Virtual Crystal Structure on Rigged Configurations

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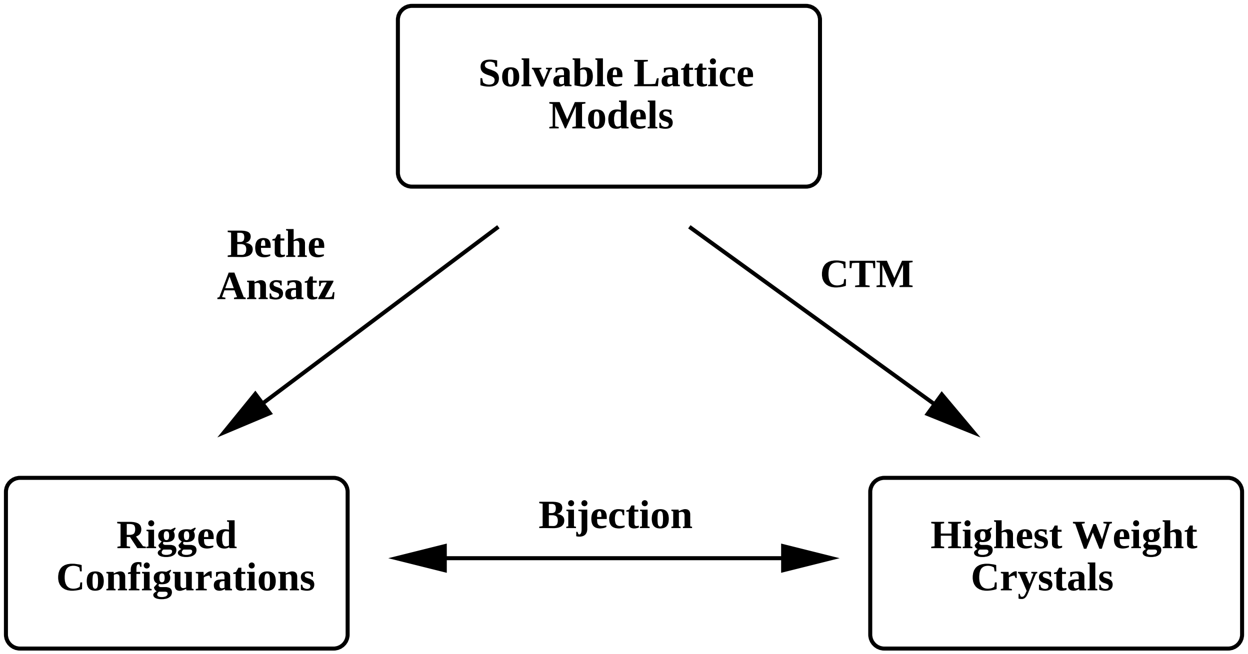
FPSAC06, San Diego
June 20, 2006

References

This talk is based on the following papers:

- A. Schilling,
  Crystal structure on rigged configurations,
  International Mathematics Research Notices,
  Volume 2006, Article ID 97376, Pages 1-27
  (math.QA/0508107)
- M. Okado, A. Schilling, M. Shimozono,
  Virtual crystals and Kleber's algorithm,
  Commun. Math. Phys. **238** (2003) 187–209
  (math.QA/0209082)

Motivation



1988 Kerov, Kirillov, Reshetikhin for Kostka polynomials

2002 Kirillov, S., Shimozono for type A

2003/2004 Okado, S., Shimozono for all nonexceptional cases

$\rightsquigarrow X = M$ conjecture of HKOTTY

Outline

- Virtual crystals
- Rigged configurations
- Virtual rigged configurations
- Crystal structure on rigged configurations
- Outlook

Embeddings of affine algebras

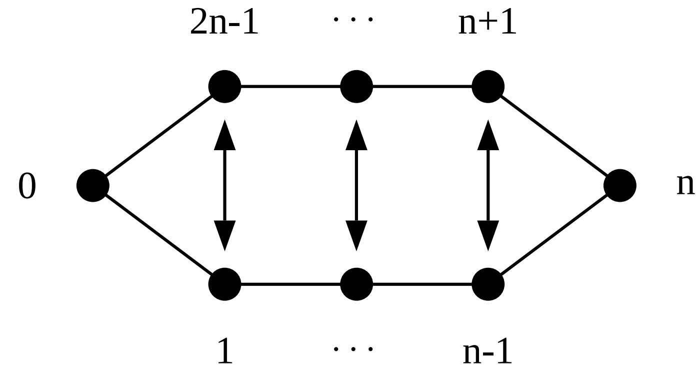
$X \hookrightarrow Y$

Graph automorphism σ of Y fixing 0

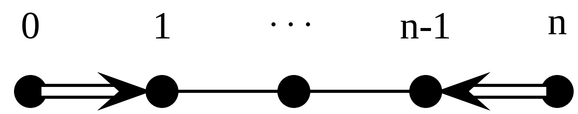
I^X, I^Y	vertex set of diagram X, Y
I^Y/σ	σ-orbits in I^Y
$I^X \xrightarrow{\iota} I^Y/\sigma$	bijection which preserves edges

