The Horn Recursion for Schur P- and Q-functions

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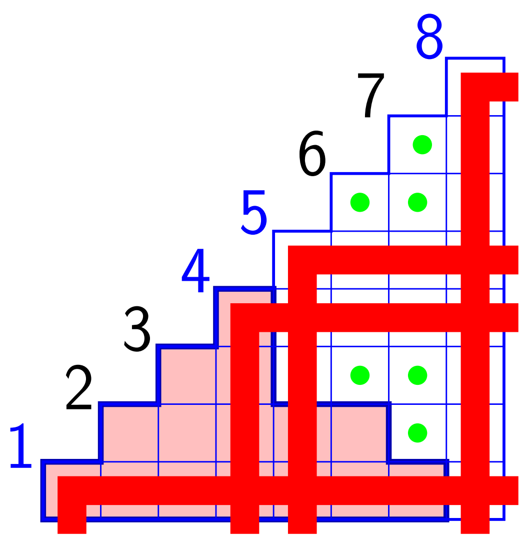
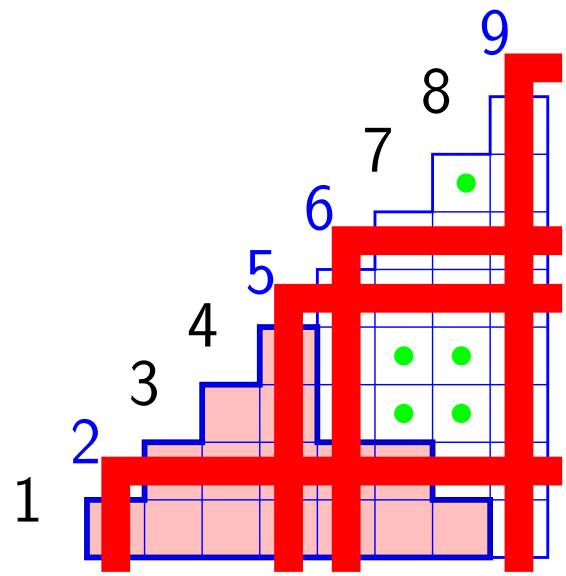
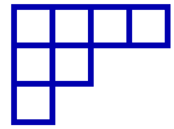




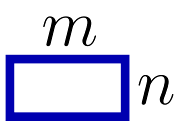
Formal Power Series & Algebraic Combinatorics
San Diego
22 June 2006

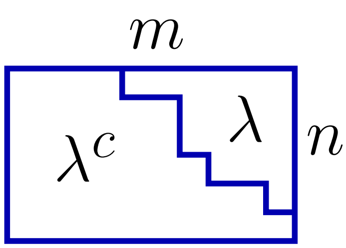
Symetric Littlewood-Richardson Numbers

Partitions $\equiv$ Young diagrams $421 \leftrightarrow$ 

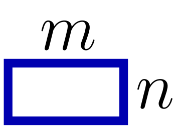
S_λ = Schur function indexed by λ

Schur functions form an intersting basis for algebra of symmetric functions, important in representation theory, combinatorics, and Schubert calculus.

For $\lambda \subset n\times m :=$  $(\lambda_1 \leq m$ & $\lambda_{n+1} = 0)$,

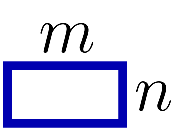
define λ^c by 

Symmetric Littlewood-Richardson numbers

For $\lambda, \mu, \nu \subset$  , define

$c_{\lambda,\mu,\nu}$:= Coefficient of $S_{n\times m}$ in $S_\lambda S_\mu S_\nu$
= Coefficient of S_{λ^c} in $S_\mu S_\nu$

Definition.

$\lambda, \mu, \nu \subset$  are *feasible* if $c_{\lambda,\mu,\nu} \neq 0$.

(Terminology from geometry.)

Horn Recursion

Let $\lambda \subset \overset{m}{\boxed{}}\,n$ and $\alpha \subset \overset{n-r}{\boxed{}}\,r$

$|\lambda|_\alpha$:= # boxes in λ that remain after crossing out rows $n{-}r{+}i{-}\alpha_i$

Example.

$n = 7$, $m = 8$, $r = 3$.

