Combinatorics (Primarily Partitions) in Ramanujan's Lost Notebook

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$$\left\{1+\frac{ax^{\frac{1}{3}}}{1-x}+\frac{a^2x}{(1-x)(1-x^2)}+\frac{[\text{illegible}]}{(1-x)(1-x^2)(1-x^3)}+\cdots\right\}$$

$$\times\left\{1+\frac{\frac{1}{a^2}x}{1-x}+\frac{\frac{1}{a^4}x^4}{(1-x)(1-x^2)}+\frac{\frac{1}{a^6}x^9}{(1-x)(1-x^2)(1-x^3)}+\cdots\right\}$$

$$+\left\{x^{\frac{1}{3}}+\frac{ax}{1-x}+\frac{a^2x^{\frac{5}{3}}}{(1-x)(1-x^2)}\cdots\right\}\left\{\frac{1}{a}+\frac{1}{a^3}\cdot\frac{x^2}{1-x}+\frac{1}{a^5}\cdot\frac{x^6}{(1-x)\cdots}\right\}$$

$$=\left\{1+\left(a+\frac{1}{a}\right)x^{\frac{1}{3}}+\left(a^2+\frac{1}{a^2}\right)x^{\frac{2}{3}}+\cdots\right\}\Big/(1-x)(1-x^2)(1-x^3)\cdots$$

$$\frac{1}{1+}\;\frac{ae^{-\epsilon}}{1+}\;\frac{ae^{-2\epsilon}}{1+\cdots}=\frac{-1+\sqrt{1+4a}}{2a}\,e^{\frac{a\epsilon}{1+4a}}-\frac{a(1-a)\epsilon^2}{2(1+4a)^{5/2}}+\frac{a(1-a)(1-14a)\epsilon^3}{6(1+4a)^4}-\cdots$$

$$(1-x)(1-x^3)\cdots\left(1+\frac{x^2}{1-x^2}+\frac{x^6}{(1-x^2)(1-x^4)}+\cdots\right)=1-\frac{x}{1-x^2}+\frac{x^3}{(1-x^2)(1-x^4)}-\cdots$$

$$(1-x)(1-x^3)\cdots\left(1+\frac{x}{1-x^2}+\frac{x^6}{(1-x^2)(1-x^4)}+\cdots\right)=1-\frac{x^2}{1-x^2}+\frac{x^6}{(1-x^2)(1-x^4)}\cdots$$

N.B. $a=1$. $\phi(x^{\frac{1}{3}})$ can be separated to $\frac{\phi(x^{\frac{1}{3}})+\phi(\omega x^{\frac{1}{3}})}{2}$ and to $\{\phi(x^{\frac{1}{3}})-\phi(\omega x^{\frac{1}{3}})\}\cdots$

$$v=\frac{x}{1+}\;\frac{x^4}{1+}\;\frac{x^{2}\cdots}{1+\cdots}\quad\frac{1}{1+}\;\frac{x}{1+}\;\frac{x^2}{1+\cdots}\quad\text{then}\quad\frac{1-v}{1+v}=\frac{\phi(x^5)}{\phi(x)}$$

$$\left\{1+\left(a+\frac{1}{a}\right)x^{\frac{1}{3}}+\left(a^2+\frac{1}{a^2}\right)x+\cdots\right\}\left\{1-\frac{ax^{\frac{1}{3}}}{1-x}+\frac{a^2x}{(1-x)(1-\cdots)}\cdots\right\}$$

$$-\left\{1-\left(a+\frac{1}{a}\right)x^{\frac{1}{3}}+\left(a^2+\frac{1}{a^2}\right)x-\cdots\right\}\left\{1+\frac{ax^{\frac{1}{3}}}{1-x}+\frac{a^2x}{(1-x)(1-x^2)}+\cdots\right\}$$

$$=2\left\{\frac{x^{\frac{1}{3}}}{a}+\frac{x^{\frac{4}{3}}}{a^3(1-x)}+\frac{x^{\frac{25}{3}}}{a^5(1-x)(1-x^2)}+\cdots\right\}(1-x)(1-x^2)(1-x^3)\cdots$$

$$\left\{1+a\frac{x^{\frac{1}{3}}}{1-x}+\frac{a^2x^{\frac{4}{3}}}{(1-x)(1-x^2)}+\frac{a^3x^{\frac{9}{3}}}{(1-x)(1-x^2)(1-x^3)}+\cdots\right\}$$

$$\times\left\{1+a^{-3}\frac{x^{\frac{1}{3}}}{1-x}+\frac{a^{-2\delta}x^{\frac{4}{3}}}{(1-x)(1-x^2)}+\cdots\right\}\quad\text{as } x\to 1\quad ??$$

$$1+ax+a^2x^4+a^3x^9+\cdots=\prod_{n=1}^{n=\infty}\left\{1+ax^{2n-1}(1+y_1+y_2+\cdots)\right\}$$

where $y_1=\dfrac{x^{n(n+1)}-x^{(n+1)(n+2)}+\cdots}{1-3x^2+5x^6-7x^{12}+\cdots}$

and $y_2=\dfrac{(n+1)x^{n(n+1)}-(n+2)x^{(n+1)(n+2)}+\cdots}{1-3x^2+5x^6-7x^{12}+\cdots}$ ~~$\dfrac{x^{n(n+1)}\cdots}{1-3x^2\cdots}$~~

If $c_0+c_1\lambda+c_2\lambda^2+\cdots=\dfrac{(1+a\lambda)(1+a\lambda x)(1+a\lambda x^2)\cdots}{(1-b\lambda)(1-b\lambda x)\cdots(1-c\lambda)(1-c\lambda x)\cdots}$

then $c_0+c_1x+c_2x^3+c_3x^6+\cdots$

$$=(1+cx)(1+cx^2)\cdots\left\{1+\frac{x(b+a)}{(1-x)(1+cx)}+\frac{x^3(b+a)(b+ax)}{(1-x)(1-x^2)(1+cx)(1+cx^2)}+\cdots\right\}$$

$$1+\frac{x(b+ax)}{(1-x)(1-ax^2)}+\frac{x^3(b+ax)(b+ax^2)}{(1-x)(1-x^2)(1-ax^2)(1-ax^4)}+\cdots$$

$$=\frac{(1+bx)(1+bx^2)(1+bx^3)\cdots}{(1-ax)(1-ax^2)(1-ax^3)\cdots}\left\{(1-ax)-ax^6\cdot\frac{1-ax}{1-x^2}\cdots\right.$$

$$\frac{(b+ax)(b+ax^2)}{(1+bx)(1+bx^2)}(1-ax^5)+a^2x^{22}\cdot\frac{(1-ax)(1-ax^3)}{(1-x^2)(1-x^4)}\cdots$$

$$\frac{(b+ax)(b+ax^2)(b+ax^3)(b+ax^4)}{(1+bx)(1+bx^2)(1+bx^3)(1+bx^4)}(1-ax\cdots)$$

$$1-\frac{q^2}{(1+aq)(1+aq^2)}\;\frac{b-a^2q^2}{1-q^2}+\cdots=\frac{(1-bq^2)(1-bq^4)\cdots}{(1-a^2q^2)(1-\cdots)}$$

$$\times\left\{(1-a^2q^2)+aq^2\cdot\frac{(1-aq)}{1-q}\cdot\frac{b-a^2q^2}{1-bq^2}\cdots\right.$$

$$\left\{\left(\frac{1-q}{1+q}\right)\left(\frac{1+q}{1-q}\right)\left(\frac{1-q}{1+q}\right) \text{[illegible]}\right\}$$

$$\times\left\{\left(\frac{1-iq'}{1+iq'}\right)\left(\frac{1+iq'^{3}}{1-iq'^{3}}\right)\left(\frac{1-iq'^{5}}{1+iq'^{5}}\right)\cdots\right\} = \text{[illegible]}$$

$$= e^{\frac{\pi}{4}} \cdot \frac{k + \left(\frac{1}{2}\right)^2 k^3 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^5 + \cdots}{1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \cdots}$$