Embeddings of affine algebras

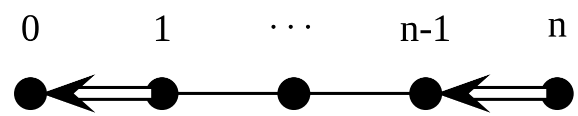
$A_{2n-1}^{(1)}$



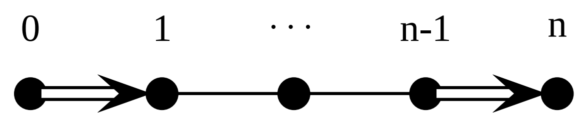
$C_n^{(1)}$



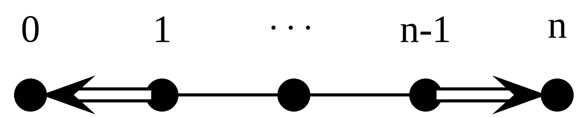
$A_{2n}^{(2)}$



$A_{2n}^{(2)\dagger}$

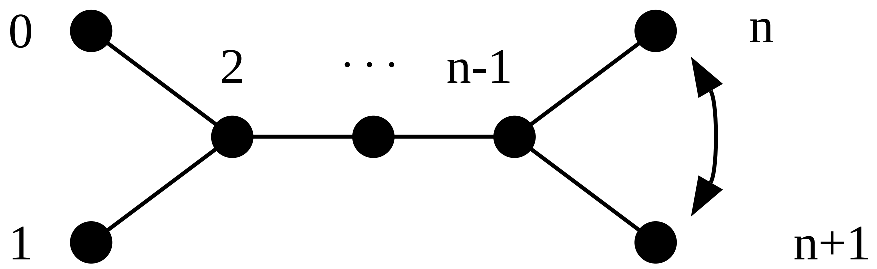


$D_{n+1}^{(2)}$

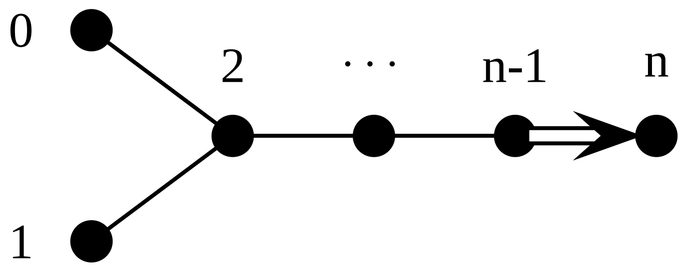


Embeddings of affine algebras

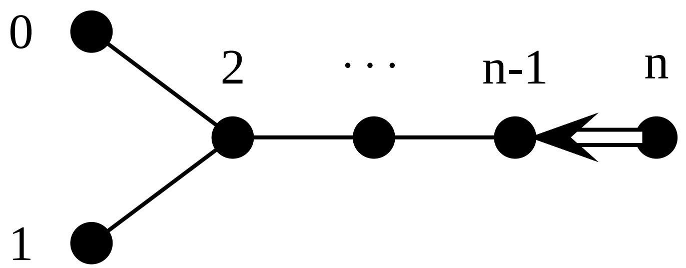
$D_{n+1}^{(1)}$



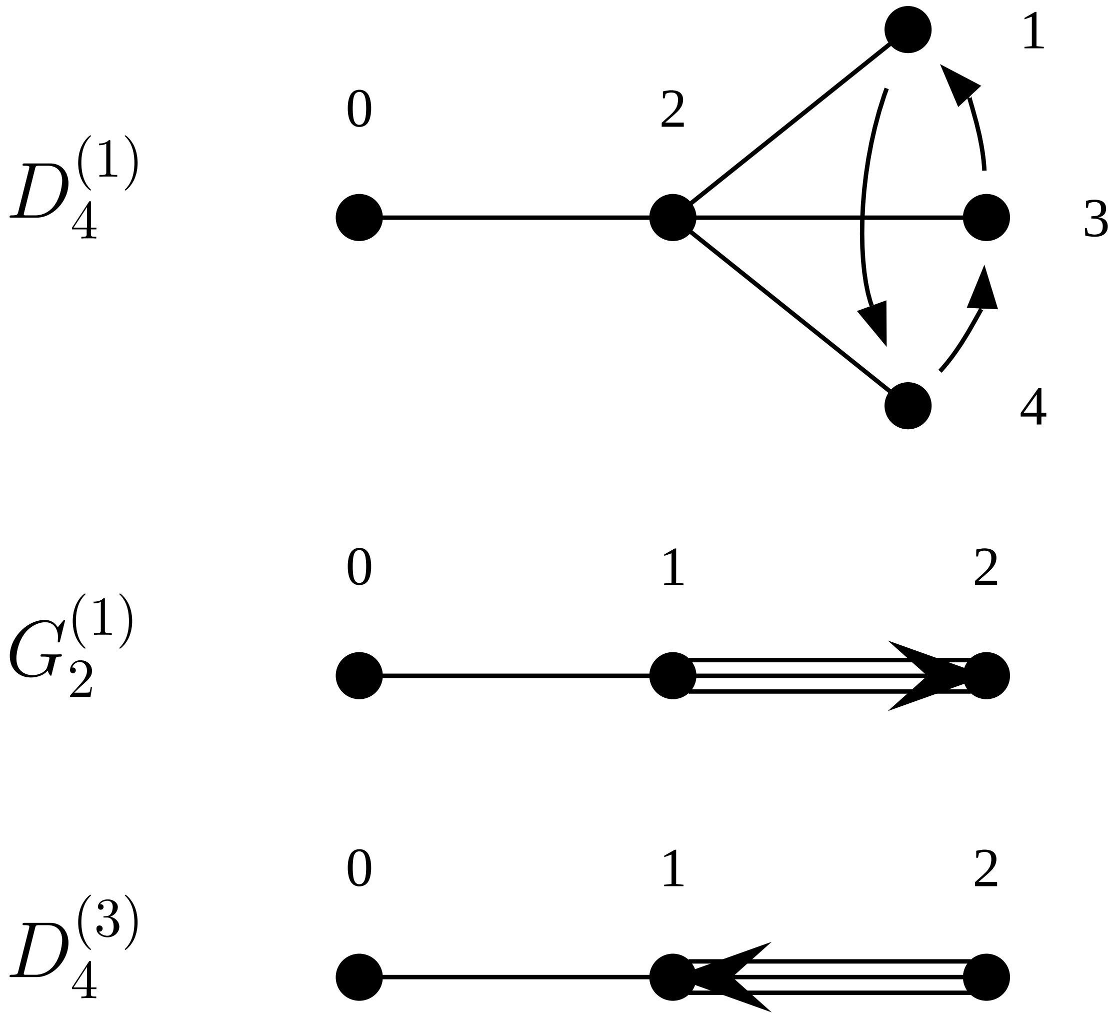
$B_n^{(1)}$



