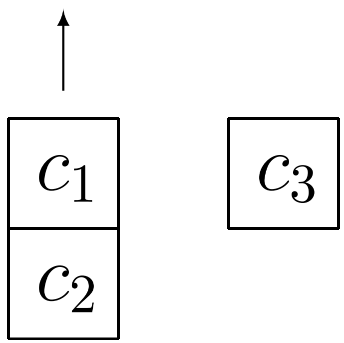
- c_1 and c_3 are in the same row



Type A triple:

a triple $\{c_1, c_2, c_3\}$ of cells such that:

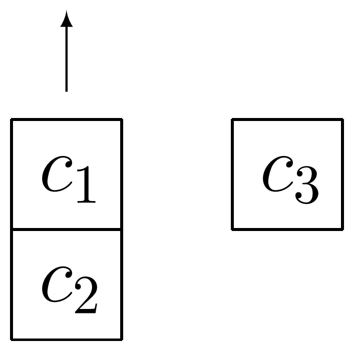
- c_1 and c_3 are in the same row
- c_2 is directly below c_1



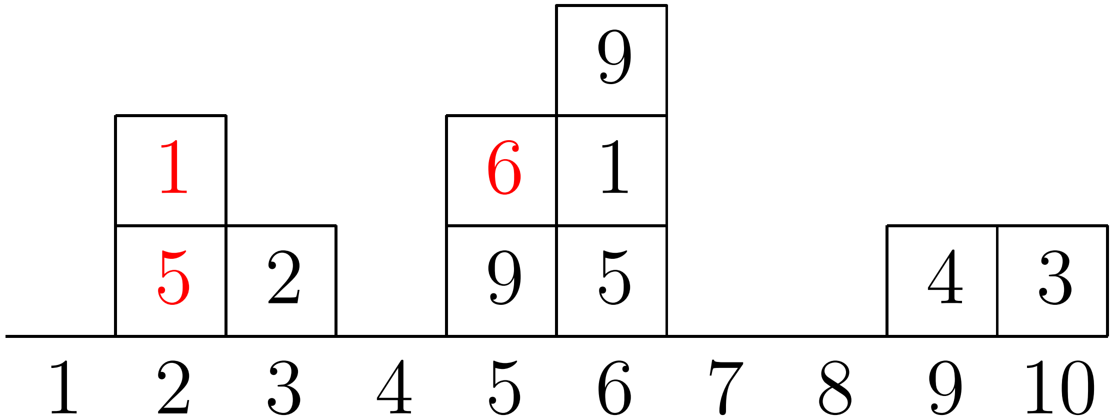
Type A triple:

a triple $\{c_1, c_2, c_3\}$ of cells such that:

- c_1 and c_3 are in the same row
- c_2 is directly below c_1
- the column containing c_1 and c_2 is weakly taller than the column containing c_3.

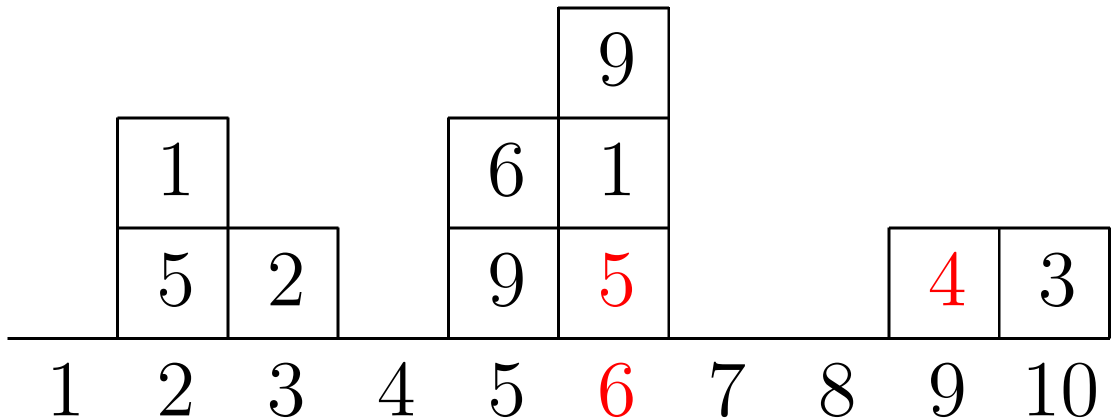


Type A inversion triple



⋆ counter-clockwise ordering from smallest to largest

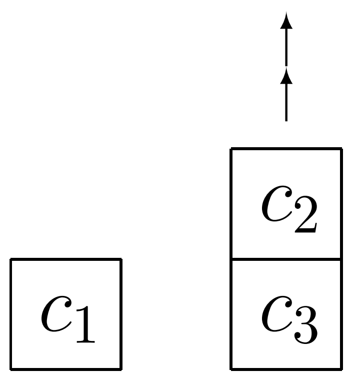
Type A inversion triple



⋆ counter-clockwise ordering from smallest to largest

Type B triple:

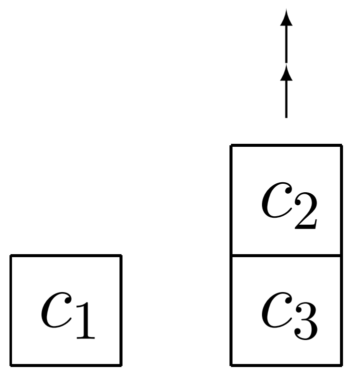
a triple $\{c_1, c_2, c_3\}$ of cells such that:



Type B triple:

a triple $\{c_1, c_2, c_3\}$ of cells such that:

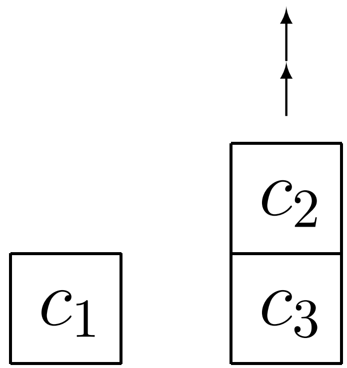
- c_1 and c_3 are in the same row



Type B triple:

a triple $\{c_1, c_2, c_3\}$ of cells such that:

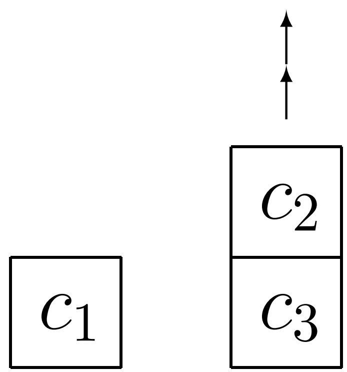
- c_1 and c_3 are in the same row
- c_2 is directly above c_3



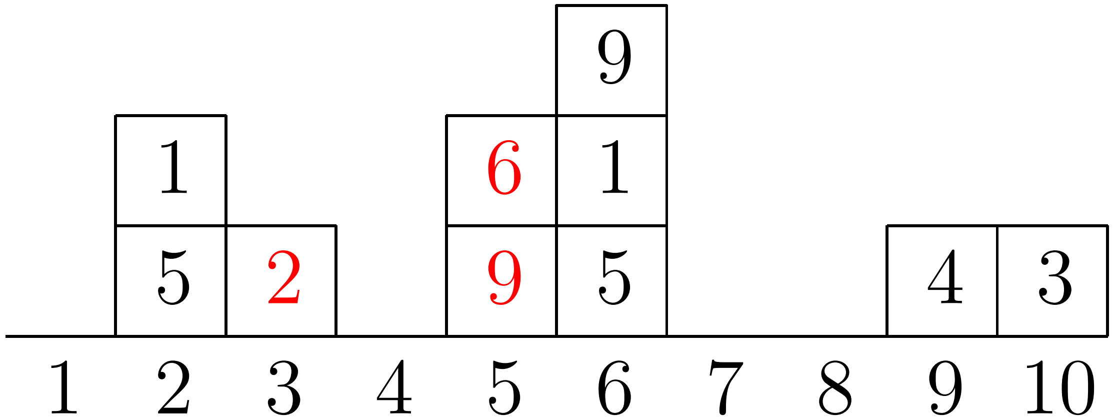
Type B triple:

a triple $\{c_1, c_2, c_3\}$ of cells such that:

- c_1 and c_3 are in the same row
- c_2 is directly above c_3
- the column containing c_2 and c_3 is **strictly** taller than the column containing c_1.