$q'^{\frac{1}{4}} = e^{\text{[illegible]} \pi \frac{k}{k'}}$

$$\frac{\psi(x^{\text{[illegible]}})}{\text{[illegible]}\,\psi(\text{[illegible]})} = 1 + u + v, \qquad \lambda = \frac{\sqrt{x}\,\psi\,\text{[illegible]}}{\text{[illegible]}}$$

$$u = \frac{\lambda - 1}{\text{[illegible]}} \ \text{[illegible]}\ \frac{\lambda^2 + 1 \text{[illegible]} \sqrt{\lambda^4 - 8\lambda^2 + 5}}{2} \ \text{[illegible]}\ (x-1)$$

$$\text{[illegible]}(x-1)\lambda^2 = 3 + \text{[illegible]}$$

$$v = \text{[illegible]}\ \lambda^3 + \text{[illegible]} - \sqrt{\lambda^4 - 2\lambda^2 + 5}\ \text{[illegible]}\ (x-1)$$

$$\text{[illegible]} = \frac{3 - \lambda^2 \text{[illegible]} \sqrt{\lambda^4 + 8\lambda^2 + 5}}{\text{[illegible]}}$$

$$\frac{\psi(\text{[illegible]})}{\psi(\text{[illegible]})} = 1 + \text{[illegible]}$$

$$u = \frac{\lambda^2 - 1}{\text{[illegible]}} \ \text{[illegible]}\ \frac{\lambda^2 + \text{[illegible]} - \sqrt{\lambda^4 \text{[illegible]} 5}}{\text{[illegible]}}$$

$5 = 3^2 + 1$

4 3 2 1 0   2. 4 2 0, 4 2 1 0,

~~17 15~~ 14 ~~13~~ 12 11 10 9 8 6

~~13 10 8 7~~ ~~12~~ 11^2 10^2 9^5 8

4 2 1 0   19 17 16 15 14 13 11

15 14 13 12 11 10

~~15 11 10 9 8 7 6~~

12 11 10

$13\ 8\ 7^2\ 6^2\ 5^2\ 4$ 4 2 0 13^2 12^2 11^2 10^2

13 11 9 7 5

~~11 9 7 5~~

~~$9 = 7 + 4 + 3^2 + 2 = 1$~~ 18 6 4

6 4

$a_9 - a_7 + a_4 + 2a_3 + a_2 - 1$

$a_3 + a_2 + a_1 + 1$

$a_6 - a_5 - a_4 + a_3 + a_1 + 1$

$+2a_2 + 2a_1 + 5$

7 5 3 2 1 0

7 6 5 4 3 2 1 0

$3 - 2 + 1 - 0$

$7 + 6 + 3 + 2 + 1 + 0$

$8 - 7 + 6 + 5 + 4 + 1 + 0$

$10\ 8\ 4\ 3\ 2^2\ 1^2\ 0\ 3$

12 11 10

[illegible]

George Andrews

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Heng Huat Chan

Song Heng Chan

O-Yeat Chan

Freeman Dyson

Frank Garvan

Dean Hickerson

Wen-Chin Liaw

Ae Ja Yee

1 Introduction to Cranks

$$\frac{1}{(q;q)_\infty} = \sum_{n=0}^{\infty} p(n)q^n$$

$$p(5n+4) \equiv 0 \pmod{5},$$
$$p(7n+5) \equiv 0 \pmod{7},$$
$$p(11n+6) \equiv 0 \pmod{11}.$$

Definition 1 *The rank of a partition is the largest part minus the number of parts.*

$N(m,n)$ = number of partitions of n with rank m.

$N(m,t,n)$ = the number of partitions of n with rank congruent to m modulo t.

$$N(k,5,5n+4) = \frac{p(5n+4)}{5}, \quad 0 \leq k \leq 4,$$
$$N(k,7,7n+5) = \frac{p(7n+5)}{7}, \quad 0 \leq k \leq 6.$$

$$\sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} N(m,n) a^m q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(aq;q)_n (q/a;q)_n}.$$

Definition 2 *For a partition π, let $\lambda(n)$ denote the largest part of π, let $\mu(\pi)$ denote the number of ones in π, and let $\nu(\pi)$ denote the number of parts of π larger than $\mu(\pi)$. The crank $c(\pi)$ is then defined to be*

$$c(\pi) = \begin{cases} \lambda(\pi), & \text{if } \mu(\pi) = 0, \\ \nu(\pi) - \mu(\pi), & \text{if } \mu(\pi) > 0. \end{cases}$$

$M(m,n)$ denotes the number of partitions of n with crank m, except that

$$M(m,n) = \begin{cases} -1, & \text{if } (m,n) = (0,1), \\ 1, & \text{if } (m,n) = (0,0), (\pm 1, 1), \\ 0, & \text{otherwise.} \end{cases}$$

$M(m,t,n)$ denotes the number of partitions of n with crank congruent to m modulo t.

$$M(k,5,5n+4) = \frac{p(5n+4)}{5}, \qquad 0 \le k \le 4,$$

$$M(k,7,7n+5) = \frac{p(7n+5)}{7}, \qquad 0 \le k \le 6,$$

$$M(k,11,11n+6) = \frac{p(11n+6)}{11}, \qquad 0 \le k \le 10.$$

2 Cranks in Ramanujan's Lost Notebook

$$\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} M(m,n)a^m q^n = \frac{(q;q)_\infty}{(aq;q)_\infty(q/a;q)_\infty}$$
$$:= F_a(q)$$

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$$F(q) := F_a(q) := \frac{(q;q)_\infty}{(aq;q)_\infty(q/a;q)_\infty} =: \sum_{n=0}^{\infty} \lambda_n q^n.$$

$$\lambda_n = \sum_{m=-\infty}^{\infty} M(m,n)a^m.$$

$$
\begin{aligned}
&1 + q(a_1 - 1) + q^2 a_2 + q^3(a_3+1) + q^4(a_4+a_2+1) \\
&+ q^5(a_5+a_3+a_1+1) + q^6(a_6+a_4+a_3+a_2+a_1+1) \\
&+ q^7(a_3+1)(a_4+a_2+1) + q^8 a_2(a_6+a_4+a_3+a_2+a_1+1) \\
&+ q^9 a_2(a_3+1)(a_4+a_2+1) + q^{10} a_2(a_3+1)(a_5+a_3+a_1+1) \\
&+ q^{11} a_1 a_2(a_8+a_5+a_4+a_3+a_2+a_1+2) \\
&+ q^{12}(a_3+a_2+a_1+1)(a_5+a_4+a_3+a_2+a_1+1) \\
&\qquad \times (a_4 - 2a_3 + 2a_2 - a_1 + 1) \\
&+ q^{13}(a_1-1)(a_2-a_1+1)(a_{10}+2a_9+2a_8+2a_7+2a_6 \\
&\qquad +4a_5+6a_4+8a_3+9a_2 \\
&\qquad +9a_1+9) \\
&+ q^{14}(a_2+1)(a_3+1)(a_4+a_2+1)(a_5-a_3+a_1+1) \\
&+ q^{15} a_1 a_2(a_5+a_4+a_3+a_2+a_1+1)(a_7-a_6+a_4+a_1) \\
&+ q^{16}(a_3+1)(a_3+a_2+a_1+1)(a_5+a_4+a_3+a_2+a_1+1) \\
&\qquad \times (a_5-2a_4+2a_3-2a_2+3a_1-3) \\
&+ q^{17}(a_2+1)(a_3+1)(a_5+a_4+a_3+a_2+a_1+1) \\
&\qquad \times (a_7-a_6+a_3+a_1-1) \\
&+ q^{18}(a_4+a_2+1)(a_3+a_2+a_1+1)(a_5+a_4+a_3+a_2+a_1+1) \\
&\qquad \times (a_6-2a_5+a_4+a_3-a_2+1) \\
&+ q^{19} a_2(a_1-1)(a_4+a_2+1)(a_3+a_2+a_1+1) \\
&\qquad \times (a_9-a_7+a_4+2a_3+a_2-1) \\
&+ q^{20}(a_2-a_1+1)(a_3+1)(a_5+a_4+a_3+a_2+a_1+1) \\
&\qquad \times (a_{10}+a_6+a_4+a_3+2a_2+2a_1+3) \\
&+ q^{21} a_1 a_2(a_3+1)(a_2-a_1+1)(a_5+a_4+a_2+a_1+1) \\
&\qquad \times (a_8-a_6+a_4+a_1+2).
\end{aligned}
$$