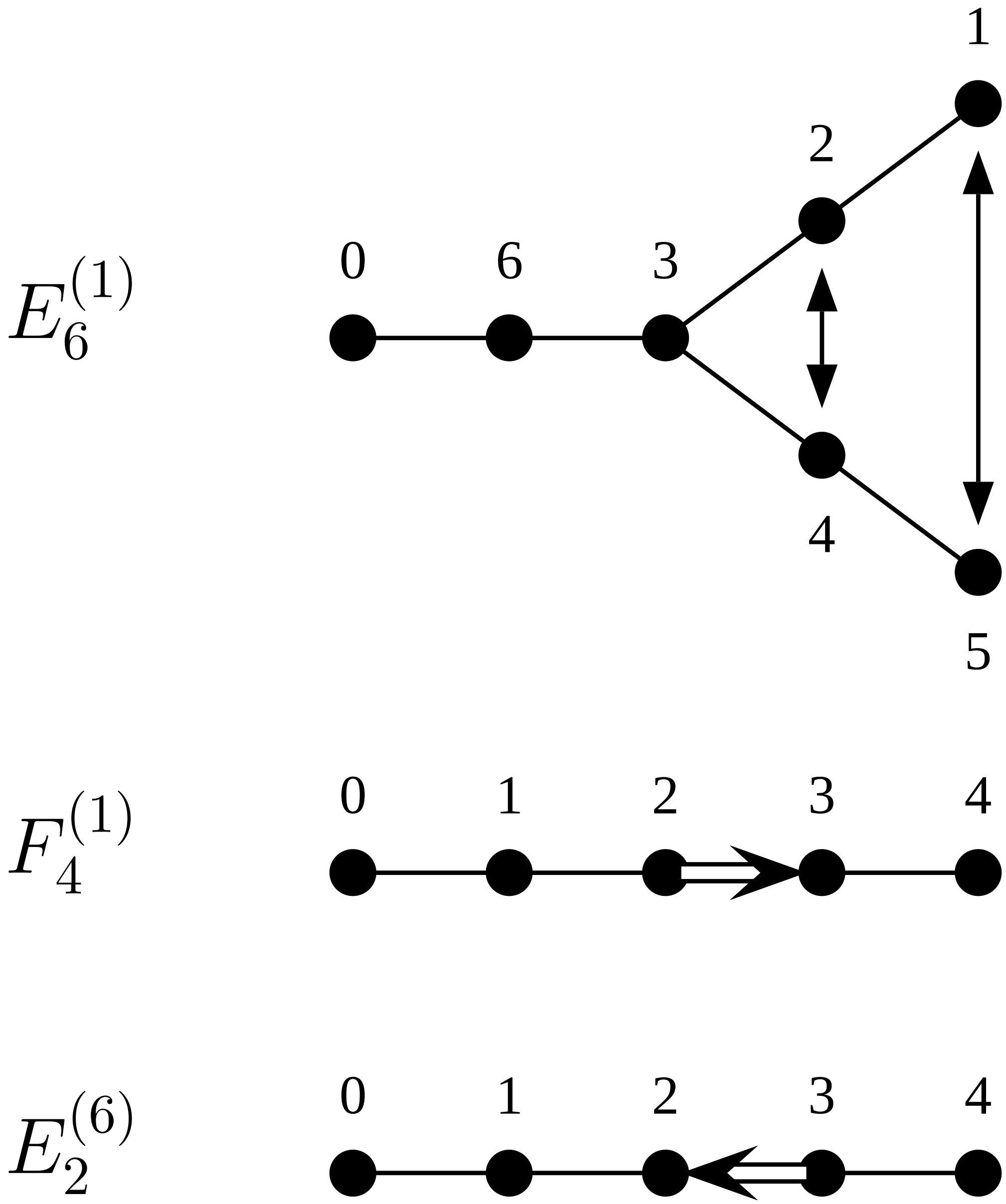
$A_{2n-1}^{(2)}$



Embeddings of affine algebras

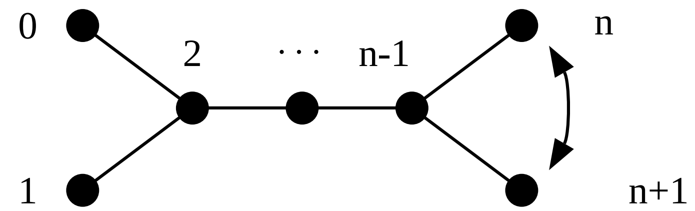


Embeddings of affine algebras



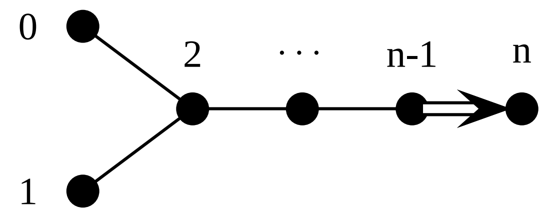
Multiplication factor γ_i

$$Y = D_{n+1}^{(1)}$$



(1) X has unique arrow
(a) arrow points away from 0-component

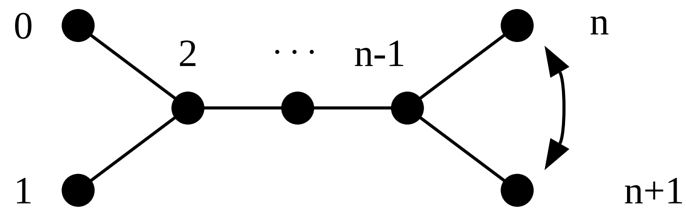
$$B_n^{(1)}$$



$$\gamma_i = \begin{cases} \text{order}(\sigma) & \text{for } i \text{ in 0-component} \\ 1 & \text{else} \end{cases}$$