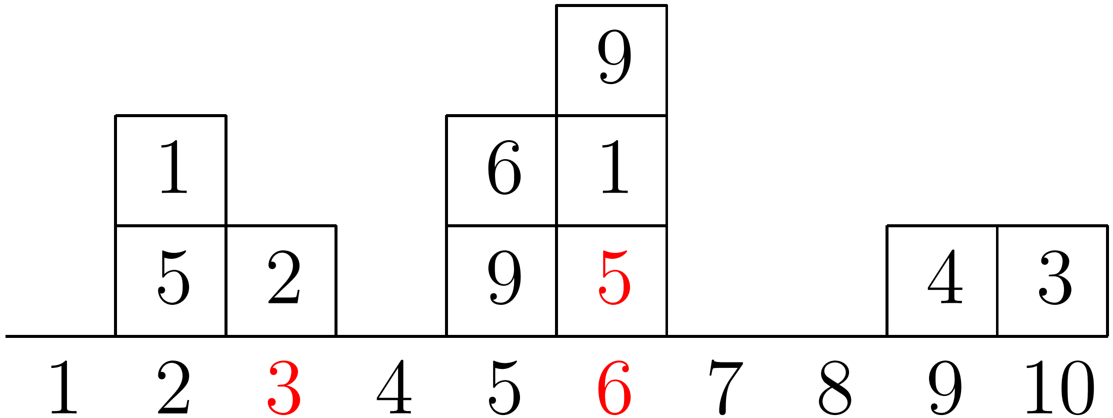


Type B inversion triple



⋆ clockwise ordering from smallest to largest

Type B inversion triple



⋆ clockwise ordering from smallest to largest

Skyline augmented fillings

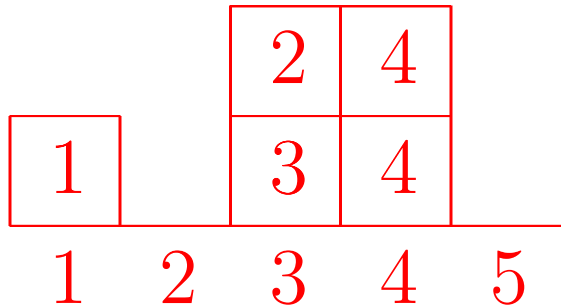
A syline augmented filling (SAF) satisfies:

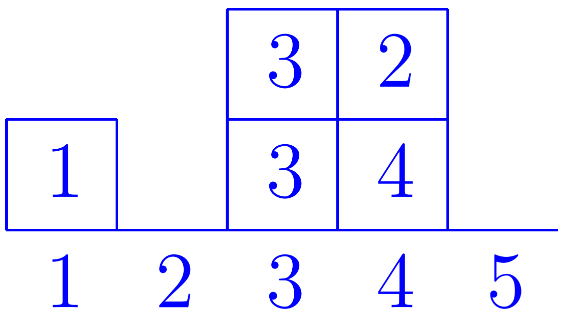
- no descents
- every triple is an inversion triple

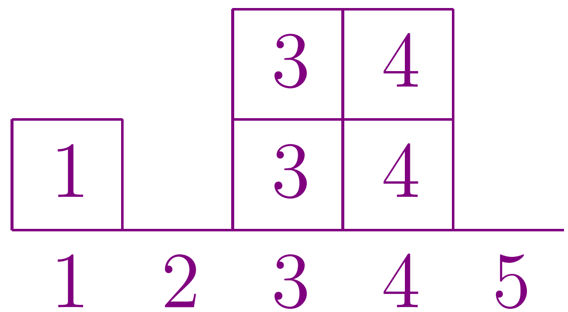
For a composition, γ, the **nonsymmetric Schur function** NS_γ is defined as the sum of the wieghts of all skyline augmented fillings of γ.

Example: $\gamma = (1, 0, 2, 2, 0)$

$$NS_\gamma = x_1 x_2 x_3^2 x_4 + x_1 x_2 x_3 x_4^2 + x_1 x_3^2 x_4^2$$







Macdonald Polynomials

The nonsymmetric Macdonald polynomial $E_\gamma(X;q,t)$ specializes to the nonsymmetric Schur function $NS_\gamma(X)$.

$$E_\gamma(X;q,t)|_{q=t=0} = NS_\gamma(X)$$

Macdonald Polynomials

$$
\begin{tikzcd}
\tilde{J}_\mu(X;q,t) = \displaystyle\sum_{\gamma^+=\mu} E_\gamma(X;q,t) \arrow[d, shift left=8em, "q=t=0"'] \arrow[d, shift right=8em, "q=t=0"] \\
s_\mu = \displaystyle\sum_{\gamma^+=\mu} NS_\gamma(X)
\end{tikzcd}
$$

The combinatorial formula of Haglund, Haiman, and Loehr for the nonsymmetric Macdonald polynomials motivates us to seek a bijective proof.

Theorem (M05)

There exists a weight-preserving bijection between SSYT and SAF which gives a combinatorial interpretation of

$$s_\mu(x_1, x_2, ..., x_n) = \sum_{\gamma^+ = \mu} NS_\gamma(x_1, x_2, ..., x_n).$$

The insertion map ρ

$$T = \begin{array}{|c|c|c|c|c|}
\cline{1-1}
10 \\
\cline{1-2}
9 & 11 \\
\cline{1-3}
8 & 10 & 10 \\
\cline{1-3}
4 & 7 & 8 \\
\cline{1-3}
2 & 5 & 5 \\
\hline
1 & 2 & 3 & 5 & 10 \\
\hline
\end{array}$$

$$Colform(T) =$$

$$10\ 9\ 8\ 4\ 2\ 1 \cdot 11\ 10\ 7\ 5\ 2 \cdot 10\ 8\ 5\ 3 \cdot 5 \cdot 10$$

$\rho : SSYT \rightarrow SAF$

- Begin with a semi-standard Young tableau T.
- Move right to left through the entries in $colform(T)$.
- Insert entry α by bumping the first entry β where $\alpha > \beta$ and β is directly on top of an entry greater than or equal to α.

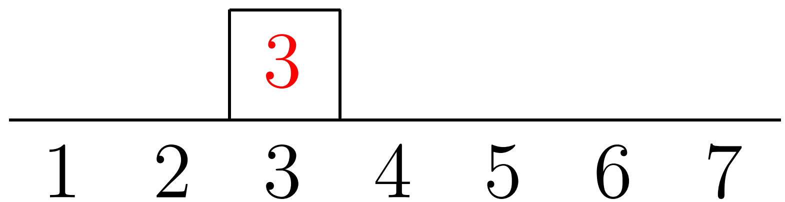
Example

Let T be the following SSYT.

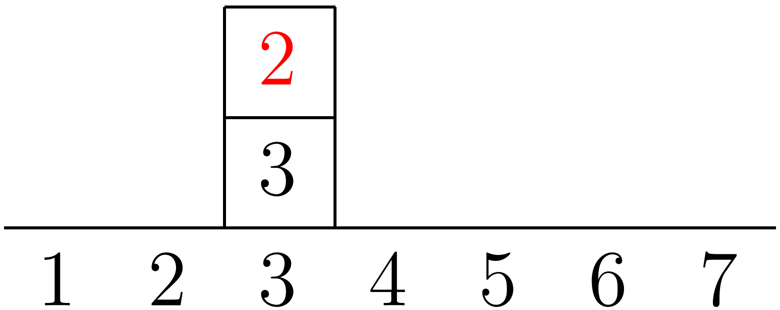
$$
\begin{array}{|c|c|c|}
\cline{1-1}
5 \\
\cline{1-1}
4 \\
\cline{1-2}
2 & 7 \\
\hline
1 & 2 & 3 \\
\hline
\end{array}
$$

$colform(T) = 5\ 4\ 2\ 1 \cdot 7\ 2 \cdot 3.$

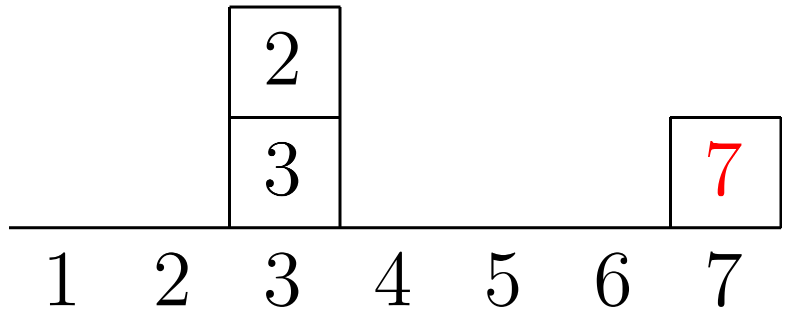
$5\,4\,2\,1 \cdot 7\,2 \cdot 3 \rightarrow$



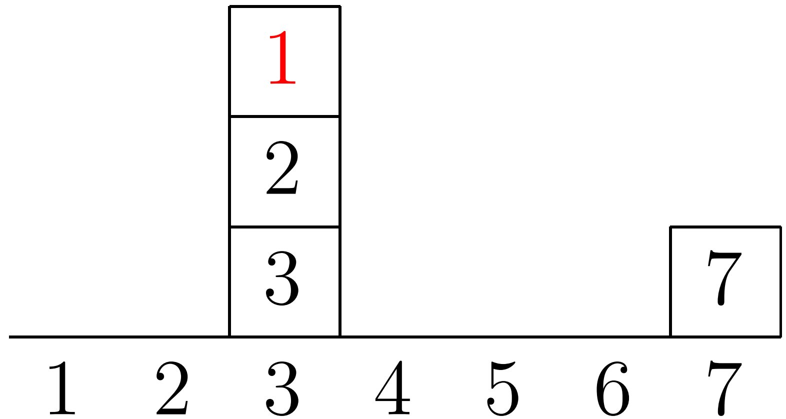
$5\,4\,2\,1 \cdot 7\,{\color{red}2} \cdot 3 \longrightarrow$