$13.\ (a_1-1)(a_2-a_1+1)$ $14.\ (a_2+1)(a_3+1)(a_4+a_2+1)$

(25) Add ... [illegible]

$.16$ $15.\ a_1 a_2 (a_5+a_4+a_3+a_2+a_1+1)$

$16.\ (a_3+1)(a_3+a_2+a_1+1)(a_5+a_4+a_3+a_2+a_1+1)$

27×17 $17.\ (a_2+1)(a_3+1)(a_5+a_4+a_3+a_2+a_1+1)$

-25 $18.\ (a_4+a_2+1)(a_3+a_2+a_1+1)(a_5+a_4+a_3+a_2+a_1+1)$

49 $19.\ a_2(a_1-1)(a_4+a_2+1)(a_3+a_2+a_1+1)$

-7.19 $20.\ (a_3+1)(a_2-a_1+1)(a_5+a_4+a_3+a_2+a_1+1)$

9 $21.\ a_1 a_2 (a_3+1)(a_2-a_1+1)(a_5+a_4+a_3+a_2+a_1+1)$

-7 $22.\ a_2(a_3+1)(a_1-1)$

-9 $23.\ (a_1-1)(a_4+a_2+1)$

$-11, 15$ $24.\ (a_3+1)(a_4+a_2+1)(a_3+a_2+a_1+1)$

-11 $25.\ a_2(a_1-1)(a_5+a_4+a_3+a_2+a_1+1)$

-4 $26.\ a_2(a_3+1)(a_3+a_2+a_1+1)$

$$\frac{1}{1+a} - \left(\frac{q}{1+aq} + \frac{q}{a+q}\right) + \frac{q^3}{1+aq^2} + \frac{q^3}{a+q^2}$$

$$1 - \{q(a_1+1) - q^2(a_2+a_1) + q^3(a_3+a_2) - {} + q^3(a_1+1) - q^5(a_2+a_1)$$

$$1 + q(a_1-1) + q^2(a_2-a_1) + q^3(a_3-a_2) + q^4(a_4-a_3) +$$
$$-\{q^3(a_1-1) + q^5(a_2-a_1) + q^7(a_3-a_2) + q^9(a_4-a_3) - \}$$
$$+\{q^6(a_1-1) + q^9(a_2-a_1) + q^{12}(a_3-a_2) + q^{15}(a_4-a_3)$$
$$-\{q^{10}(a_1-1) + q^{14}(a_2-a_1) + q^{18}(a_3-a_2) + q^{22}(a_4-a_3)$$
$$+\{q^{15}(a_1-1) + q^{20}(a_2-a_1) + q^{25}(a_3-a_2)$$
$$-\{q^{21}(a_1-1) + \}$$

$$1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + q^{22}$$

$$1 + q(a_1-1) + a_2 q^2 + (a_3+1)q^3 + q^4(a_4 \ldots) + q^5(a_5+a_3+a_1+1)$$

$a_5+1+a_2+a_4$ $a_5+a_3+1+a_3+x$

863 3 62 8
531 864 55 66
1 3 0 573 243

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$$
\begin{aligned}
&1 + q(a_1 - 1) + q^2 a_2 + q^3(a_3 + 1) \\
&+ q^4(a_4 + a_2 + 1) + q^5(a_5 + a_3 + a_1 + 1) \\
&+ q^6(a_6 + a_4 + a_3 + a_2 + a_1 + 1) \\
&+ q^7(a_3 + 1)(a_4 + a_2 + 1) \\
&+ q^8 a_2(a_6 + a_4 + a_3 + a_2 + a_1 + 1) \\
&+ q^9 a_2(a_3 + 1)(a_4 + a_2 + 1) \\
&+ q^{10} a_2(a_3 + 1)(a_5 + a_3 + a_1 + 1) \\
&+ q^{11} a_1 a_2(a_8 + a_5 + a_4 + a_3 + a_2 + a_1 + 2) \\
&+ q^{12}(a_3 + a_2 + a_1 + 1) \\
&\times (a_5 + a_4 + a_3 + a_2 + a_1 + 1) \\
&\times (a_4 - 2a_3 + 2a_2 - a_1 + 1)
\end{aligned}
$$

$$
\begin{aligned}
&+ q^{13}(a_1 - 1)(a_2 - a_1 + 1)(a_{10} + 2a_9 + 2a_8 \\
&+ 2a_7 + 2a_6 + 4a_5 + 6a_4 + 8a_3 + 9a_2 + 9a_1 + 9) \\
&+ q^{14}(a_2 + 1)(a_3 + 1)(a_4 + a_2 + 1) \\
&\times (a_5 - a_3 + a_1 + 1) \\
&+ q^{15}a_1a_2(a_5 + a_4 + a_3 + a_2 + a_1 + 1) \\
&\times (a_7 - a_6 + a_4 + a_1) \\
&+ q^{16}(a_3 + 1)(a_3 + a_2 + a_1 + 1) \\
&\times (a_5 + a_4 + a_3 + a_2 + a_1 + 1) \\
&\times (a_5 - 2a_4 + 2a_3 - 2a_2 + 3a_1 - 3) \\
&+ q^{17}(a_2 + 1)(a_3 + 1) \\
&\times (a_5 + a_4 + a_3 + a_2 + a_1 + 1) \\
&\times (a_7 - a_6 + a_3 + a_1 - 1)
\end{aligned}
$$

$$
\begin{aligned}
&+ q^{18}(a_4 + a_2 + 1)(a_3 + a_2 + a_1 + 1) \\
&\times (a_5 + a_4 + a_3 + a_2 + a_1 + 1) \\
&\times (a_6 - 2a_5 + a_4 + a_3 - a_2 + 1) \\
&+ q^{19}a_2(a_1 - 1)(a_4 + a_2 + 1)(a_3 + a_2 + a_1 + 1) \\
&\times (a_9 - a_7 + a_4 + 2a_3 + a_2 - 1) \\
&+ q^{20}(a_2 - a_1 + 1)(a_3 + 1) \\
&(a_5 + a_4 + a_3 + a_2 + a_1 + 1) \\
&\times (a_{10} + a_6 + a_4 + a_3 + 2a_2 + 2a_1 + 3) \\
&+ q^{21}a_1a_2(a_3 + 1)(a_2 - a_1 + 1) \\
&\times (a_5 + a_4 + a_3 + a_2 + a_1 + 1) \\
&\times (a_8 - a_6 + a_4 + a_1 + 2) + \cdots .
\end{aligned}
$$

$$a_n := a^n + a^{-n}$$

13. $(a_1 - 1)(a_2 - a_1 + 1)$

14. $(a_2 + 1)(a_3 + 1)(a_4 + a_2 + 1)$

15. $a_1 a_2 (a_5 + a_4 + a_3 + a_2 + a_1 + 1)$

16. $(a_3 + 1)(a_3 + a_2 + a_1 + 1)$
$\times (a_5 + a_4 + a_3 + a_2 + a_1 + 1)$

17. $(a_2 + 1)(a_3 + 1)(a_5 + a_4 + a_3 + a_2 + a_1 + 1)$

18. $(a_4 + a_2 + 1)(a_3 + a_2 + a_1 + 1)$
$\times (a_5 + a_4 + a_3 + a_2 + a_1 + 1)$

19. $a_2 (a_1 - 1)(a_4 + a_2 + 1)(a_3 + a_2 + a_1 + 1)$

20. $(a_3 + 1)(a_2 - a_1 + 1)$
$\times (a_5 + a_4 + a_3 + a_2 + a_1 + 1)$

21. $a_1 a_2 (a_3 + 1)(a_2 - a_1 + 1)$
$\times (a_5 + a_4 + a_3 + a_2 + a_1 + 1)$

22. $a_2 (a_3 + 1)(a_1 - 1)$

23. $(a_1 - 1)(a_4 + a_2 + 1)$

24. $(a_3 + 1)(a_4 + a_2 + 1)(a_3 + a_2 + a_1 + 1)$

25. $a_2(a_1 - 1)(a_5 + a_4 + a_3 + a_2 + a_1 + 1)$

26. $a_2(a_3 + 1)(a_3 + a_2 + a_1 + 1)$.