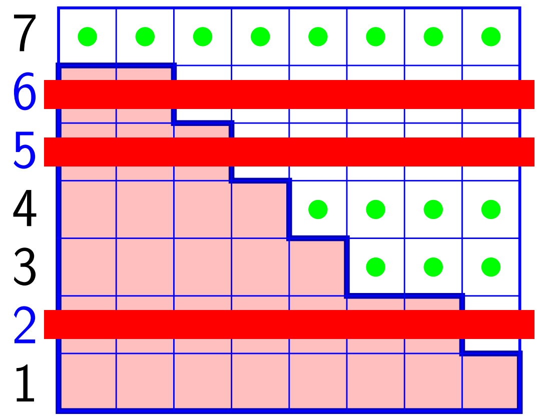
$\lambda = 8654310$

$\alpha = 311 \qquad n{-}r{+}i{-}\alpha_i$

$4+1-3 = 2$

$4+2-1 = 5$

$4+3-1 = 6$



$|\lambda|_\alpha = 15$

Horn Recursion. (Klyachko, Knutson-Tao)

$\lambda, \mu, \nu \subset \overset{m}{\boxed{}}\,n$ are feasible iff

$|\lambda| + |\mu| + |\nu| \;=\; mn$, and

$$|\lambda|_\alpha + |\mu|_\beta + |\nu|_\gamma \;\leq\; m(n-r)$$

for all feasible $\alpha, \beta, \gamma \subset \overset{n-r}{\boxed{}}\,r$ with $0 < r < n$.

Remark. Could cross out *columns* instead of rows.

Symetric Horn Recursion

Given $\lambda \subset \overset{m}{\square}\, n$ and $\alpha \subset \overset{n-r}{\square}\, r$ as before,
and $\alpha' \subset \overset{m-r}{\square}\, r$, define $|\lambda|_{\alpha,\alpha'}$ by
crossing out rows using α and columns using α'

Example

$n = 7$, $m = 8$, $r = 3$.

$\lambda = 8654310$

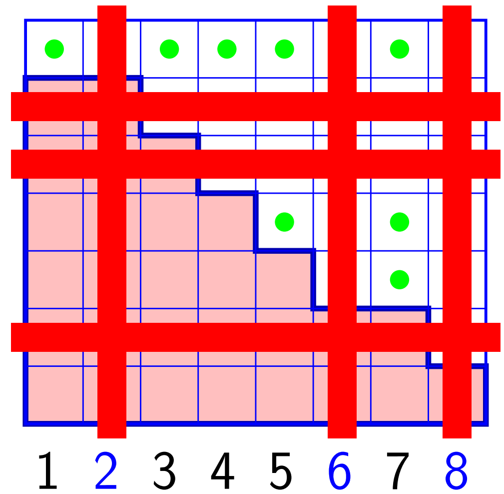
$\alpha = 311$ rows $2, 5, 6$

$\alpha' = 410$ $\quad m-r+i-\alpha'_i$

$5 + 1 - 4 = 2$

$5 + 2 - 1 = 6$

$5 + 3 - 0 = 8$



$|\lambda|_{\alpha,\alpha'} = 8.$

Symmetric Horn Recursion. (P.-Sottile)

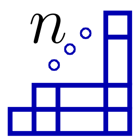
$\lambda, \mu, \nu \subset \overset{m}{\square}\, n$ are feasible iff
$|\lambda| + |\mu| + |\nu| \;=\; mn$ and

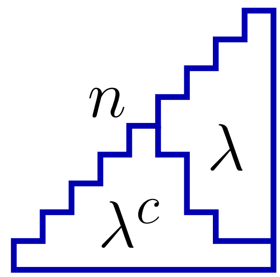
$$|\lambda|_{\alpha,\alpha'} + |\mu|_{\beta,\beta'} + |\nu|_{\gamma,\gamma'} \;\leq\; (m-r)(n-r)$$

for all feasible $\alpha, \beta, \gamma \subset \overset{n-r}{\square}\, r$
and all feasible $\alpha', \beta', \gamma' \subset \overset{m-r}{\square}\, r$
with $0 < r < \min(m, n)$ (same r).

Schur P- and Q- Functions

Q_λ = Schur Q-function indexed by strict partition λ
These form an interesting basis for odd subalgebra of combinatorial Hopf algebra of symmetric functions.
Important for representation theory and isotropic Schubert calculus.