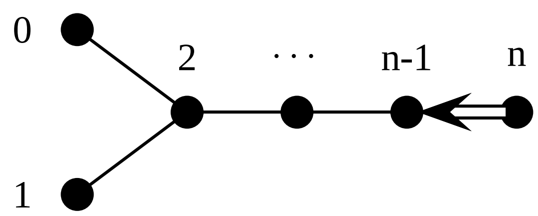
Multiplication factor γ_i

$$Y = D^{(1)}_{n+1}$$



(1) X has unique arrow
(b) arrow points towards 0-component

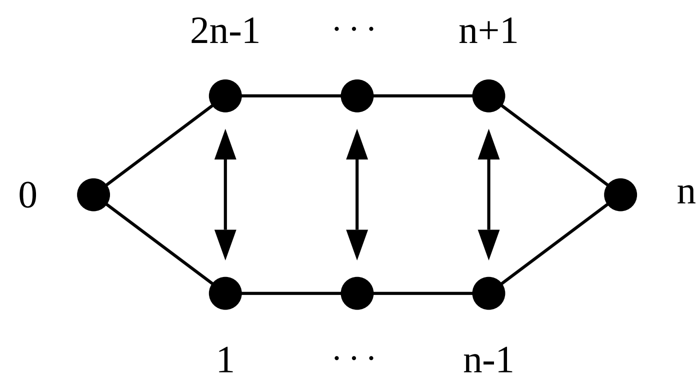
$$A^{(2)}_{2n-1}$$



$$\gamma_i = 1 \qquad \text{for all } i$$

Multiplication factor γ_i

$$Y = A^{(1)}_{2n-1}$$



(2) X has two arrows, i.e. $Y = A^{(1)}_{2n-1}$

$$\gamma_i = 1 \quad \text{if } 1 \leq i \leq n-1$$

$$\gamma_i = 2 \quad \text{if } i = 0, n, \text{ arrow points away from } i$$

$$\gamma_i = 1 \quad \text{else}$$

Embedding

$$P^X \xrightarrow{\Psi} P^Y$$

$$\Lambda_i^X \mapsto \gamma_i \sum_{j \in \iota(i)} \Lambda_j^Y$$

Multiplication factor $\widetilde{\gamma}_i$

$$\widetilde{\gamma}_i = \begin{cases} 1 & \text{if } i = n \text{ for } A_{2n}^{(2)} \\ \gamma_i & \text{else} \end{cases}$$

Virtual crystals

$\widehat{V}$ is Y-crystal

Virtual crystal operator $\widehat{f}_i$ for $i \in I^X$

$$\widehat{f}_i = \prod_{j \in \iota(i)} f_j^{\gamma_i}$$

Virtual crystals

$\widehat{V}$ is Y-crystal

Virtual crystal operator $\widehat{f}_i$ for $i \in I^X$

$$\widehat{f}_i = \prod_{j \in \iota(i)} f_j^{\gamma_i}$$

A virtual crystal is a pair $(V, \widehat{V})$ such that:

1. $\widehat{V}$ is a Y-crystal.
2. $V \subset \widehat{V}$ is closed under $\widehat{f}_i$ for $i \in I^X$.
3. There is an X-crystal B and an X-crystal isomorphism $\Psi : B \to V$ such that

$$\widehat{f}_i \Psi(b) = \Psi(f_i b)$$

Virtual KR crystals

$$\widehat{V}^{r,s} = \bigotimes_{j \in \iota(r)} B_Y^{j,\gamma_r s}$$

Def $V^{r,s}$ subset of $\widehat{V}^{r,s}$ generated from $u(\widehat{V}^{r,s})$ using virtual crystal operator $\widehat{f}_i$ for $i \in I^X$.

Virtual KR crystals

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Conj. [OSS] There is an isomorphism of X-crystals

$$\Psi : B_X^{r,s} \cong V^{r,s}$$

such that f_i corresponds to $\widehat{f}_i$ for all $i \in I^X$.

Virtual KR crystals

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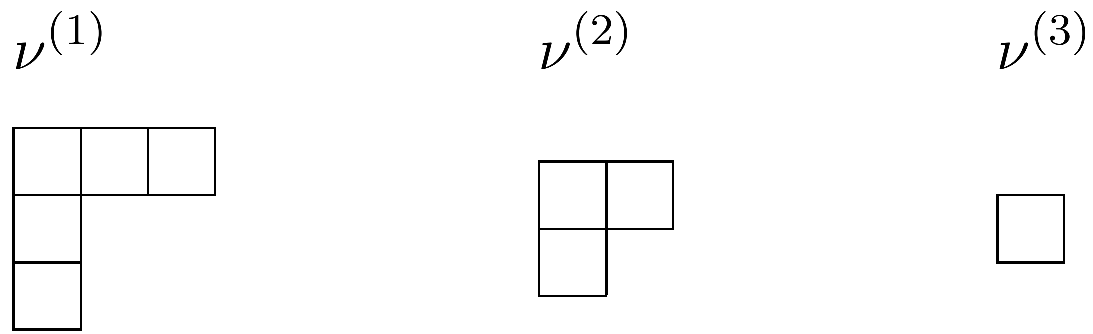
Proven for:
- $C_n^{(1)}, A_{2n}^{(2)}, D_{n+1}^{(2)} \hookrightarrow A_{2n-1}^{(1)}$ and $s = 1$
- nonexceptional cases, $r = 1$

Rigged configurations



(L,Λ)-configuration

$$\sum_{(a,i)\in\mathcal{H}} i m_i^{(a)} \alpha_a = \sum_{(a,i)\in\mathcal{H}} i L_i^{(a)} \Lambda_a - \Lambda$$