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$$\frac{1}{1+a} + \sum_{n=1}^{\infty} \left(\frac{q^{2n-1}}{1+aq^n} + \frac{q^{2n-1}}{a+q^n} \right)$$

$$1 + \sum_{m=1,n=0}^{\infty} (-1)^{m+n} q^{m(m+1)/2+mn} (a_{n+1} + a_n)$$

Theorem 3 *If*

$$A_n = a^n + a^{-n}, \tag{2.1}$$

then

$$\frac{(q;q)_\infty^2}{(aq;q)_\infty(q/a;q)_\infty}$$
$$= 1 - \sum_{m=1,n=0}^{\infty} (-1)^m q^{m(m+1)/2+mn}(A_{n+1} - A_n).$$

Theorem 4 (R. J. Evans; V. G. Kač & M. Wakimoto)

Let

$$\alpha_k = (-1)^k q^{k(k+1)/2}.$$

Then

$$\frac{(q;q)_\infty^2}{(q/a;q)_\infty(qa;q)_\infty} = \sum_{k=-\infty}^{\infty} \frac{\alpha_k(1-a)}{(1-aq^k)}.$$

3 Dissections of Cranks

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \qquad |ab| < 1. \tag{3.1}$$

$$f(-q,-q^2) =: f(-q) = (q;q)_\infty \tag{3.2}$$

2-Dissection of $F(q)$

Theorem 5

$$F_a(\sqrt{q}) \equiv \frac{f(-q^3,-q^5)}{(-q^2;q^2)_\infty} + \left(a - 1 + \frac{1}{a}\right) \times \sqrt{q}\frac{f(-q,-q^7)}{(-q^2;q^2)_\infty} \pmod{a^2 + \frac{1}{a^2}}.$$

3-Dissection of $F(q)$

Theorem 6

$$F_a(q^{1/3}) \equiv \frac{f(-q^2,-q^7)f(-q^4,-q^5)}{(q^9;q^9)_\infty}$$

$$+\left(a-1+\frac{1}{a}\right)q^{1/3}\frac{f(-q,-q^8)f(-q^4,-q^5)}{(q^9;q^9)_\infty}$$

$$+\left(a^2+\frac{1}{a^2}\right)q^{2/3}\frac{f(-q,-q^8)f(-q^2,-q^7)}{(q^9;q^9)_\infty}$$

$$(\operatorname{mod} a^3+1+\frac{1}{a^3}).$$

$$F(q)=\frac{(1-q)(1-q^2)(1-q^3)\cdots}{\left(1-2q\cos\frac{2n\pi}{5}+q^2\right)\left(1-2q^2\cos\frac{4n\pi}{5}+q^4\right)\cdots}$$

$$f(q)=1+\frac{q}{1-2q\cos\frac{2n\pi}{5}+q^2}+\frac{q^4}{\left(1-2q\cos\frac{2n\pi}{5}+q^2\right)\left(1-2q^2\cos\frac{4n\pi}{5}+q^4\right)}+\cdots, \quad n=1 \text{ or } 2.$$

$$F(q^{\frac15})=A(q)-4q^{\frac15}\cos^2\frac{2n\pi}{5}\cdot B(q)+2q^{\frac25}\cos\frac{4n\pi}{5}\cdot C(q)-2q^{\frac35}\cos\frac{2n\pi}{5}\,D(q);$$

$$f(q^{\frac15})=\left\{A(q)-4\sin^2\frac{n\pi}{5}\,\phi(q)\right\}+q^{\frac15}B(q)+2q^{\frac25}\cos\frac{2n\pi}{5}\,C(q)-2q^{\frac35}\cos\frac{2n\pi}{5}\left\{D(q)+4\sin^2\frac{2n\pi}{5}\,\frac{\psi(q)}{q}\right\}.$$

$$A(q)=\frac{1-q^2-q^3+q^9+\cdots}{(1-q)^2(1-q^4)^2(1-q^6)^2\cdots}$$

$$B(q)=\frac{(1-q^5)(1-q^{10})(1-q^{15})\cdots}{(1-q)(1-q^4)(1-q^6)\cdots}$$

$$C(q)=\frac{(1-q^5)(1-q^{10})(1-q^{15})\cdots}{(1-q^2)(1-q^3)(1-q^7)\cdots}$$

$$D(q)=\frac{1-q-q^4+q^7+\cdots}{(1-q^2)^2(1-q^3)^2(1-q^7)^2\cdots}$$

$$\phi(q)=-1+\left\{\frac{1}{1-q}+\frac{q^5}{(1-q)(1-q^4)(1-q^6)}+\frac{q^{20}}{(1-q)(1-q^4)(1-q^6)(1-q^9)(1-q^{11})}+\cdots\right\}.$$

$$\psi(q)=-1+\left\{\frac{1}{1-q^2}+\frac{q^5}{(1-q^2)(1-q^3)(1-q^7)}+\frac{q^{20}}{(1-q^2)(1-q^3)(1-q^7)(1-q^8)(1-q^{12})}+\cdots\right\}$$

$$\frac{q}{1-q}+\frac{q^3}{(1-q^2)(1-q^3)}+\frac{q^5}{(1-q^3)(1-q^4)(1-q^5)}+\cdots$$
$$=3\phi(q)+1-A(q).$$

$$\frac{q}{1-q}+\frac{q^2}{(1-q^2)(1-q^3)}+\frac{q^3}{(1-q^3)(1-q^4)(1-q^5)}+\cdots$$
$$=3\psi(q)+qD(q).$$

$$\frac{q^2}{1-q}+\frac{q^8}{(1-q)(1-q^3)}+\frac{q^{18}}{(1-q)(1-q^3)(1-q^5)}+\cdots$$
$$=\phi(q)-q\cdot\frac{1+q^5+q^{15}+\cdots}{(1-q^4)(1-q^6)(1-q^{14})\cdots}$$

$$\frac{q}{1-q}+\frac{q^6}{(1-q)(1-q^3)}+\frac{q^{13}}{(1-q)(1-q^3)(1-q^5)}+\cdots$$
$$=\psi(q)+q\cdot\frac{1+q^5+q^{15}+\cdots}{(1-q^2)(1-q^8)(1-q^{12})\cdots}$$

5-Dissection of $F(q)$

Theorem 7 (First Version, p. 20) *Recall that $f(-q)$ is defined by* **(??)**. *For $n = 1$ or 2,*

$$F_a(q^{1/5}) = \frac{f(-q^2,-q^3)}{f^2(-q,-q^4)} f^2(-q^5)$$
$$- 4\cos^2(2n\pi/5) q^{1/5} \frac{f^2(-q^5)}{f(-q,-q^4)}$$
$$+ 2\cos(4n\pi/5) q^{2/5} \frac{f^2(-q^5)}{f(-q^2,-q^{15})}$$
$$- 2\cos(2n\pi/5) q^{3/5} \frac{f(-q,-q^4)}{f^2(-q^2,-q^3)} f^2(-q^5).$$

$$S_n(a) := \sum_{k=-n}^{n} a^k. \tag{3.3}$$

Theorem 8 (Second Version) *With $f(-q)$ and S_2 as defined above and A_n defined by* **(??)**,

$$F_a(q) \equiv \frac{f(-q^{10},-q^{15})}{f^2(-q^5,-q^{20})} f^2(-q^{25})$$

$$+ (A_1 - 1)q \frac{f^2(-q^{25})}{f(-q^5,-q^{20})}$$

$$+ A_2 q^2 \frac{f^2(-q^{25})}{f(-q^{10},-q^{15})}$$

$$- A_1 q^3 \frac{f(-q^5,-q^{20})}{f^2(-q^{10},-q^{15})} f^2(-q^{25}) \pmod{S_2}.$$

Add the q^{4}'s

130

81 106

$$\frac{(4,7)\ (11)}{(1,10)(2,9)} \qquad \frac{(2,9)\ (11)}{(1,10)(5,6)} \qquad \frac{(5,6)\ (11)}{(2,9)(4,7)}$$

$$\frac{q}{1+q} - \frac{q^{3}(1-q)}{(1+q)(1+q^{2})^{2}}$$

$$1 + \frac{q(1-q)}{(1+q)^{2}} + \frac{q^{2}(1-q)(1-q^{2})}{(1+q)^{2}(1+q^{2})^{2}}$$