For $\lambda \subset \Delta_n :=$ ,

define $\lambda^c := \Delta_n - \lambda$ or pictorially 

Symmetric Littlewood-Richardson numbers

For $\lambda, \mu, \nu \subset \Delta_n$, define

$$q_{\lambda,\mu,\nu} := \text{Coefficient of } Q_{\Delta_n} \text{ in } Q_\lambda Q_\mu Q_\nu$$
$$= \text{Coefficient of } Q_{\lambda^c} \text{ in } Q_\mu Q_\nu$$

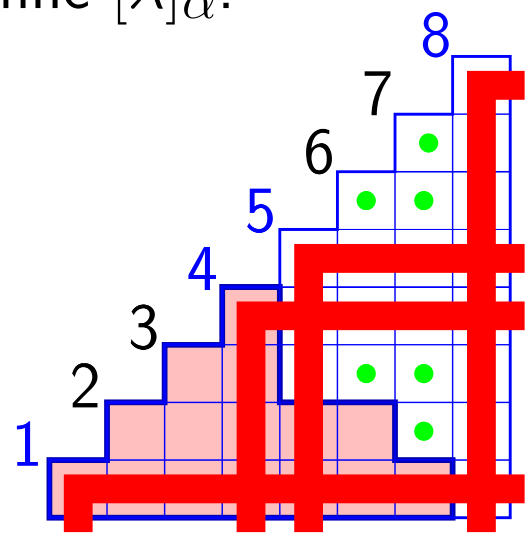
Definition.
$\lambda, \mu, \nu \subset \Delta_n$ are *feasible* if $q_{\lambda,\mu,\nu} \neq 0$.

Schur P-functions P_λ are another basis.
Have symmetric Littlewood-Richardson coeffients $p_{\lambda,\mu,\nu}$ with the same feasible triples.

Two measures, $[\lambda]_\alpha$ and $\{\lambda\}_\alpha$

Given $\lambda \subset \Delta_n$ and $\alpha \subset \overset{n-r}{\square}\, r$, cross out hooks at inner corners given by α to define $[\lambda]_\alpha$:

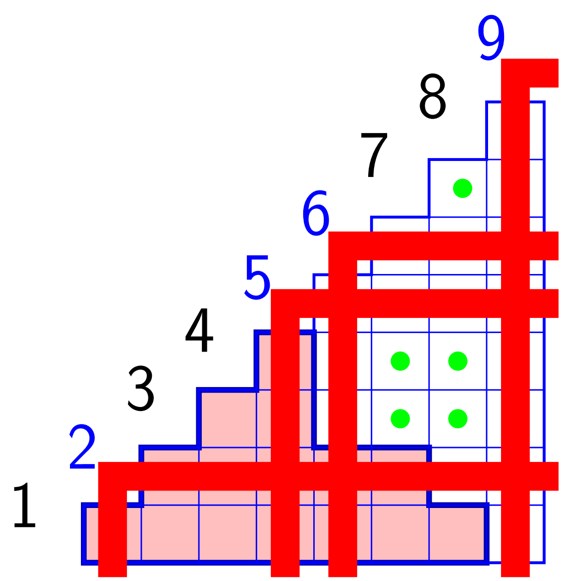
$n = 8$, $r = 4$.
$\lambda = 8643$
$\alpha = 4220 \qquad n{-}r{+}i{-}\alpha_i$
$4 + 1 - 4 = 1$
$4 + 2 - 2 = 4$
$4 + 3 - 2 = 5$
$4 + 4 - 0 = 8$



$[\lambda]_\alpha = 6$

Given instead $\alpha \subset \overset{n+1-r}{\square}\, r$, cross out hooks at outer corners given by α and $n+1$ to define $\{\lambda\}_\alpha$:

$n = 8$, $r = 4$.
$\lambda = 8643$
$\alpha = 4220 \quad n{+}1{-}r{+}i{-}\alpha_i$
$5 + 1 - 4 = 2$
$5 + 2 - 2 = 5$
$5 + 3 - 2 = 6$
$5 + 4 - 0 = 9$



$\{\lambda\}_\alpha = 5$

Horn Recursion. (P.-Sottile)

$\lambda, \mu, \nu \subset \triangle_n$ are feasible iff

$|\lambda| + |\mu| + |\nu| \;=\; \binom{n+1}{2} \;=\; |\triangle_n|$

and either

(1) $[\lambda]_\alpha + [\mu]_\beta + [\nu]_\gamma \;\leq\; \binom{n+1-r}{2} \;=\; |\triangle_{n-r}|$