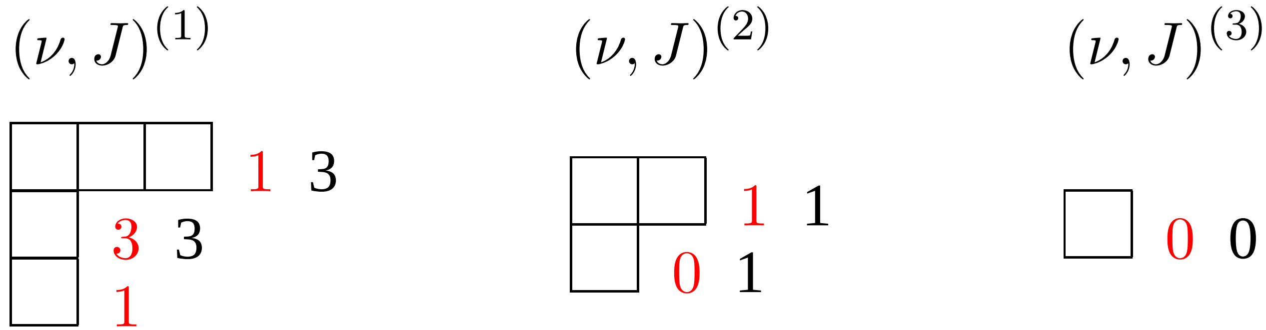
where $\mathcal{H} = \{1,2,\ldots,n\} \times \mathbb{Z}_{>0}$

$L = (L_i^{(a)} \mid (a,i) \in \mathcal{H})$ nonnegative integers

$m_i^{(a)}$ number of parts of size i in $\nu^{(a)}$

and Λ dominant weight, Λ_a fundamental weight, α_a simple root

Rigged configurations



Vacancy numbers

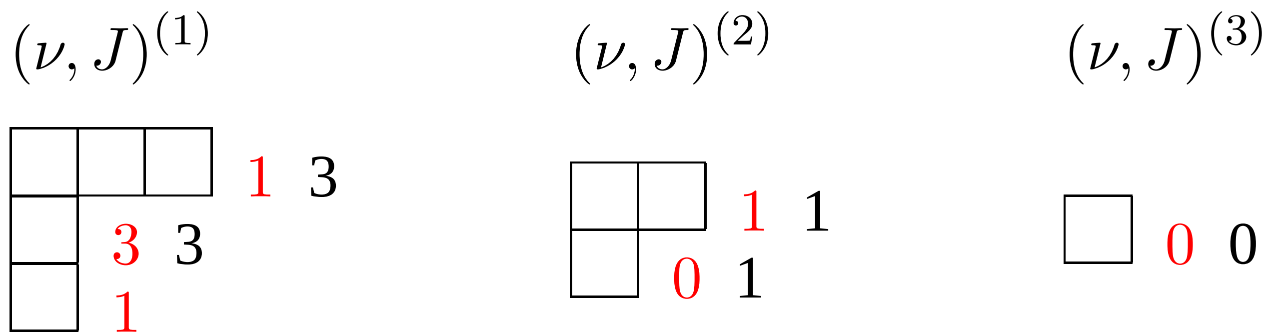
$$p_i^{(a)} = \sum_{j \geq 1} \min(i,j) L_j^{(a)} - \sum_{(b,j) \in \mathcal{H}} (\alpha_a | \alpha_b) \min(i,j) m_j^{(b)}$$

Admissible (L, Λ)-configuration

$p_i^{(a)} \geq 0$ for all $(a,i) \in \mathcal{H}$

$\overline{C}(L,\Lambda)$ set of admissible (L,Λ)-configurations

Rigged configurations



Rigged configuration

Attach a label x to each part i of $\nu^{(a)}$ s.t.

$$0 \leq x \leq p_i^{(a)}$$

$\overline{\mathrm{RC}}(L, \Lambda)$ set of all (L, Λ)-rigged configurations

Virtual rigged configurations

Def $X \hookrightarrow Y$

$\widehat{L}^{(b)}_{\gamma_a i} = L^{(a)}_i, \quad b \in \iota(a)$

$\mathrm{RC}^v(L,\lambda)$ set of $(\widehat{\nu},\widehat{J}) \in \mathrm{RC}(\widehat{L},\Psi(\lambda))$ such that:

$$1. \quad \widehat{m}^{(a)}_i = \widehat{m}^{(b)}_i$$

$$\widehat{J}^{(a)}_i = \widehat{J}^{(b)}_i$$

$$2. \quad \widehat{m}^{(b)}_j = 0 \quad \text{if } j \notin \widetilde{\gamma}_a \mathbb{Z}$$

$$\text{parts of } \widehat{J}^{(b)}_i \in \gamma_a \mathbb{Z}$$

if a, b are in the same σ-orbit in I^Y

Virtual rigged configurations

Theorem [OSS]
There exists a bijection

$$\mathrm{RC}(L,\lambda) \longrightarrow \mathrm{RC}^v(\widehat{L},\lambda)$$
$$(\nu,J) \mapsto (\widehat{\nu},\widehat{J})$$

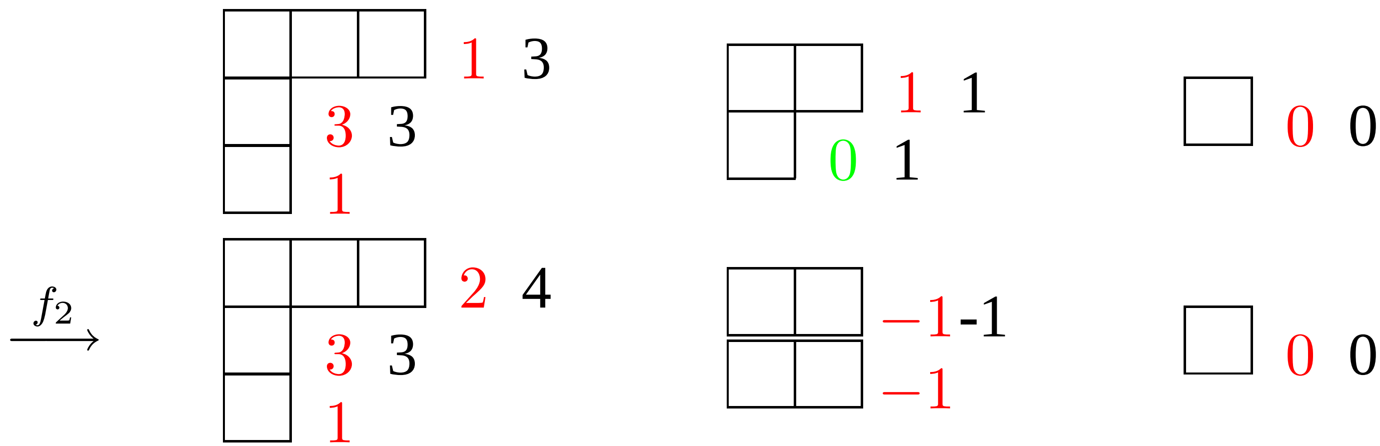
where
$$\widehat{m}^{(b)}_{\widetilde{\gamma}_a i} = m^{(a)}_i$$
$$\widehat{J}^{(b)}_{\widetilde{\gamma}_a i} = \gamma_a J^{(a)}_i \quad \text{for } b \in \iota(a) \subset I^Y$$