$5\,4\,2\,1 \cdot 7\,2 \cdot 3 \longrightarrow$



$5\ 4\ 2\ 1 \cdot 7\ 2 \cdot 3 \longrightarrow$



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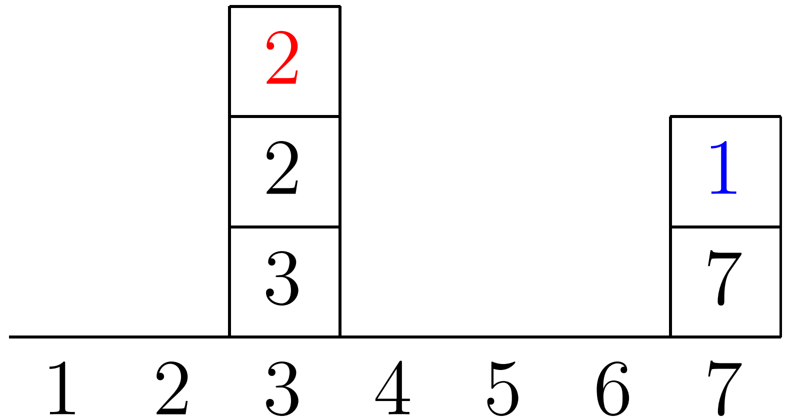

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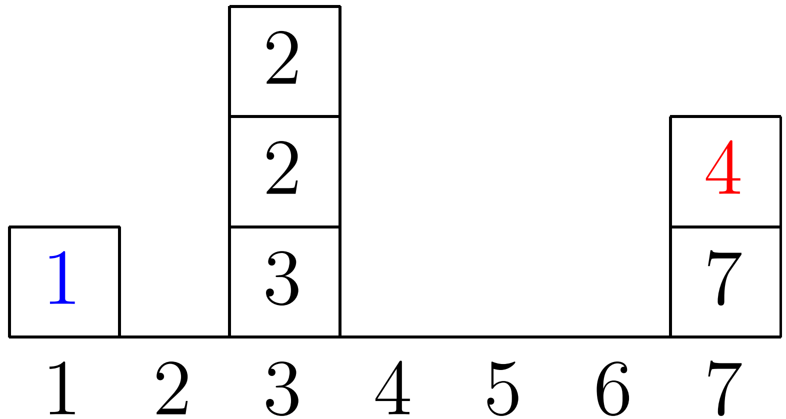
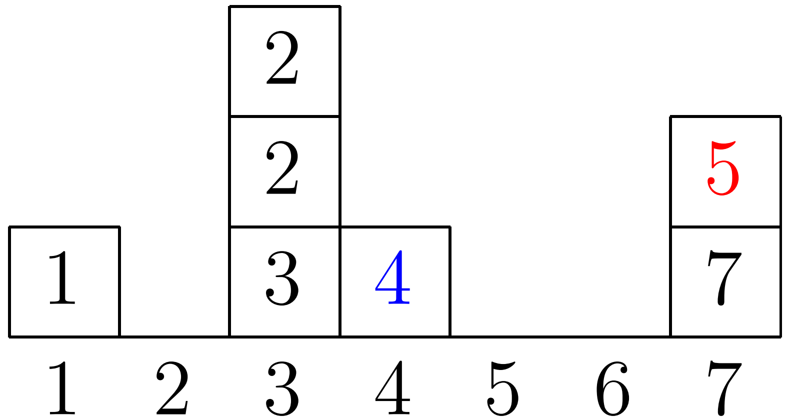
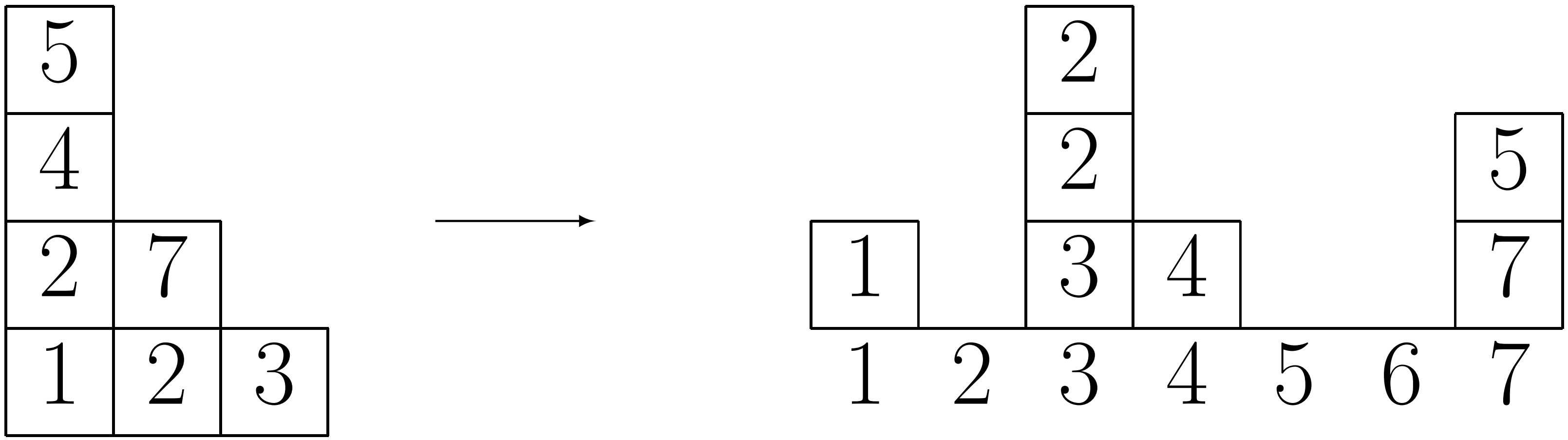


$5\ 4\ 2\ 1 \cdot 7\ 2 \cdot 3 \rightarrow$



$\rho : SSYT \to SAF$



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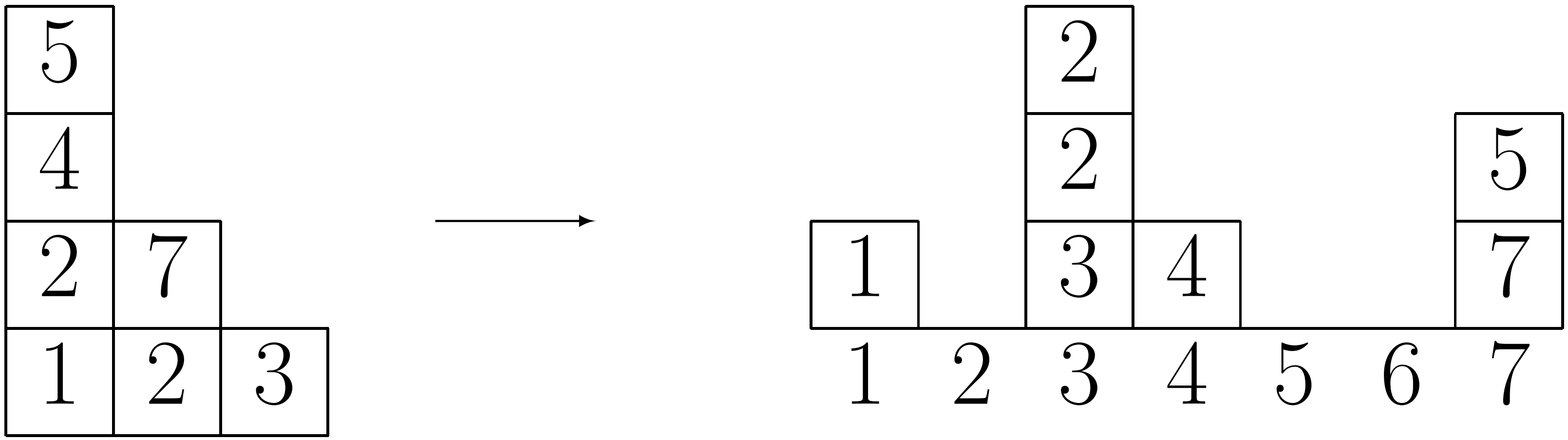

- This map preserves weight.