$q - 3q^{2} + 5q^{3} - 7q^{4}$

$q^{2} - 3q^{3} + 3q^{4}$

$3q - 5q^{2} + 7q^{3} - 9q^{4} + 11q^{5}$

$+ q - 3q^{2} + 2q^{3} + 2q^{4} - q^{5} - 5q^{6}$

$$\frac{q}{1+q} - \frac{q^{2}}{1+q^{2}} + \frac{q^{3}}{1+q^{3}} - \frac{q^{4}}{1+q^{4}} + \frac{q^{5}}{1+q^{5}}$$

$1 - 3 + 5 - 7 + 9$

$q - 2q^{2} + 2q^{3} - q^{4} + 2q^{5}$

$$4\left(\frac{1}{2} - \frac{q}{1+q} + \frac{q^{2}}{1+q^{2}} - \frac{q^{3}}{1+q^{3}}\right)$$

$$= 1 + \frac{q(1-q)}{(1+q)^{2}} +$$

$+1 - 5q + 10q^{2} - 11q^{3} + 11q^{4}$

$$\frac{1 - 4q + 4q^{2} + 4q^{4}}{1 + q - 2q^{2} - 9q^{3}}$$

$$q \cdot \frac{1-q}{(1+q)^{2}} - q^{3} + 2q^{4}$$

$$-q^{3} \cdot \frac{(1-q)(1-q^{2})^{2}}{(1+q)^{2}(1+q^{2})^{2}}$$

$q + q^{3} + q^{4} + q^{5}$

The 7-Dissection for $F(q)$

Theorem 9 *With $f(a,b)$ defined by* (3.1), $f(-q)$ *defined by* (3.2), A_n *defined by* (2.1), *and* S_m *defined by* (3.3),

$$\frac{(q;q)_\infty}{(qa;q)_\infty(q/a;q)_\infty} \equiv \frac{1}{f(-q^7)}\Big(A^2 + A_2q^2B^2 + (A_1-1)qAB + (A_3+1)q^3AC - A_1q^4BC - (A_2+1)q^6C^2\Big) \pmod{S_3},$$

where

$$A = f(-q^{21}, -q^{28}),$$
$$B = f(-q^{35}, -q^{14}),$$
$$C = f(-q^{42}, -q^{7}).$$

$\frac{(1-q^{11})(1-q^{11})}{(1-q)(1-q^{10})} + q^{\frac{1}{11}} \frac{}{(1-q)(1-q^2)(1-q^3)\cdots} \qquad \frac{(11)}{(1,10)}$

$1-q^2$

$1+q^2+q^3+q^4+q^6+q^8+q^9+q^{10}+q^{14} \qquad q^{\frac{1}{11}}\frac{(5,6)\ (11)}{(2,9)\ (3\ 8)}$ 1

$-q^{11}$

$1-q-q^2+q^5+q^7$ $\quad 1-q^2 \mid q^7$

$1-q-q^2$ $\quad 1-q^3 \mid q^8$

$\frac{1}{11}-\frac{1}{264}=\frac{11}{12}+\frac{-7}{88}-49-25$

$1+q^2-q^5-q^8-q^{11}$

$-q^6$

$\frac{1}{12}-\frac{2}{88} \quad \frac{81}{88}$

$+q^{\frac{2}{11}}\frac{(3,8)\ (11)}{(1,10)(4,7)}$ 2

$1+q+q^2+q^4+q^5+q^6+q^7$

$1-q^3+q^4 \qquad 1-q^3$

$+q^{\frac{3}{11}}\frac{(2,9)\ (11)}{(1,10)\ (3,8)}$ 3

$1-q^4-q^7$

$+q^{\frac{4}{11}}\frac{(11)}{(2,9)}$ 5

$1+q+q^3+q^4$

$1-q^2+q^3+q^4-q^5$ $\quad 1.8 \quad 1.28 \quad 1.58$

$+q^{\frac{5}{11}}\frac{(4,7)\ (11)}{(2,9)\ (5,6)}$ 7

$1+q^2+q^4+q^6$

$+q^{\frac{7}{11}}\frac{(11)}{(3,8)}$ 4

$q^{\frac{19}{11}}=qX + q^{\frac{8}{11}}\frac{(1,10)\ (11)}{(5,6)}$

$1+q^2+q^5$

$+q^{\frac{8}{11}}\frac{(1,10)\ (11)}{(4,7)\ (5,6)}$ 6

$1+q^3+q^6$

$1+1-q+q^4$

$+q^{\frac{9}{11}}\frac{(1,10)\ (11)}{(4,7)}$ 8

$1+q^4+q^2$

$+q^{\frac{10}{11}}\frac{(1,10)(11)}{(5,6)}$ 9

11712

$1+q^5$

$1+q+2q^2+2q^3+2q^4$

$(1-q)(1-q^2)^{11}$

$-4q^5+4q^7+5q^8$

$-3q^9-2q^{10}$

$1+2q+5q^2\ (3)$ $\quad 4 \quad 6 \quad 8 \quad 9$

5

$3 \quad 4$

$1-q^4$

[illegible]

$-B_6+B_2-1$ $\qquad -(1+B_2+B+1)$

$B_2+B_4+B_3+1$

The 11-Dissection for $F(q)$

Theorem 10 *With* A_m *defined by* (2.1) *and* S_5 *defined by* (3.3), *we have*

$$F_a(q) \equiv \frac{1}{(q^{11};q^{11})_\infty (q^{121};q^{121})_\infty^2}$$
$$\times \Big(ABCD + \{A_1 - 1\}\, qA^2BE$$
$$+ A_2\, q^2AC^2D + \{A_3 + 1\}\, q^3ABD^2$$
$$+ \{A_2 + A_4 + 1\}\, q^4ABCE$$
$$- \{A_2 + A_4\}\, q^5B^2CE$$
$$+ \{A_1 + A_4\}\, q^7ABDE$$
$$- \{A_2 + A_5 + 1\}\, q^{19}CDE^2$$
$$- \{A_4 + 1\}\, q^9ACDE$$
$$- \{A_3\}\, q^{10}BCDE\Big) \pmod{S_5},$$

where

$$A = f(-q^{55}, -q^{66}),$$
$$B = f(-q^{77}, -q^{44}),$$
$$C = f(-q^{88}, -q^{33}),$$
$$D = f(-q^{99}, -q^{22}),$$
$$E = f(-q^{110}, -q^{11}).$$

$$\alpha_k = (-1)^k q^{k(k+1)/2}.$$

$$\sum_{k=-\infty}^{\infty} \alpha_k \frac{q^k - 1}{1 + q^{4k}} = q \frac{(q;q)_\infty}{(-q^4;q^4)_\infty} f(-q^2, -q^{14}).$$

$$\sum_{k=-\infty}^{\infty} \alpha_k \frac{q^k + 1}{1 + q^{4k}} = \frac{(q;q)_\infty}{(-q^4;q^4)_\infty} f(-q^6, -q^{10}).$$

$$\begin{aligned}&\sum_{k=-\infty}^{\infty} \alpha_k \frac{q^k - 1}{1 + q^{3k} + q^{6k}} \\ =&q(q;q)_\infty \frac{f(-q^3, -q^{24}) f(-q^{12}, -q^{15})}{(q^{27};q^{27})_\infty}.\end{aligned}$$

Table 1. $\lambda_n \equiv 0 \,(\mathrm{mod}\, a^2 + \dfrac{1}{a^2})$

47 values

$$2, 8, 9, 10, 11, 15, 19, 21, 22, 25, 26, 27, 28,$$
$$30, 31, 34, 40, 42, 45, 46, 47, 50, 55, 57, 58, 59,$$
$$62, 66, 70, 74, 75, 78, 79, 86, 94, 98, 106, 110,$$
$$122, 126, 130, 142, 154, 158, 170, 174, 206.$$

Table 2. $\lambda_n \equiv 1 \,(\mathrm{mod}\, a^2 + \dfrac{1}{a^2})$

$$\frac{f(-q^6, -q^{10})}{(-q^4; q^4)_\infty}.$$

27 values

$$14, 16, 18, 24, 32, 48, 56, 72, 82, 88, 90, 104,$$
$$114, 138, 146, 162, 178, 186, 194, 202, 210,$$
$$218, 226, 234, 242, 250, 266.$$