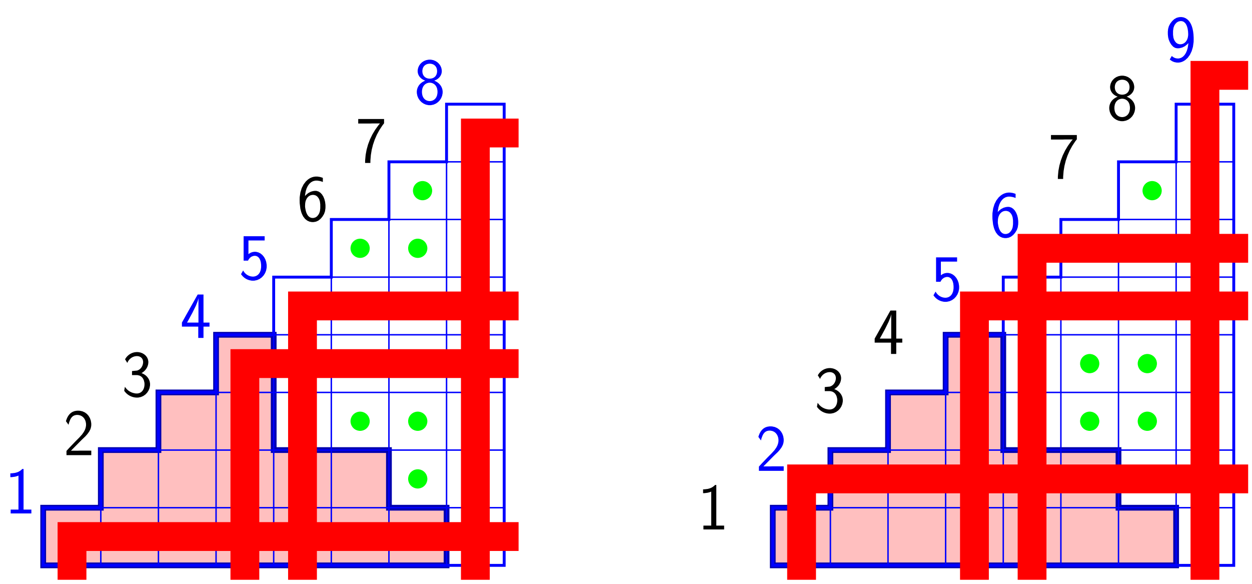
for all feasible $\alpha, \beta, \gamma \subset \overset{n-r}{\boxed{\qquad}}\, r$ with $0 < r < n$

or else

(2) $\{\lambda\}_\alpha + \{\mu\}_\beta + \{\nu\}_\gamma \;\leq\; \binom{n+1-r}{2} \;=\; |\triangle_{n-r}|$

for all feasible $\alpha, \beta, \gamma \subset \overset{n+1-r}{\boxed{\qquad}}\, r$

with $0 < r < n{+}1$ and r even.



Underlying Geometry

$G/P =$ cominuscule flag variety
$L =$ Levi subgroup of P, acts on $\mathfrak{g}/\mathfrak{p}$ (finitely many orbits)
$\lambda \rightsquigarrow T_\lambda \subset \mathfrak{g}/\mathfrak{p}$ (tangent space to Schubert variety)

For $x \in \mathfrak{g}/\mathfrak{p}$, consider quotient map:

$$\phi_x : \mathfrak{g}/\mathfrak{p} \to (\mathfrak{g}/\mathfrak{p})\big/(T_x(L \cdot x))$$

Idea. λ, μ, ν are feasible

$\iff \; a \cdot T_\lambda,\; b \cdot T_\mu,\; c \cdot T_\nu \subset \mathfrak{g}/\mathfrak{p}$ are transverse

for generic $a, b, c \in L$

$$\iff \text{codim}\, \phi_x(a \cdot T_\lambda) + \text{codim}\, \phi_x(b \cdot T_\mu) + \text{codim}\, \phi_x(c \cdot T_\nu) \le \text{rank}(\phi_x)$$

for all $x \in \mathfrak{g}/\mathfrak{p}$.

Simplified, these become the Horn inequalities.

	G	$\mathfrak{g}/\mathfrak{p}$	L	L-orbits on $\mathfrak{g}/\mathfrak{p}$
Schur fcns	$GL(m+n)$	Mat$(m\times n)$	$GL(m)\times GL(n)$	rank r
Schur Q-fcns	$Sp(2n)$	Sym(n)	$GL(n)$	rank r
Schur P-fcns	$SO(2n+2)$	SkewSym$(n+1)$	$GL(n+1)$	(even) rank $2r$