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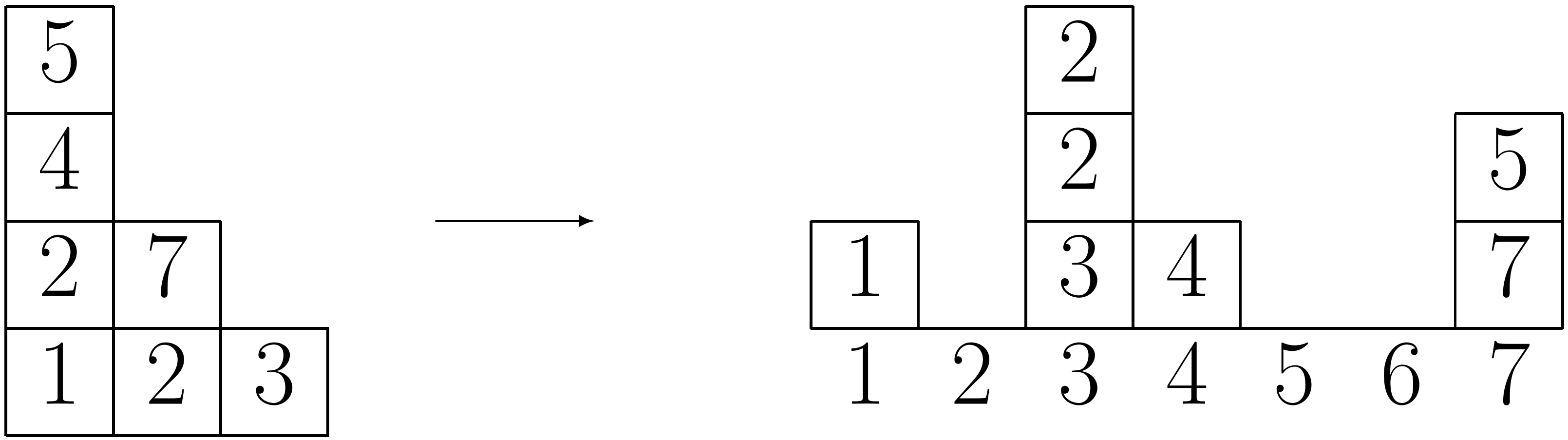
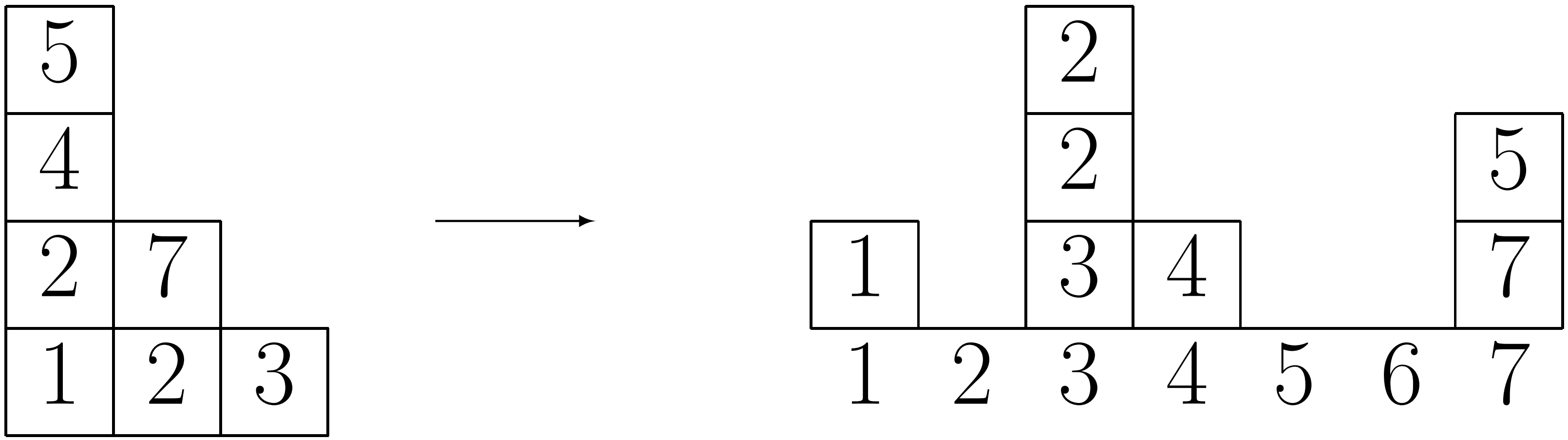


- This map preserves weight.
- The shape of $\rho(T)$ is always a rearrangement of the shape of T.

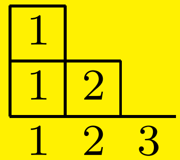
$\rho : SSYT \rightarrow SAF$



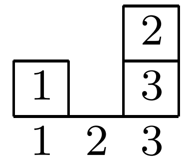
- This map preserves weight.
- The shape of $\rho(T)$ is always a rearrangement of the shape of T.
- ρ is invertible.

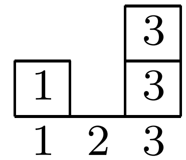
Example: $\lambda = (2,1)$

$$NS_{(2,1,0)} = x_1^2 x_2$$

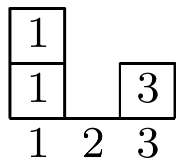


$$NS_{(1,0,2)} = x_1x_2x_3 + x_1x_3^2$$

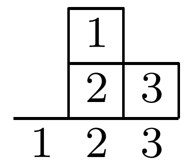


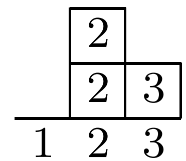


$$NS_{(2,0,1)} = x_1^2 x_3$$

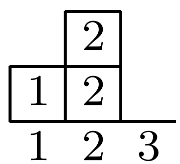


$$NS_{(0,2,1)} = x_1x_2x_3 + x_2^2x_3$$

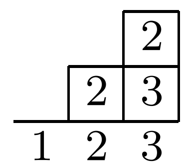




$$NS_{(1,2,0)} = x_1x_2^2$$



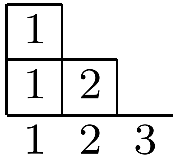
$$NS_{(0,1,2)} = x_2x_3^2$$



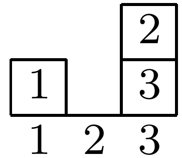
$s_{(2,1)} = x_1^2x_2 +$

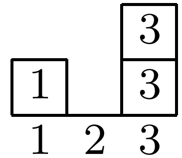
Example: $\lambda = (2,1)$

$$NS_{(2,1,0)} = x_1^2 x_2$$

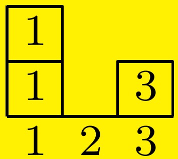


$$NS_{(1,0,2)} = x_1 x_2 x_3 + x_1 x_3^2$$

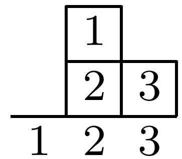


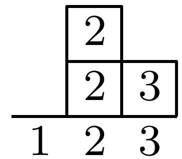


$$NS_{(2,0,1)} = x_1^2 x_3$$

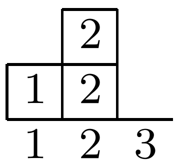


$$NS_{(0,2,1)} = x_1 x_2 x_3 + x_2^2 x_3$$

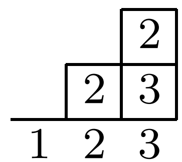




$$NS_{(1,2,0)} = x_1 x_2^2$$



$$NS_{(0,1,2)} = x_2 x_3^2$$



$s_{(2,1)} = x_1^2 x_2 + x_1^2 x_3$

Example: $\lambda = (2,1)$

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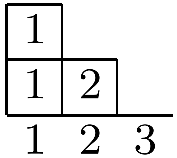
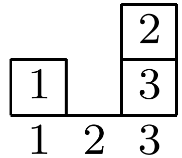
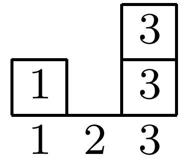


$$NS_{(1,0,2)} = x_1 x_2 x_3 + x_1 x_3^2$$





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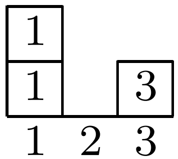
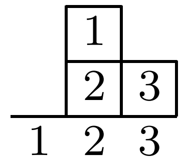
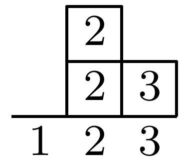
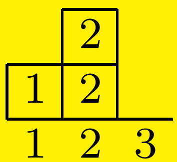


$$NS_{(0,2,1)} = x_1 x_2 x_3 + x_2^2 x_3$$

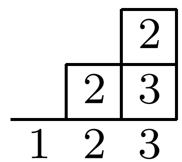




$$\boxed{NS_{(1,2,0)} = x_1 x_2^2}$$



$$NS_{(0,1,2)} = x_2 x_3^2$$



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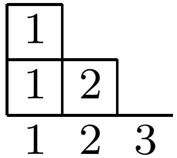

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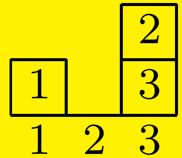
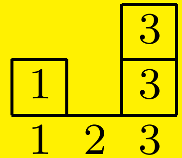
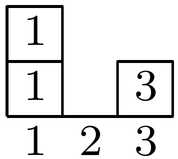
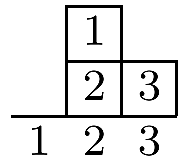


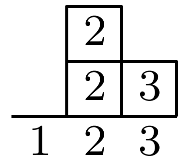


$$NS_{(2,0,1)} = x_1^2 x_3$$



$$NS_{(0,2,1)} = x_1 x_2 x_3 + x_2^2 x_3$$





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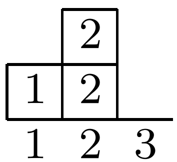
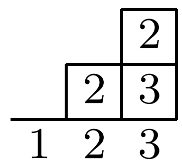


$$NS_{(0,1,2)} = x_2 x_3^2$$



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Example: $\lambda = (2,1)$

$$NS_{(2,1,0)} = x_1^2 x_2$$

$$\begin{array}{ccc} \boxed{1} & & \\ \boxed{1} & \boxed{2} & \\ \hline 1 & 2 & 3 \end{array}$$

$$NS_{(1,0,2)} = x_1 x_2 x_3 + x_1 x_3^2$$

$$\begin{array}{ccc} & & \boxed{2} \\ \boxed{1} & & \boxed{3} \\ \hline 1 & 2 & 3 \end{array} \qquad \begin{array}{ccc} & & \boxed{3} \\ \boxed{1} & & \boxed{3} \\ \hline 1 & 2 & 3 \end{array}$$

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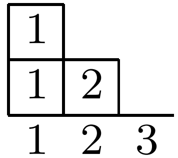

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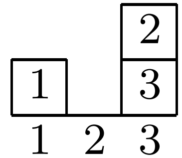

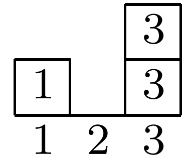

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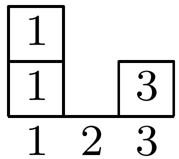

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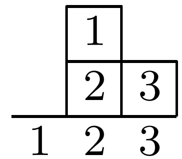

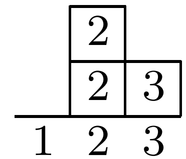

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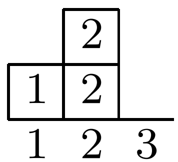
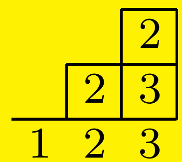


$$NS_{(0,1,2)} = x_2 x_3^2$$



$$s_{(2,1)} = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + 2x_1 x_2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2$$

The RSK Algorithm

The Robinson-Schensted-Knuth (RSK) Algorithm provides a bijection between $\mathbb{N}$-matrices with finite support and pairs (P, Q) of SSYT of the same shape.

$$\mathbb{N} - \text{matrices} \quad \overset{RSK}{\longleftrightarrow} \quad (P, Q)$$

$$\text{permutations} \quad \overset{RSK}{\longleftrightarrow} \quad (S, T),$$

for S and T standard Young tableaux.