Table 3. $\lambda_n \equiv -1 \; (\mathrm{mod}\, a^2 + \frac{1}{a^2})$

27 (not 26) values

$$4, 6, 12, 20, 36, 38, 44, 52, 54, 60, 68,$$
$$76, 92, 102, 118, 134, 150, 166, 182,$$
$$190, 214, 222, 238, 254, 270, 286, 302.$$

Table 4. $\lambda_n \equiv a - 1 + \frac{1}{a} \; (\mathrm{mod}\, a^2 + \frac{1}{a^2})$

$$q\frac{f(-q^2, -q^{14})}{(-q^4; q^4)_\infty}$$

22 values

$$1, 7, 17, 23, 33, 39, 41, 49, 63, 71, 73, 81,$$
$$87, 89, 95, 105, 111, 119, 121, 127, 143, 159.$$

Table 5. $\lambda_n \equiv -\left(a - 1 + \frac{1}{a}\right)$ (mod $a^2 + \frac{1}{a^2}$)

23 values

$$3, 5, 13, 29, 35, 37, 43, 51, 53, 61, 67, 69, 77,$$
$$83, 85, 91, 93, 99, 107, 115, 123, 139, 155.$$

Table 6. $\lambda_n \equiv 0$ (mod $a + \frac{1}{a}$)

3 values

$$11, 15, 21,$$

Table 7. $\lambda_n \equiv 0$ (mod $a - 1 + \frac{1}{a}$)

19 values

$$1, 6, 8, 13, 14, 17, 19, 22, 23, 25,$$
$$33, 34, 37, 44, 46, 55, 58, 61, 82.$$

Table 8. $\lambda_n \equiv 1 \,(\text{mod}\, a - 1 + \frac{1}{a})$

26 values

$$5, 7, 10, 11, 12, 18, 24, 29, 30, 31, 35, 41, 42, 43,$$
$$47, 49, 53, 54, 59, 67, 71, 73, 85, 91, 97, 109.$$

Table 9. $\lambda_n \equiv -1 \,(\text{mod}\, a - 1 + \frac{1}{a})$

26 values

$$2, 3, 4, 9, 15, 16, 20, 21, 26, 27, 28, 32, 38, 39,$$
$$40, 52, 56, 62, 64, 68, 70, 76, 94, 106, 118, 130.$$

Table 10. $\lambda_n \equiv 0 \,(\text{mod}\, a + 1 + \frac{1}{a})$

2 values

$$14, 17.$$

$$\frac{f(-q^6,-q^{10})}{(-q^4;q^4)_\infty}, \qquad \frac{f(-q^2,-q^{14})}{(-q^4;q^4)_\infty},$$

$$\frac{f(-q)f(-q^2)}{f(-q^4)}, \qquad \frac{f(-q^2)f(-q^3)}{f(-q^6)}, \qquad \frac{f^2(-q)}{f(-q^3)}.$$

Conjecture 11 *Each component of each of the dissections for the five products given above has monotonic coefficients for powers of q above 1400.*

Conjecture 12 *For any positive integers α and β, each component of the $(\alpha+\beta+1)$-dissection of the product*

$$\frac{f(-q^\alpha)f(-q^\beta)}{f(-q^{\alpha+\beta+1})}$$

has monotonic coefficients for sufficiently large powers of q.

18 16 15 14^2 13^3 12^4 11^6 1x8

$1 - q^{11} - q^{22} + 2q^{33} + q^{44} + 2q^{55} - 2q^{66} - q^{88} - 2q^{99} +$

$1 - q - q^{10} + q^{13} + q^{21} - q^{36} - q^{63} + q^{70} + q^{106}$

$1 + q + q^2 + q^3 + q^4 + q^5 + q^6 + q^7 + q^8 + q^9 + 2q^{10} + q^{11}$

$+ 2q^{12} + 2q^{13} + 2q^{14} + 2q^{15} + 2q^{16} + 2q^{17} + 2q^{18} + 2q^{19}$

$+ 3q^{20} + 3q^{21} + 3q^{22} + 3q^{23} + 4q^{24} + 4q^{25} + 4q^{26} + 4q^{27}$

$+ 4q^{28} + 4q^{29} + 5q^{30} + 5q^{31} + 5q^{32} + 6q^{33} + 6q^{34} + 6q^{35}$

$+ 7q^{36} + 7q^{37} + 7q^{38} + 7q^{39} + 8q^{40} + 8q^{41} + 9q^{42} + 9q^{43}$

$+ 10q^{44} + 10q^{45} + 11q^{46} + 11q^{47} + 12q^{48} + 12q^{49} + 13q^{50} + 13q^{51}$

$+ 14q^{52} + 14q^{53} + 15q^{54} + 16q^{55} + 17q^{56} + 17q^{57} + 18q^{58} + 18q^{59} + 20q^{60}$

$+ 20q^{61} + 21q^{62} + 22q^{63} + 23q^{64} + 24q^{65} + 25q^{66} + 26q^{67}$

$+ 27q^{68} + 28q^{69} + 30q^{70} + 30q^{71} + 32q^{72} + 33q^{73} + 34q^{74}$

$+ 35q^{75} + 37q^{76} + 38q^{77} + 40q^{78} + 41q^{79} + 43q^{80}$

$+ 44q^{81} + 46q^{82} + 47q^{83} + 50q^{84} + 51q^{85}$

$+ 53q^{86} + 55q^{87} + 57q^{88} + 59q^{89}$

$+ 62q^{90} + 63q^{91} + 66q^{92} + 68q^{93}$

$+ 71q^{94} + 72q^{95} + 76q^{96}$

$+ 78q^{97} + 81q^{98} + 83q^{99}$

$+ 87q^{100}$

~~11+7-6-2+1~~
11+7-6-2+2
~~12+7-7-2+2~~
12+8-7-2+2
~~13+8-7-3+2~~
~~14+10-8-3+2~~
15+10-9-4+2+2
18+12-11-4+3
18+13-11-4+4
20+13-12-5+4
20+14-12-5+4

13+9-7-3+2
~~14+9-8-3+2~~
~~16+11-9-4+3~~
17+11-10-4+3
17+12-10-4+3
21+14-13-5+4+1
22+15-13-6+4+1
23+16-14-6+4+1
24+17-14-6+5+1

25+17-15-7+5+1

$1 - q - q^2 + q^5 + q^7$

$1 - 2q - q^2 + 2q^3 + q^4 + 2q^5 - 2q^6$

8.7^2

$$\frac{1 - 2q^{11} - q^{22} + 2q^{33} + q^{44} + 2q^{55} - 2q^{66}}{1 - q^2 - q^9 + q^{15} + q^{29} - q^{39} - q^{60}}$$

$1 + q^2 + q^4 + q^6 + q^8 + q^9 + q^{10} + q^{12} + q^{13} + q^{14} + q^{15}$
$+ q^{16} + q^{17} + 2q^{18} + q^{19} + 2q^{20} + q^{21} + 2q^{22} + q^{20} + 2q^{24} + q^{25}$
$+ 3q^{26} + 2q^{27} + 2q^{28} + 2q^{29} + 2q^{30} + 2q^{31} + 2q^{32} + 2q^{33}$

202 224 246 268 ~~290~~ 301 323 345

7 + 5 = 2 − 1

1 − 2 − 9 + 7
1 − 1 − 1 − 9 + 8

1 − 1 − 4 + 5 1 − 2 − 9 + 9
~~1 − 5 − 4 + 9~~
1 − 1 − 4 + 5
~~1 − 1 − 6~~ ~~1 − 3 + 5~~ ~~1 − 3 + 4~~
~~1 − 1 − 4 + 4~~
~~1 − 3 + 4~~ ~~1 − 1 − 4 + 4~~ ~~1 − 1 − 7~~

7 − 6 − 5 + 4 + 3² + 1² + 0
1 + 0
8 − 6 + 4 + 1 + 0²
8 − 7 + 6 − 5 + 4 + 1 ...
15 11 10 9 8 7³
21 19 18 17 16 15 14 13
6 4 3 2 1 0²
7 5 3 2 1 0
21 1918 17² 16² 15−4 14⁵ 10⁷
32
6 3 0
1 2 3
11 + 10 + 7
8 7
15 11 10 9 8 7
−13 −11
16 14
15 14 13
17 16
3 1

4 Power Series on Pages 63, 64

$$L_{p,r}(q) := \frac{(q^p;q^p)_\infty}{(q^r;q^p)_\infty (q^{p-r};q^p)_\infty}$$
$$= \sum_{j=0}^{\infty} \frac{q^{rj}}{(q^p;q^p)_j (q^{pj+p-r};q^p)_\infty}.$$

Ramanujan's series on pages 63 and 64 are $L_{11,1}(q)$ and $L_{11,2}(q)$, respectively.