The cocharge changes by

$$\mathrm{cc}(\widehat{\nu},\widehat{J}) = \gamma_0 \, \mathrm{cc}(\nu,J)$$

Crystal structure on RCs

Action of f_a:



$f_a(\nu, J)$:

- add γ_a boxes to string of length k in $(\nu, J)^{(a)}$
- leave all colabels fixed, decrease the new label by 1

k is length of string with smallest nonpositive rigging of largest length

Crystal structure on RCs

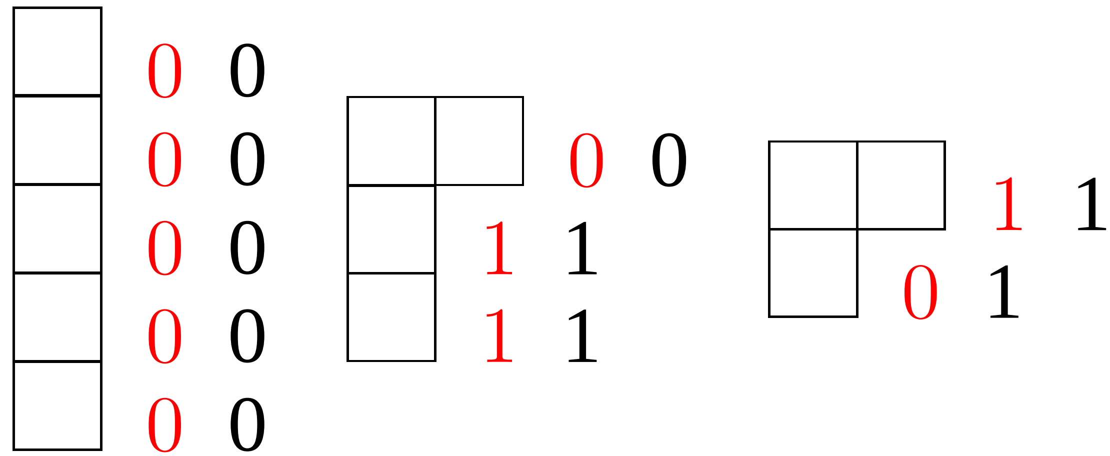
Theorem [S] The operators f_a are Kashiwara crystal operators.

Proof:
For simply-laced types uses Stembridge's local characterization of crystals.
For nonsimply-laced types uses virtual crystal method.