Theorem (M)

There exists a bijection between $\mathbb{N} - matrices$ with finite support and pairs (F, G) of SAFs whose shapes rearrange the same partition.

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There exists a bijection between $\mathbb{N} - matrices$ with finite support and pairs (F, G) of SAFs whose shapes rearrange the same partition.

The fundamental operation of this bijection is the insertion process ρ described above.

Example: $A = (a_{ij})$

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Corresponding array:

$$A \longrightarrow \begin{pmatrix} 1 & 1 & 2 & 2 & 2 & 3 & 4 \\ 2 & 4 & 1 & 1 & 4 & 2 & 3 \end{pmatrix}$$

Procedure

1. Begin with the rightmost entry, α, in the bottom row of the array.

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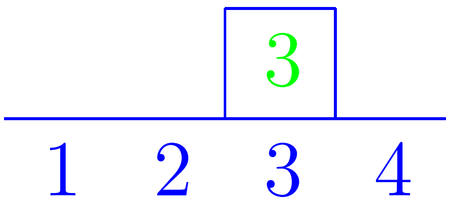
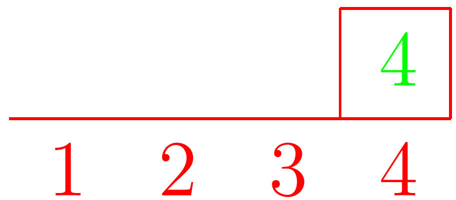
5. The entry β immediately above α in the array marks in G the position of α in F.
6. Repeat with the new F and G, with the entry immediately to the left of α.

Example

$$\begin{pmatrix} 1 & 1 & 2 & 2 & 2 & 3 & {\color{red}4} \\ 2 & 4 & 1 & 1 & 4 & 2 & {\color{blue}3} \end{pmatrix}$$



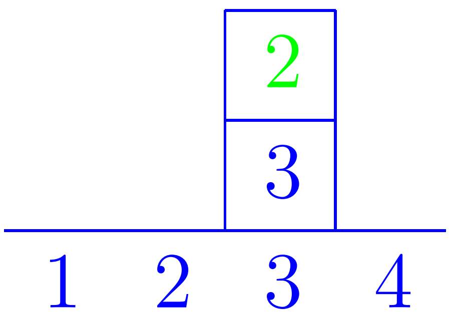
F



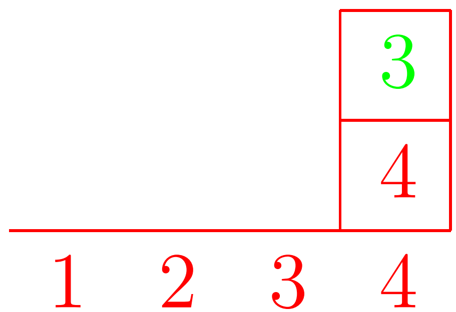
G

Example

$$\begin{pmatrix} 1 & 1 & 2 & 2 & 2 & 3 & 4 \\ 2 & 4 & 1 & 1 & 4 & 2 & 3 \end{pmatrix}$$



F



G

Example

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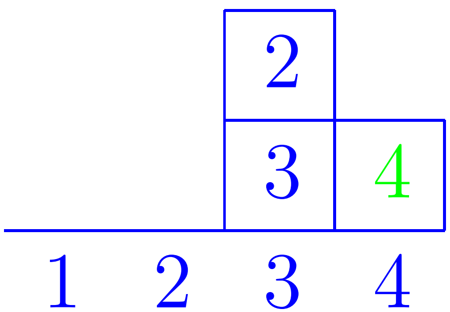
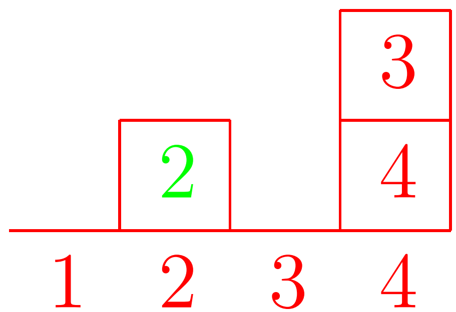


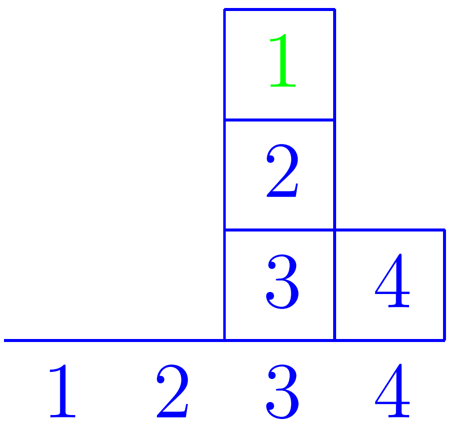
F



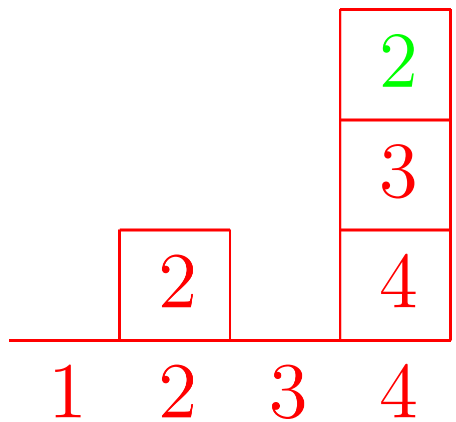
G

Example

$$\begin{pmatrix} 1 & 1 & 2 & \color{red}{2} & 2 & 3 & 4 \\ 2 & 4 & 1 & \color{blue}{1} & 4 & 2 & 3 \end{pmatrix}$$



F



G

Example

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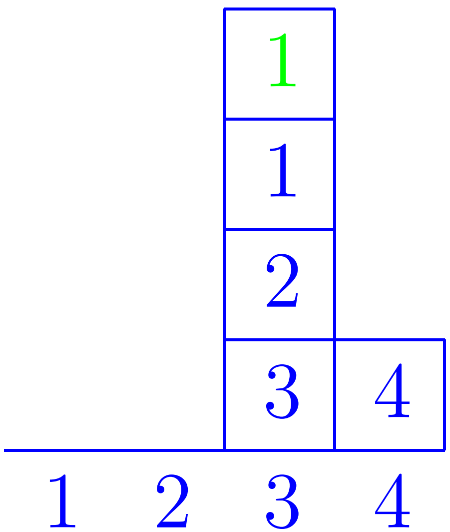
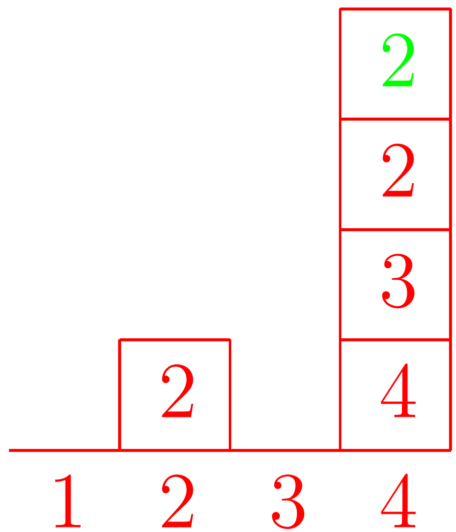


F



G

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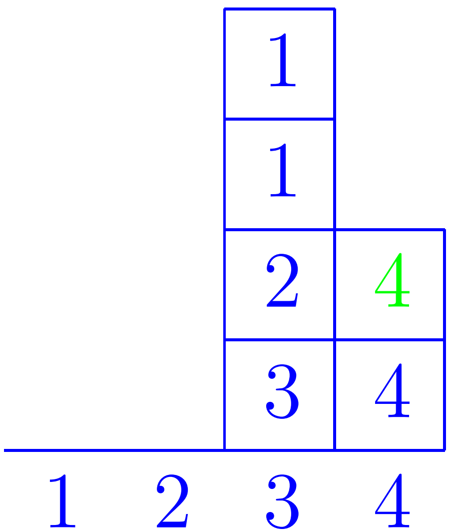
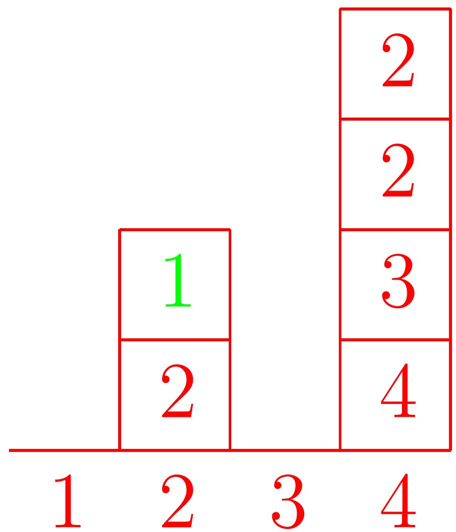