$L_{p,r}(q)$ is the generating function of partitions into r's and parts congruent to 0 or $-r$ modulo p, and the largest part that is a multiple of p is no more than p times the number of r's, which in turn is not greater than the smallest part that is congruent to $-r$ modulo p.

Theorem 13 *Let p and r be positive integers with $p \geq 2$ and $r < p$. Let*

$$L_{p,r}(q) = \frac{(q^p;q^p)_\infty}{(q^r;q^p)_\infty(q^{p-r};q^p)_\infty} := \sum_{n=0}^{\infty} b_{p,r}(n)q^n. \tag{4.1}$$

Then $b_{p,r}(n) \geq 0$ for all n. Moreover, we let

$$L_{p,r}(q)+q^p := \sum_{n=0}^{\infty} c_{p,r}(n)q^n := \Sigma_0+\Sigma_1+\cdots+\Sigma_{r-1}, \tag{4.2}$$

where the exponents in Σ_j are congruent to j modulo r, $0 \leq i \leq r-1$, i.e.,

$$\Sigma_i = \sum_{n=0}^{\infty} c_{p,r}(nr+i)q^{nr+i}.$$

Then for each i the coefficient sequence $\{c_{p,r}(nr+i)\}_{n=0}^{\infty}$ is non-decreasing.

i.e.

$\rho_1 = a - 1 + \frac{1}{a}$; $\rho = a^2 - a + 1 - \frac{1}{a} + \frac{1}{a^2}$, $\rho_2 = a^2 + \frac{1}{a^2}$;

$\rho_3 = a^3 + 1 + \frac{1}{a^3}$; $\rho_4 = (a + \frac{1}{a})(a^2 + \frac{1}{a^2})$;

$\rho_5 = a^4 + a^2 + 1 + \frac{1}{a^2} + \frac{1}{a^4}$;

$\rho_7 = a^3 + a^2 + a + 1 + \frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3}$

$\rho_9 = (a^2 + 1 + \frac{1}{a^2})(a^3 + 1 + \frac{1}{a^3})$

$\rho_{11} = a^5 + a^4 + a^3 + a^2 + a + 1 + \frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3} + \frac{1}{a^4} + \frac{1}{a^5}$

and $\sum_0^\infty p_n q^n = \frac{1}{(1-q)(1-q^2)(1-q^3)\cdots}$, then

$p_1 = 1$; $\lambda_1 = \rho_1$

$p_2 = 2$; $\lambda_2 = \rho_2$

$p_3 = 3$; $\lambda_3 = \rho_3$.

$p_4 = 5$; $\lambda_4 = \rho_5$

$p_5 = 7$; $\lambda_5 = \rho_7 \rho$

$p_6 = 11$; $\lambda_6 = \rho_1 \rho_{11}$

$p_7 = 15$; $\lambda_7 = \rho_3 \rho_5$

$p_8 = 22$; $\lambda_8 = \rho_1 \rho_2 \rho_{11}$

$p_9 = 30$; $\lambda_9 = \rho_2 \rho_3 \rho_5$

$p_{10} = 42$; $\lambda_{10} = \rho \rho_2 \rho_3 \rho_7$

$p_{11} = 56$; $\lambda_{11} = \rho_4 \rho_7 \times (a^5 - a^4 + a^2 + \frac{1}{a^2} - \frac{1}{a^4} - \frac{1}{a^5})$

$p_{12} = 77$; $\lambda_{12} = \rho_7 \rho_{11} \times (a^4 - 2a^3 + 2a^2 - a + 1 - \frac{1}{a} + \frac{2}{a^2} - \frac{2}{a^3} + \frac{1}{a^4})$

$p_{13} = 101$; $\lambda_{13} = \rho \rho_1 \times (a^{10} + 2a^9 + 2a^8 + 2a^7 + 3a^6 + 4a^5 + 6a^4 + 8a^3 + 9a^2 + 9a + 9 + \frac{9}{a} + \frac{9}{a^2} + \frac{8}{a^3} + \frac{6}{a^4} + \frac{4}{a^5} + \frac{3}{a^6} + \frac{2}{a^7} + \frac{2}{a^8} + \frac{2}{a^9} + \frac{1}{a^{10}})$.

$p_{14} = 135$; $\lambda_{14} = \rho_5 \rho_9 \times (a^5 - a^3 + a + 1 + \frac{1}{a} - \frac{1}{a^3} - \frac{1}{a^5})$

$p_{15} = 176$; $\lambda_{15} = \rho_4 \rho_{11} (a^7 - a^6 + a^4 + a + \frac{1}{a} + \frac{1}{a^4} - \frac{1}{a^6} + \frac{1}{a^7})$

$p_{16} = 231$; $\lambda_{16} = \rho_3 \rho_7 \rho_{11} \times (a^5 - 2a^4 + 2a^3 - 2a^2 + 3a - 3 + \frac{3}{a} - \frac{2}{a^2} + \frac{2}{a^3} - \frac{2}{a^4} + \frac{1}{a^5})$.

$p_{17} = 297$; $\lambda_{17} = \rho_9 \rho_{11} \times (a^7 - a^6 + a^3 + a - 1 + \frac{1}{a} + \frac{1}{a^3} - \frac{1}{a^6} + \frac{1}{a^7})$

$p_{18} = 385$; $\lambda_{18} = \rho_5 \rho_7 \rho_{11} \times (a^6 - 2a^5 + a^4 + a^3 - a^2 + 1 - \frac{1}{a^2} + \frac{1}{a^3} + \frac{1}{a^4} - \frac{2}{a^5} + \frac{1}{a^6})$

$p_{19} = 490$; $\lambda_{19} = \rho_1 \rho_2 \rho_5 \rho_7 \times (a^9 - a^7 + a^4 + 2a^3 + a^2 - 1 + \frac{1}{a^2} + \frac{2}{a^3} + \frac{1}{a^4} - \frac{1}{a^7} + \frac{1}{a^9})$

$p_{20} = 627$; $\lambda_{20} = \rho \rho_3 \rho_{11} \times (a^{10} + a^6 + a^4 + a^3 + 2a^2 + 2a + 3 + \frac{2}{a} + \frac{2}{a^2} + \frac{1}{a^3} + \frac{1}{a^4} + \frac{1}{a^6} + \frac{1}{a^{10}})$

$p_{21} = 792$; $\lambda_{21} = \rho \rho_3 \rho_4 \rho_{11} \times (a^8 - a^6 + a^4 + a + 2 + \frac{1}{a} + \frac{1}{a^4} - \frac{1}{a^6} + \frac{1}{a^8})$.