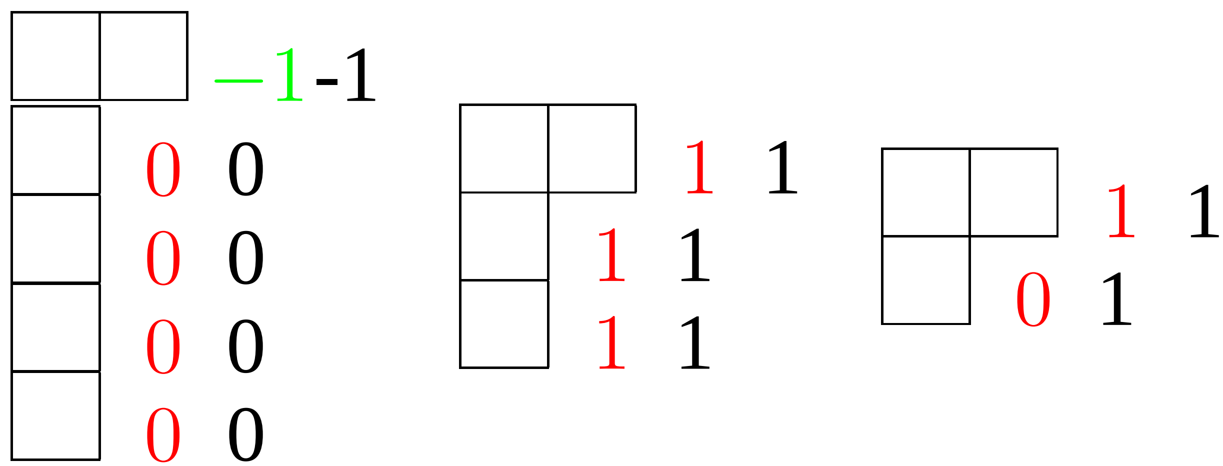
Example

RC of type $A_6^{(2)}$, $\Lambda = \Lambda_1 + \Lambda_3$, $L_1^{(1)} = 7$

$(\nu, J) =$



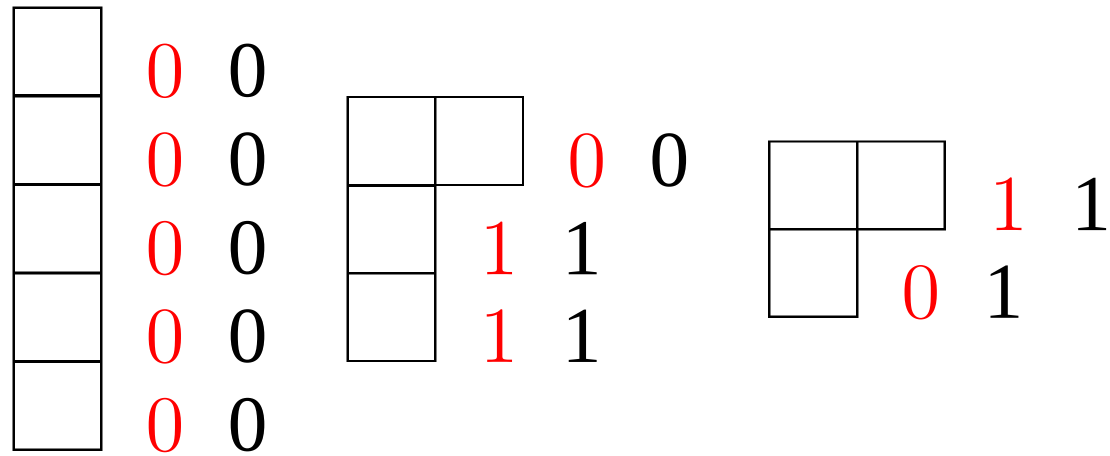
$f_1(\nu, J) =$



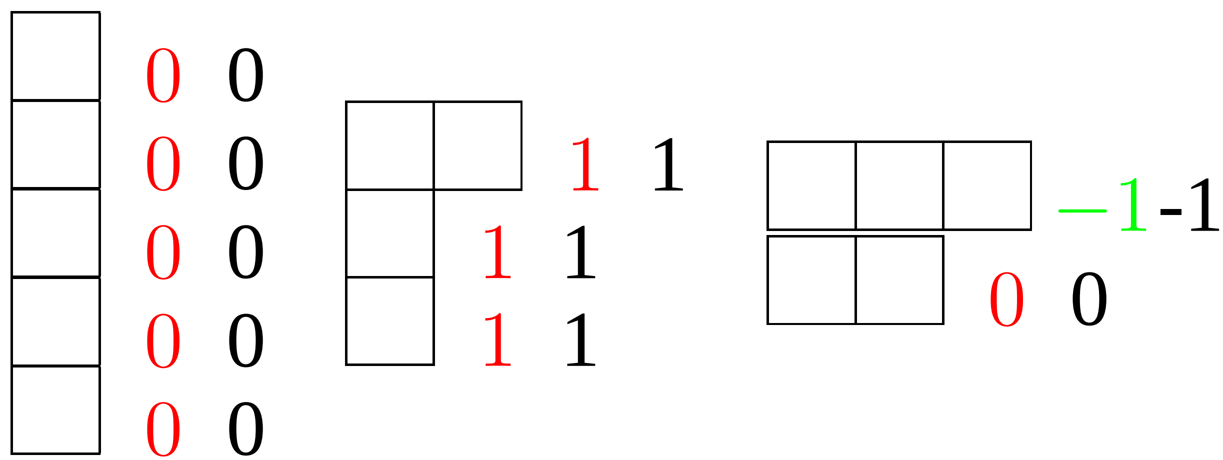
Example

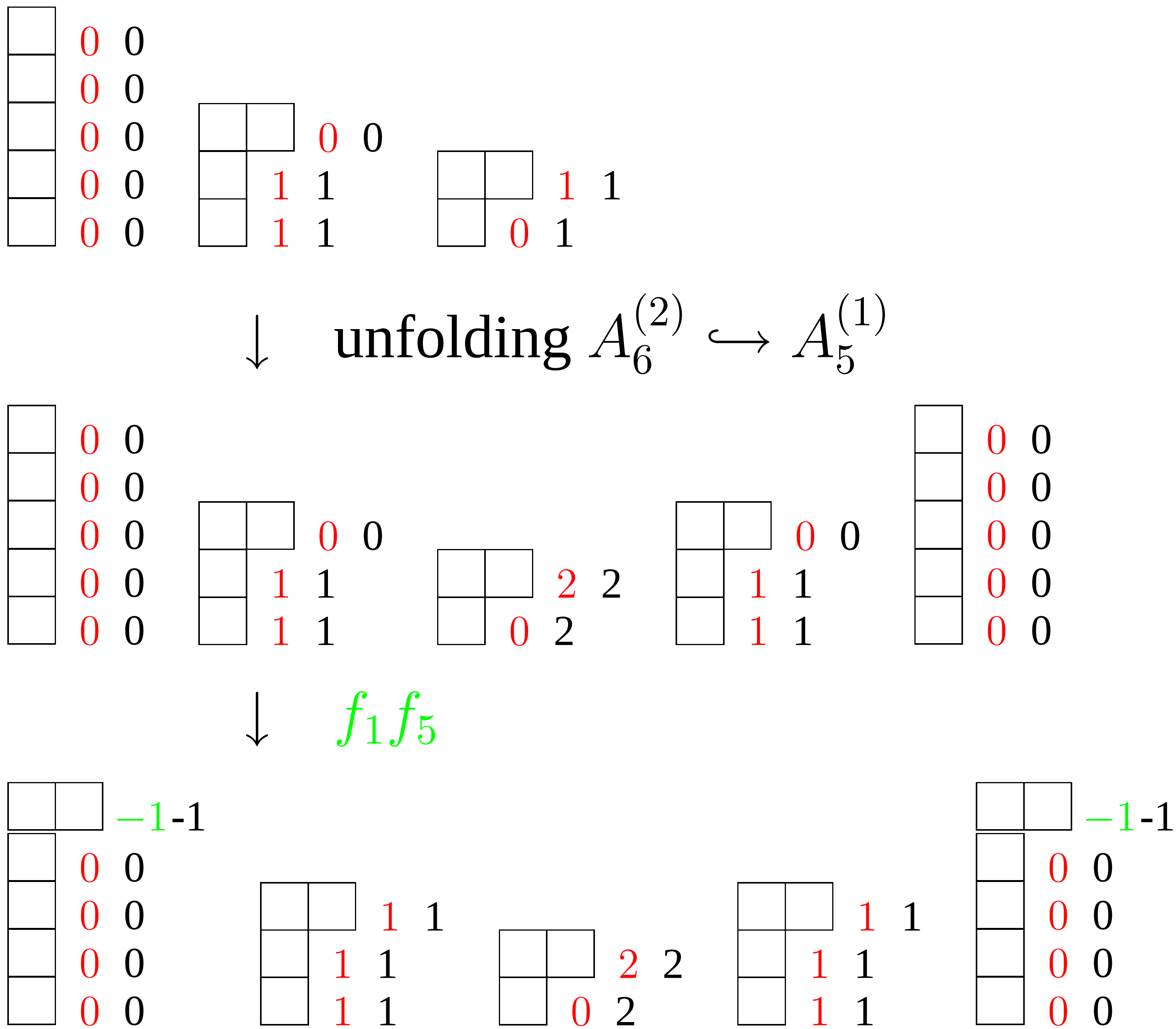
RC of type $A_6^{(2)}$, $\Lambda = \Lambda_1 + \Lambda_3$, $L_1^{(1)} = 7$