F



G

Example

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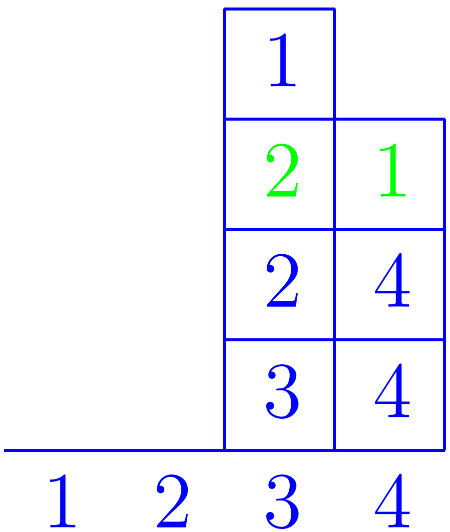
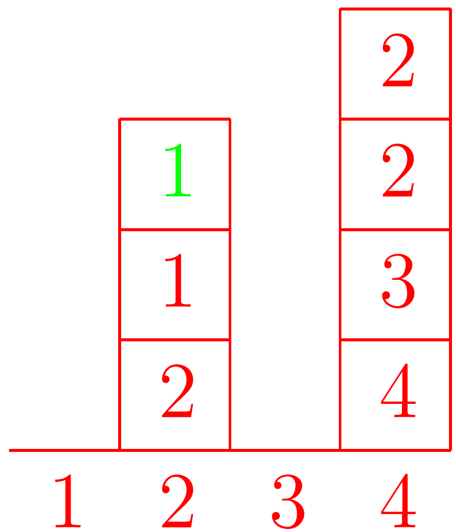


F



G

Schubert polynomials

- introduced by Lascoux and Schützenberger in 1982
- combinatorial tool to approach questions in algebraic geometry
- constructed from standard bases $\mathfrak{U}(\pi, \lambda)$ (π a permutation, λ a partition)

Definition

A key is a semi-standard Young tableau such that the entries in the $(j+1)^{th}$ column are a subset of the entries in the j^{th} column.

Example

11							
9							
8	9						
7	8	9	9				
4	7	7	7	9	9	9	
2	4	4	4	7	7	7	
1	1	1	1	4	4	4	7

Keys $\leftrightarrow$ compositions

Each key K maps to the composition denoting its content, $\gamma(K)$:

$$\gamma(K) = (\gamma_1, \gamma_2, \ldots),$$

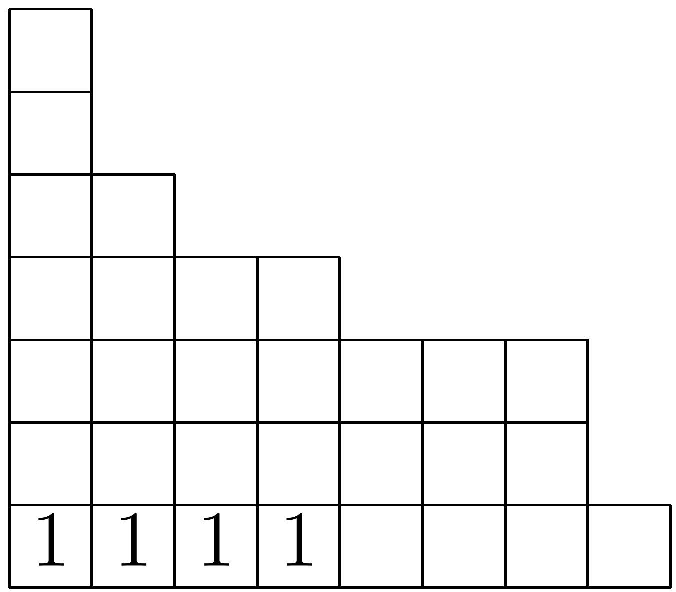
where γ_i is the number of i's in K.

$$\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
11 \\
\cline{1-1}
9 \\
\cline{1-2}
8 & 9 \\
\cline{1-4}
7 & 8 & 9 & 9 \\
\cline{1-7}
4 & 7 & 7 & 7 & 9 & 9 & 9 \\
\cline{1-7}
2 & 4 & 4 & 4 & 7 & 7 & 7 \\
\cline{1-8}
1 & 1 & 1 & 1 & 4 & 4 & 4 & 7 \\
\hline
\end{array} \longrightarrow (4, 1, 0, 7, 0, 0, 8, 2, 7, 0, 1)$$

Keys $\leftrightarrow$ *compositions*

A composition γ maps to the key $key(\gamma)$ with content γ:

$$\gamma = (\textcolor{red}{4}, 1, 0, 7, 0, 0, 8, 2, 7, 0, 1)$$



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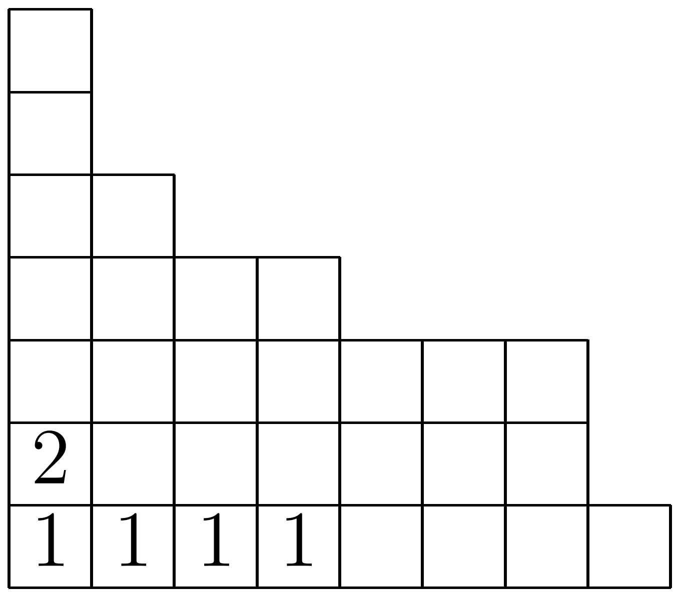

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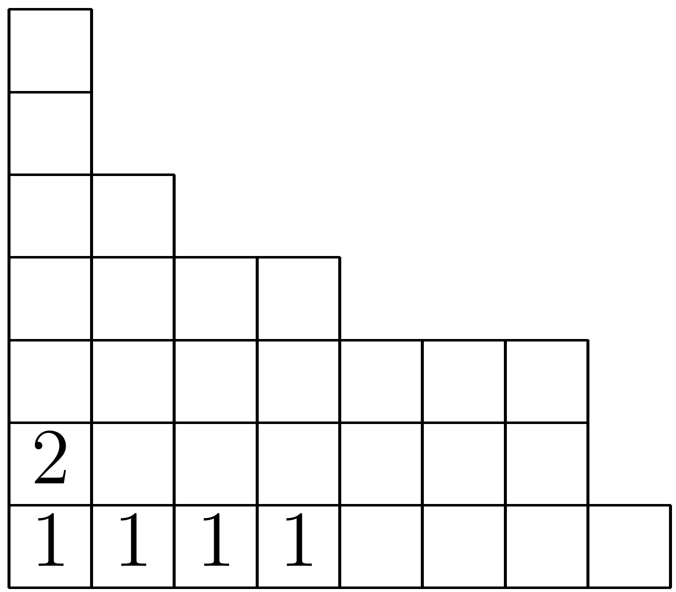

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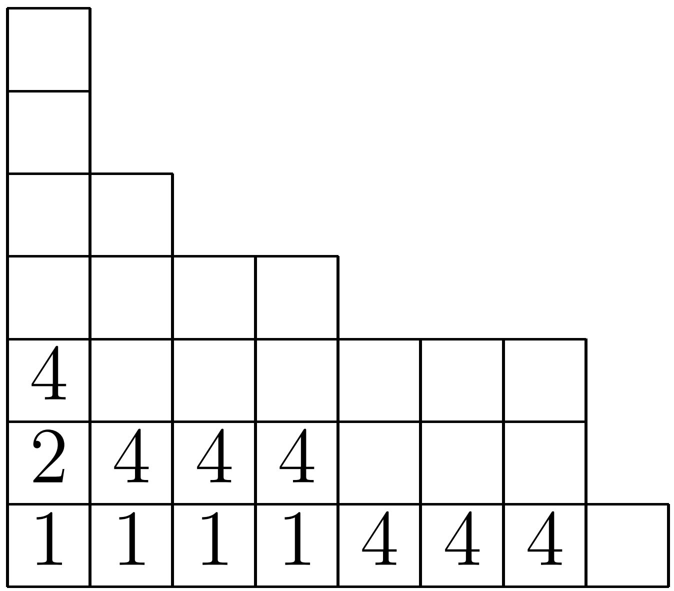