5 Divisors of $p(n)$

$$
\begin{aligned}
\rho_1 &= a_1 - 1, \\
\rho &= a_2 - a_1 + 1, \\
\rho_2 &= a_2, \\
\rho_3 &= a_3 + 1, \\
\rho_4 &= a_1 a_2, \\
\rho_5 &= a_4 + a_2 + 1, \\
\rho_7 &= a_3 + a_2 + a_1 + 1, \\
\rho_9 &= (a_2 + 1)(a_3 + 1), \\
\rho_{11} &= a_5 + a_4 + a_3 + a_2 + a_1 + 1.
\end{aligned}
$$

$n =$(as a sum of powers of a) the number of terms with positive coefficients minus the number of terms with negative coefficients in the representation of ρ_n.

$$p(10) = 1 \cdot 2 \cdot 3 \cdot 7 = 42.$$

$$
\begin{aligned}
p(1) =&1, \quad \lambda_1 = \rho_1,\\
p(2) =&2, \quad \lambda_2 = \rho_2,\\
p(3) =&3, \quad \lambda_3 = \rho_3,\\
p(4) =&5, \quad \lambda_4 = \rho_5,\\
p(5) =&7, \quad \lambda_5 = \rho_7\rho,\\
p(6) =&11, \quad \lambda_6 = \rho_1\rho_{11},\\
p(7) =&15, \quad \lambda_7 = \rho_3\rho_5,\\
p(8) =&22, \quad \lambda_8 = \rho_1\rho_2\rho_{11},\\
p(9) =&30, \quad \lambda_9 = \rho_2\rho_3\rho_5,\\
p(10) =&42, \quad \lambda_{10} = \rho\rho_2\rho_3\rho_7,\\
p(11) =&56, \quad \lambda_{11} = \rho_4\rho_7(a_5 - a_4 + a_2),\\
p(12) =&77, \quad \lambda_{12} = \rho_7\rho_{11}(a_4 - 2a_3 + 2a_2 - a_1 + 1)\\
p(13) =&101, \quad \lambda_{13} = \rho\rho_1(a_{10} + 2a_9 + 2a_8\\
&\qquad + 2a_7 + 3a_6 + 4a_5 + 6a_4
\end{aligned}
$$

$$
\begin{aligned}
& && + 8a_3 + 9a_2 + 9a_1 + 9), \\
p(14) =& 135, && \lambda_{14} = \rho_5\rho_9(a_5 - a_3 + a_1 + 1), \\
p(15) =& 176, && \lambda_{15} = \rho_4\rho_{11}(a_7 - a_6 + a_4 + a_1), \\
p(16) =& 231, && \lambda_{16} = \rho_3\rho_7\rho_{11}(a_5 - 2a_4 + 2a_3 \\
& && - 2a_2 + 3a_1 - 3), \\
p(17) =& 297, && \lambda_{17} = \rho_9\rho_{11}(a_7 - a_6 + a_3 + a_1 - 1), \\
p(18) =& 385, && \lambda_{18} = \rho_5\rho_7\rho_{11}(a_6 - 2a_5 \\
& && + a_4 + a_3 - a_2 + 1), \\
p(19) =& 490, && \lambda_{19} = \rho_1\rho_2\rho_5\rho_7(a_9 - a_7 \\
& && + a_4 + 2a_3 + a_2 - 1), \\
p(20) =& 627, && \lambda_{20} = \rho\rho_3\rho_{11}(a_{10} + a_6 + a_4 \\
& && + a_3 + 2a_2 + 2a_1 + 3), \\
p(21) =& 792, && \lambda_{21} = \rho\rho_3\rho_4\rho_{11}(a_8 - a_6 + a_4 \\
& && + a_1 + 2).
\end{aligned}
$$

6 Mock Theta Functions

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n^2} = \frac{1}{(q;q)_\infty}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2}$$ third order mock theta function

Definition 14 *$f(q)$ is a mock theta function if:*

(1) *For ever root of unity ζ, there is a theta function $\theta_\zeta(q)$ such that the difference $f(q) - \theta_\zeta(q)$ is bounded as $q \to \zeta$ radially.*

(2) *There is no single theta function that works for all ζ, i.e., for every theta function $\theta(q)$ there is some root of unity ζ for which $f(q) - \theta_\zeta(q)$ is unbounded as $q \to \zeta$ radially.*

There are no proofs that any mock theta functions exist!

Definition 15 *The rank of a partition equals the largest part minus the number of parts.*

Definition 16 $N(a, b, n) = \#$ *of partitions of* n *with rank congruent to* a *modulo* b.

Definition 17 $\rho(n) = \#$ *of partitions of* n *with unique smallest part and all other parts* $\leq$ *double the smallest part.*

Conjecture 18 (First Mock Theta Conjecture)

$$N(1, 5, 5n) = N(0, 5, 5n) + \rho(n)$$

Example 19 *Let* $n = 5$. $p(25) = 1958$.

$$\begin{aligned} N(1, 5, 25) &= 393, \\ N(0, 5, 25) &= 390, \\ \rho(5) &= 3. \end{aligned}$$

$$5, 2 + 3, 1 + 2 + 2$$

7 Stacks

Definition 20 *A stack with summit is a subset of S of*

$$L := \{(x, y) : x \in \mathbf{Z}, y \in \mathbf{Z}_0\}$$

such that

(1) all elements of L lying on a vertical or horizontal line connecting 2 elements of S are also in S,

(2) if $(x, y) \in S, (x, z) \in L \to (x, z) \in S, 0 \leq z \leq y$,

(3) if $y_0 = \sup_{(x,y) \in S} y$, then $(0, y_0) \in S$.

The point $(0, y_0)$ is called the summit of S.

$\sigma\sigma(n) = \#$ of stacks with summits of size n.

$$\sum_{n=0}^{\infty} \sigma\sigma(n) q^n = \sum_{n=0}^{\infty} \frac{q^n}{(q)_n^2}$$

Theorem 21 (p. 10)

$$\sum_{n=0}^{\infty} \frac{q^n}{(q)_n^2} = \frac{1}{(q)_\infty^2} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2}.$$

Gradual Stacks

Theorem 22 (p. 10) *If*

$$\varphi(-q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2},$$

then

$$\varphi(-q) \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q;q)_n^2} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(q^2;q^2)_n}.$$

8 Identities Arising from the Rogers-Fine Identity

$$\sum_{n=0}^{\infty} \frac{(\alpha;q)_n}{(\beta;q)_n} \tau^n$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha;q)_n (\alpha\tau q/\beta;q)_n \beta^n \tau^n q^{n^2-n}(1-\alpha\tau q^{2n})}{(\beta;q)_n (\tau;q)_{n+1}}$$

Examples Arising from Franklin Involution

Theorem 23 (p. 37)

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(-q;q)_n} = \sum_{n=0}^{\infty} q^{n(3n+1)/2}(1-q^{2n+1}).$$

Theorem 24 (p. 37)

$$\sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(-q;q)_{2n+1}} = \sum_{n=0}^{\infty} q^{n(3n+1)/2}(1-q^{2n+1}).$$

Durfee Squares

Theorem 25 (p. 36)

$$\sum_{n=0}^{\infty} \frac{(-1)^n a^{2n} q^{n^2}}{(a^2q^2;q^2)_n} = 1 - a \sum_{n=1}^{\infty} \frac{a^n q^n}{(-aq;q)_n}.$$

A Difficult Entry

Theorem 26 (p. 29)

$$\sum_{n=0}^{\infty} \frac{(-aq;q^2)_n(-aq)^n}{(-aq^2;q^2)_n} = \sum_{n=0}^{\infty} (-a)^n q^{n(n+1)/2}.$$

9 Identities Involving Theta Functions

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \qquad |ab| < 1.$$

Theorem 27 (p. 31)

$$\sum_{n=0}^{\infty} \frac{q^n}{(q)_{2n}} = \frac{f(q^5, q^3)}{(q;q)_\infty}$$

and

$$\sum_{n=0}^{\infty} \frac{q^n}{(q)_{2n+1}} = \frac{f(q^7, q)}{(q;q)_\infty}.$$

Topics in Ramanujan's Lost Notebook

q-series
Mock theta functions
Theta functions
Partial theta function expansions
False theta functions
Identities connected with the Rogers–Fine identity
Theory of partitions
Eisenstein series
modular equations
Rogers–Ramanujan continued fraction
Other q-continued fractions
Asymptotic expansions of q-series and q-continued fractions
Integrals of theta functions
Integrals of q-products
Incomplete elliptic integrals
Other continued fractions
Other integrals
Infinite series identities
Dirichlet series
Approximations
Arithmetic functions
Numerical calculations
Diophantine equations
Elementary mathematics

$R(q)$

$L(s,\chi)$

circle

Freeman J. Dyson

University of Illinois

June 1, 1987

9–10 a.m.

coeffs. of Eisenstein series

congruences for coeffs. of recip

new results on cong. for $p(n)$

integrals of eta-funcs., Eisenstein series

$\frac{1}{\pi}$ series

---

mention rank & crank

I give thanks to Ramanujan for two things, for discovering congruence properties of partitions and for not discovering the criterion for dividing them into equal classes. That was the wonderful thing about Ramanujan. He discovered so much, and yet he left so much more in his garden for other people to discover. In the 44 years since that happy day, I have intermittently been coming back to Ramanujan's garden. Every time when I come back, I find fresh flowers blooming.