$(\nu, J) \quad =$

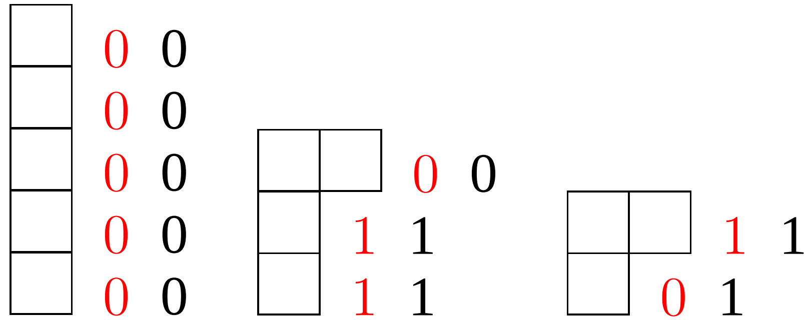


$f_3(\nu, J) \quad =$

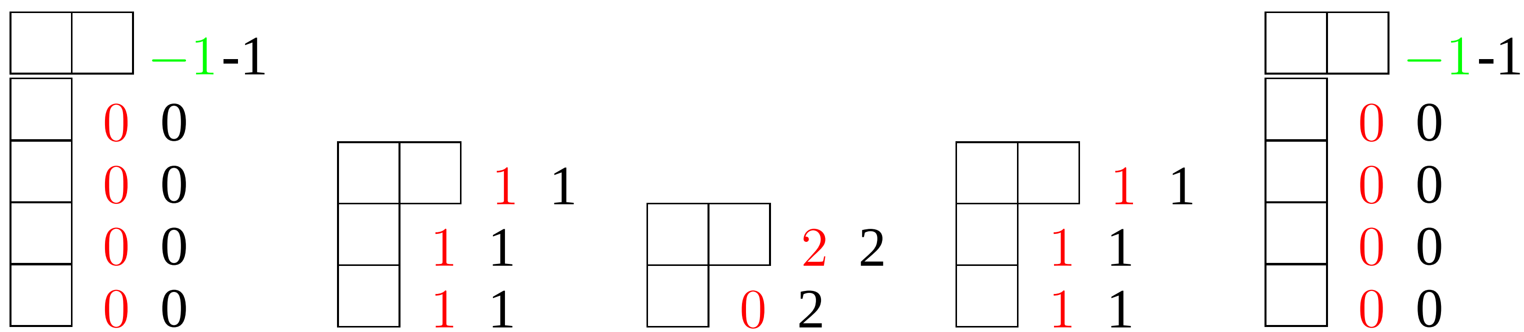




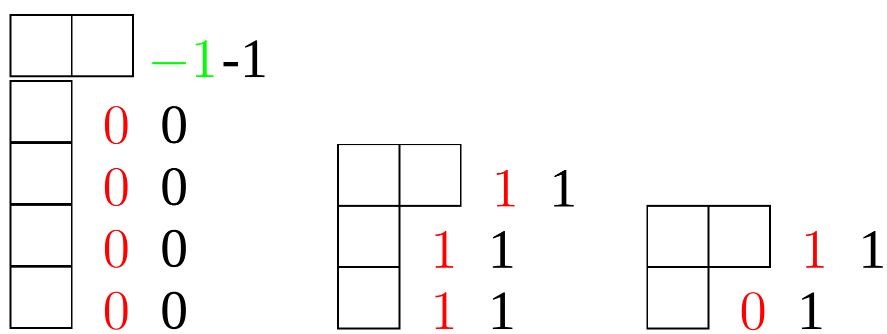
0 0
0 0
0 0
0 0
0 0
0 0
1 1
1 1
1 1
0 1
↓ unfolding $A_6^{(2)} \hookrightarrow A_5^{(1)}$
0 0
0 0
0 0
0 0
0 0
0 0
1 1
1 1
2 2
0 2
0 0
1 1
1 1
0 0
0 0
0 0
0 0
0 0
↓ $f_1 f_5$
-1 -1
0 0
0 0
0 0
0 0
1 1
1 1
1 1
2 2
0 2
1 1
1 1
1 1
-1 -1
0 0
0 0
0 0
0 0



$\downarrow$ unfolding $A_6^{(2)} \hookrightarrow A_5^{(1)}$, $f_1 f_5$



$\downarrow$ folding $A_6^{(2)} \hookrightarrow A_5^{(1)}$



Outlook

- Affine crystal structure (done for type $A_{n-1}^{(1)}$)
- Characterization of unrestricted rigged configurations (done for type $A_{n-1}^{(1)}$)
- Fermionic formulas for unrestricted Kostka polynomials
  Relation to fermionic formulas of [HKKOTY]?
- Relation to other rigged configurations [S]
  $\rightsquigarrow$ LLT polynomials
- Relation to box ball systems, description in terms of R-matrices
- Extension of Bailey lemma
- Level restriction