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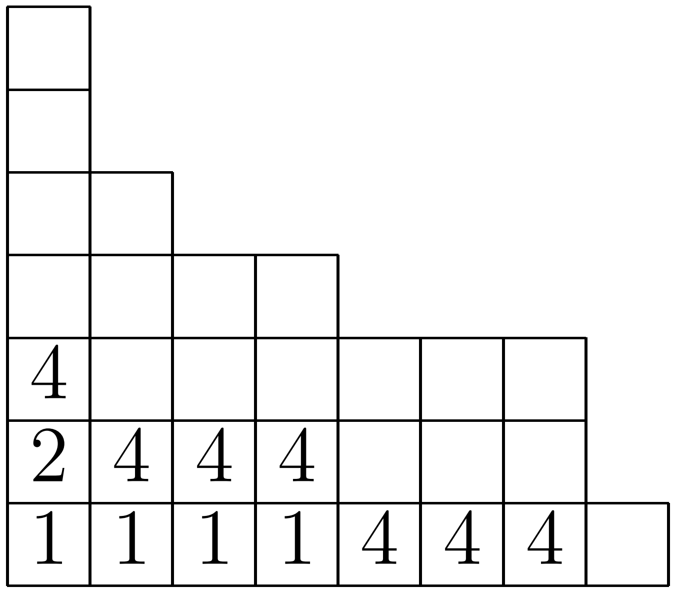
$$\begin{ytableau}
\, \\
\, \\
\, & \, \\
\, & \, & \, & \, \\
4 & \, & \, & \, & \, & \, & \, \\
2 & 4 & 4 & 4 & \, & \, & \, \\
1 & 1 & 1 & 1 & 4 & 4 & 4 & \,
\end{ytableau}$$

Keys $\leftrightarrow$ compositions

A composition γ maps to the key $key(\gamma)$ with content γ:

$$\gamma = (4, 1, 0, 7, 0, 0, 8, 2, 7, 0, 1)$$



Keys $\leftrightarrow$ compositions

A composition γ maps to the key $key(\gamma)$ with content γ:

$$\gamma = (4, 1, 0, 7, 0, 0, {\color{red}8}, 2, 7, 0, 1)$$

7							
4	7	7	7				
2	4	4	4	7	7	7	
1	1	1	1	4	4	4	7

Keys $\leftrightarrow$ compositions

A composition γ maps to the key $key(\gamma)$ with content γ:

$$\gamma = (4, 1, 0, 7, 0, 0, 8, {\color{red}2}, 7, 0, 1)$$

8							
7	8						
4	7	7	7				
2	4	4	4	7	7	7	
1	1	1	1	4	4	4	7

Keys $\leftrightarrow$ compositions

A composition γ maps to the key $key(\gamma)$ with content γ:

$$\gamma = (4, 1, 0, 7, 0, 0, 8, 2, 7, 0, 1)$$

9							
8	9						
7	8	9	9				
4	7	7	7	9	9	9	
2	4	4	4	7	7	7	
1	1	1	1	4	4	4	7

Keys $\leftrightarrow$ compositions

A composition γ maps to the key $key(\gamma)$ with content γ:

$$\gamma = (4, 1, 0, 7, 0, 0, 8, 2, 7, 0, 1)$$

9							
8	9						
7	8	9	9				
4	7	7	7	9	9	9	
2	4	4	4	7	7	7	
1	1	1	1	4	4	4	7

Keys $\leftrightarrow$ compositions

A composition γ maps to the key $key(\gamma)$ with content γ:

$$\gamma = (4, 1, 0, 7, 0, 0, 8, 2, 7, 0, 1)$$

11							
9							
8	9						
7	8	9	9				
4	7	7	7	9	9	9	
2	4	4	4	7	7	7	
1	1	1	1	4	4	4	7

Right keys

Each semi-standard Young tableau T has an associated right key denoted $K_+(T)$

$\star K_+(T)$ is determined by examining words which are Knuth equivalent to T.

Example

$$T = \begin{array}{lll}
\cline{1-1}
\multicolumn{1}{|c|}{6} & & \\
\cline{1-2}
\multicolumn{1}{|c|}{4} & \multicolumn{1}{c|}{5} & \\
\cline{1-2}
\multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{3} & \\
\cline{1-3}
\multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{4} \\
\cline{1-3}
\end{array}$$

Example

$$T = \begin{array}{|c|c|c|} \cline{1-1} 6 \\ \cline{1-2} 4 & 5 \\ \cline{1-2} 2 & 3 \\ \hline 1 & 2 & 4 \\ \hline \end{array}$$

$$w_1 = 4\ 2\ 1 \cdot 3 \cdot 6\ 5\ 4\ 2$$

$$K_+(T) = \begin{array}{|c|c|c|} \cline{1-1} 6 \\ \cline{1-2} 5 & \\ \cline{1-2} 4 & \\ \hline 2 & & \\ \hline \end{array}$$

Example

$$T = \begin{array}{|c|c|c|} \cline{1-1} 6 \\ \cline{1-2} 4 & 5 \\ \cline{1-2} 2 & 3 \\ \hline 1 & 2 & 4 \\ \hline \end{array}$$

$$w_1 = 4\ 2\ 1 \cdot 3 \cdot 6\ 5\ 4\ 2$$

$$w_2 = 6\ 4\ 2\ 1 \cdot 3 \cdot 5\ 4\ 2$$

$$K_+(T) = \begin{array}{|c|c|c|} \cline{1-1} 6 \\ \cline{1-2} 5 & 5 \\ \cline{1-2} 4 & 4 \\ \hline 2 & 2 & \\ \hline \end{array}$$

Example

$$T = \begin{array}{|c|cc}\hline 6 & \multicolumn{1}{|c}{} & \\ \hline 4 & \multicolumn{1}{c|}{5} & \\ \hline 2 & \multicolumn{1}{c|}{3} & \\ \hline 1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{4} \\ \hline \end{array}$$

$$w_1 = 4\ 2\ 1 \cdot 3 \cdot {\color{red}6\ 5\ 4\ 2}$$
$$w_2 = 6\ 4\ 2\ 1 \cdot 3 \cdot {\color{red}5\ 4\ 2}$$
$$w_3 = 6\ 4\ 2\ 1 \cdot 5\ 3\ 2 \cdot {\color{red}4}$$

$$K_+(T) = \begin{array}{|c|cc}\hline 6 & \multicolumn{1}{|c}{} & \\ \hline 5 & \multicolumn{1}{c|}{5} & \\ \hline 4 & \multicolumn{1}{c|}{4} & \\ \hline 2 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{4} \\ \hline \end{array}$$

Definition

Given:

- partition λ
- permutation π

There exists an associated key $K(\lambda, \pi)$ such that the i^{th} column of $K(\lambda, \pi)$ contains the first λ_j letters of π.

Note: The shape of $K(\lambda, \pi)$ is λ'.

Example

- $\lambda = (4, 3, 1)$
- $\pi = 61524738$

$$K(\lambda, \pi) = \begin{array}{|c|c|c|}\cline{1-1} 6 & \multicolumn{2}{c}{} \\ \cline{1-2} 5 & 6 & \multicolumn{1}{c}{} \\ \cline{1-2} 2 & 5 & \multicolumn{1}{c}{} \\ \hline 1 & 1 & 6 \\ \hline \end{array}$$

Keys and standard bases

The standard basis $\mathfrak{U}(\lambda, \pi)$ is given by the weights of all SSYT with right key $K(\lambda, \pi)$:

$$\mathfrak{U}(\lambda, \pi) = \sum_{K_+(T)=K(\lambda,\pi)} x^T$$

Theorem (M05)

The standard basis corresponding to partition λ and permutation π is equal to the nonsymmetric Schur function corresponding to the composition $\pi(\lambda)$:

$$\mathfrak{U}(\lambda, \pi) = NS_{\pi(\lambda)}$$

Remark

This theorem gives a non-inductive construction of the standard basis $\mathfrak{U}(\mu, I)$ given μ and I. The proof also uses the insertion process similar to Schensted insertion.

Sketch of proof

Each partition, $\lambda = (\lambda_1, \lambda_2, ...\lambda_k)$, has an associated *dominant monomial*

$$x^{\lambda} = (x_1^{\lambda_1} x_2^{\lambda_2} \dots x_k^{\lambda_k})$$

(The monomial x^{λ} is the weight of the super tableau of shape λ.)

Example: $\lambda = (5, 3, 3, 3, 2, 1)$

$$
\begin{array}{|c|c|c|c|c|}
\cline{1-1}
6 \\
\cline{1-2}
5 & 5 \\
\cline{1-3}
4 & 4 & 4 \\
\cline{1-3}
3 & 3 & 3 \\
\cline{1-3}
2 & 2 & 2 \\
\hline
1 & 1 & 1 & 1 & 1 \\
\hline
\end{array}
$$

$$X^{\lambda} = x_1^5 x_2^3 x_3^3 x_4^3 x_5^2 x_6$$

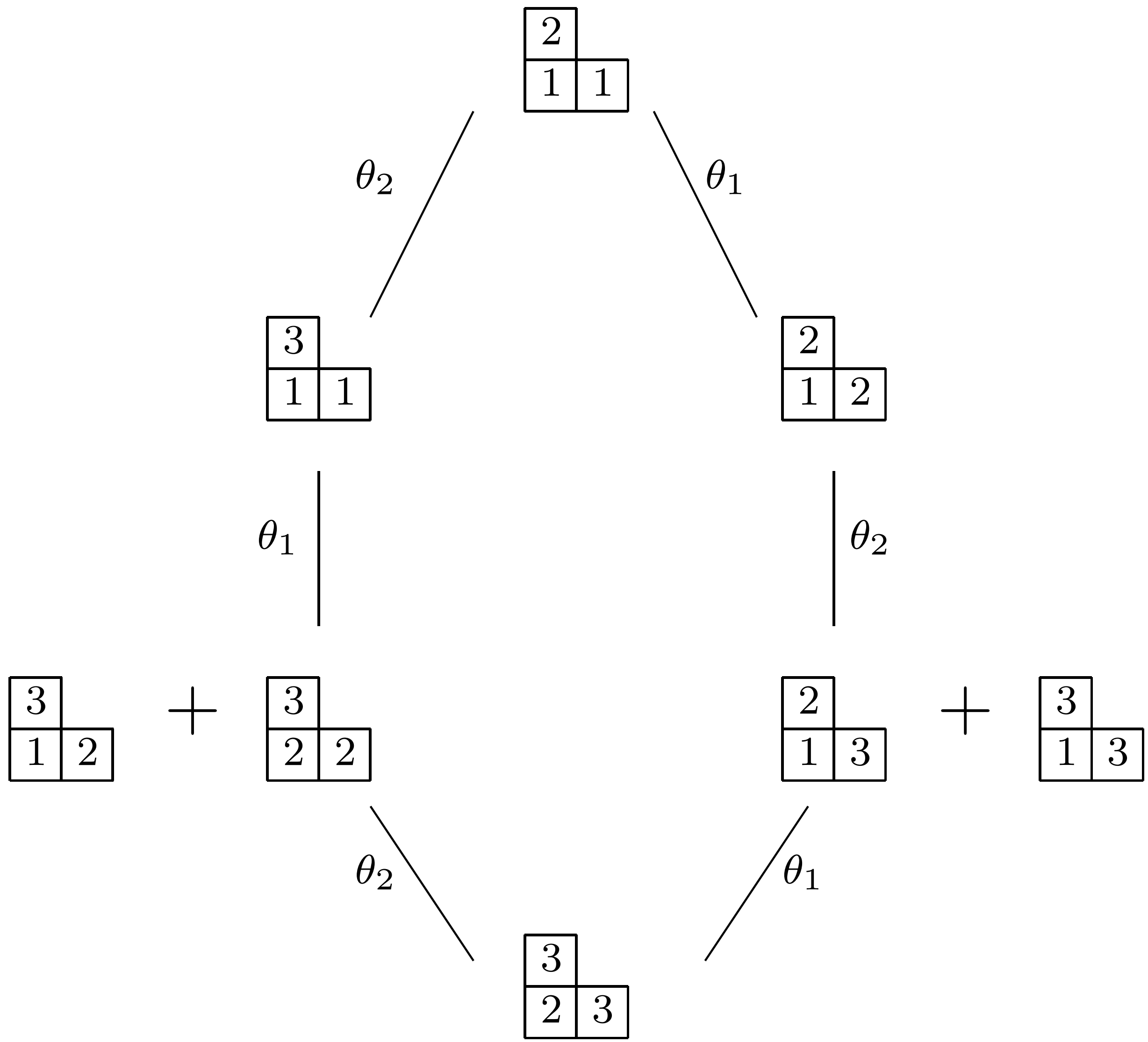
Theorem

(Lascoux, Schützenberger)

Let x^λ be a dominant monomial and $\sigma_i\sigma_j\ldots\sigma_k$ be any reduced decomposition of a permutation π. Then $\mathfrak{U}(\mu,\lambda) = x^\lambda\theta_i\theta_j\ldots\theta_k$.

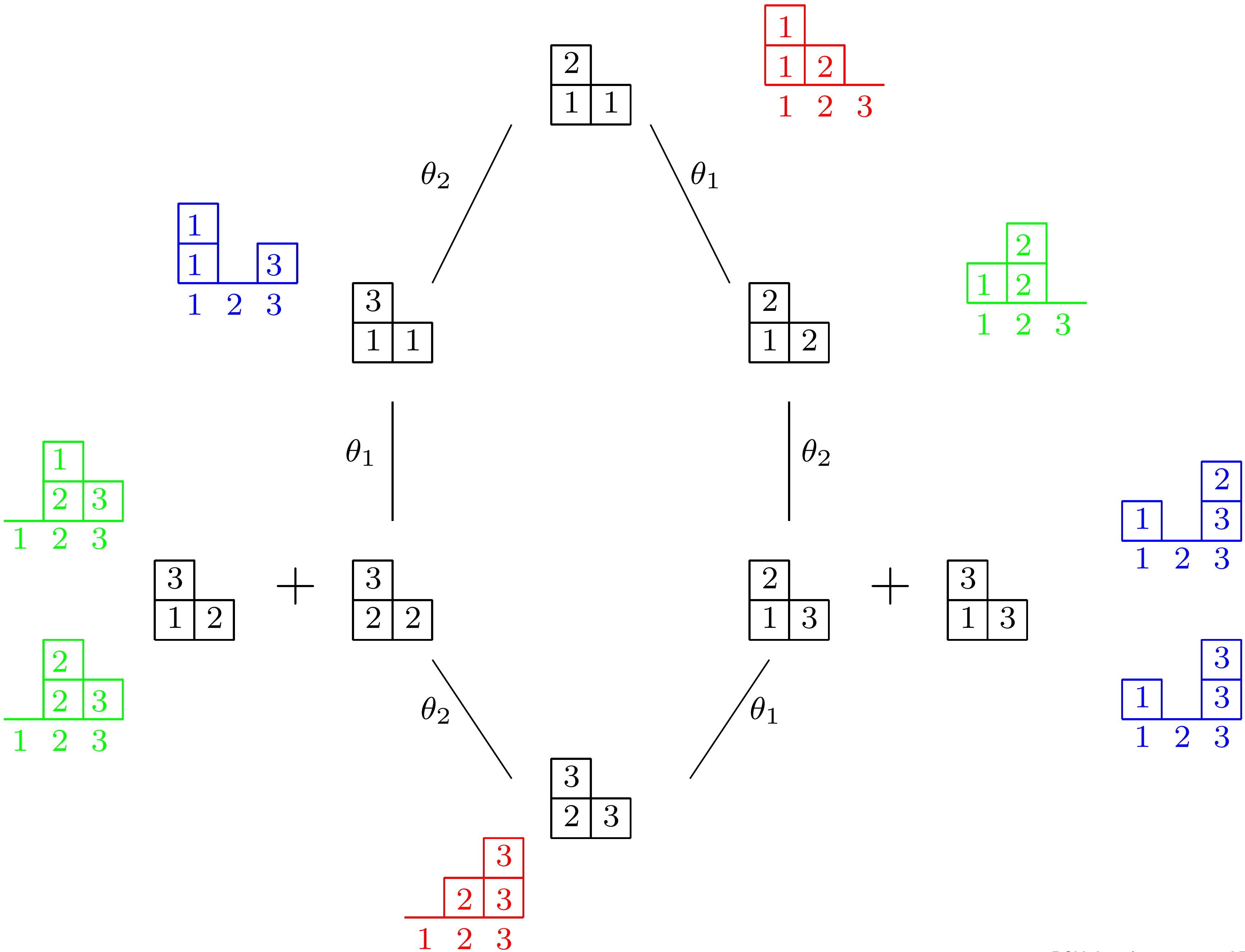
(Here θ_i is a certain action on words corresponding to the elementary transposition σ_i.)

$\lambda = (2, 1)$



2
1 1
θ_2
θ_1
3
1 1
2
1 2
θ_1
θ_2
3
1 2
+
3
2 2
2
1 3
+
3
1 3
θ_2
θ_1
3
2 3

Example: $\lambda = (2, 1)$



Corollary

The right key of a semi-standard Young tableau T is the key corresponding to the shape of $\Psi(T)$.

Proof.

- Each SSYT T maps to an SAF $\Psi(T)$.
- $\Psi(T) \in NS_\gamma$ for some γ.
- The key $key(\gamma)$ is the super SAF in NS_γ.

Example

$$T = \begin{array}{|c|c|c|}
\cline{1-1}
6 \\
\cline{1-2}
4 & 5 \\
\cline{1-2}
2 & 3 \\
\hline
1 & 2 & 4 \\
\hline
\end{array}$$

Example

$$T = \begin{array}{|c|c|c|} \hline 6 & \multicolumn{2}{c}{} \\ \cline{1-2} 4 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2} 2 & 3 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 4 \\ \hline \end{array}$$

$$\Psi(T) = \begin{array}{cccccc} & & & \boxed{2} & & \\ & \boxed{1} & & \boxed{4} & \boxed{3} & \\ & \boxed{2} & & \boxed{4} & \boxed{5} & \boxed{6} \\ \hline 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

$$sh(\Psi(T)) = (0, 2, 0, 3, 2, 1)$$

Example

$$T = \begin{array}{|c|c|c}\cline{1-1} 6 & \multicolumn{2}{c}{} \\ \cline{1-2} 4 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2} 2 & 3 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & \multicolumn{1}{c|}{4} \\ \hline \end{array}$$

$$\Psi(T) = \begin{array}{cccccc} & & & \boxed{2} & & \\ & \boxed{1} & & \boxed{4} & \boxed{3} & \\ & \boxed{2} & & \boxed{4} & \boxed{5} & \boxed{6} \\ \hline 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

$$sh(\Psi(T)) = (0, 2, 0, 3, 2, 1)$$

$$K_+(T) = \begin{array}{|c|c|c}\cline{1-1} 6 & \multicolumn{2}{c}{} \\ \cline{1-2} 5 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2} 4 & 4 & \multicolumn{1}{c}{} \\ \hline 2 & 2 & \multicolumn{1}{c|}{4} \\ \hline \end{array}$$

Where to find this...

Slides available at:

www.math.upenn.edu/ sarahm2

Pre-print:

arXiv: math.CO/0604430