Computer Algebra and Power Series with Positive Coefficients

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For some r, obviously *no.*
For some r, obviously *yes.*
For some r, it seems that *yes,* but nobody knows how to prove this.

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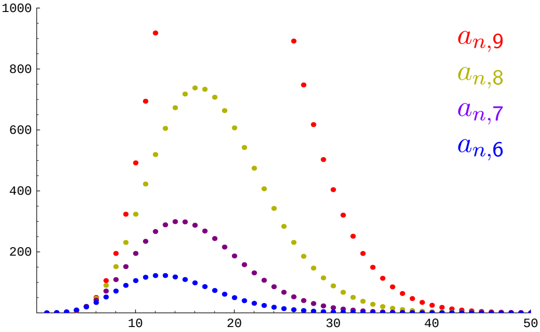

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These proofs are really complicated!

Open problems

Gillis/Reznick/Zeilberger conjectured in 1982 that

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These conjectures are still open.

Can The Computer Help?

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Such an algorithm could be used for solving Diophantine equations, because

$$rat(x_1, \ldots, x_r) = \sum_{n_1,\ldots,n_r} poly(n_1, \ldots, n_r)^2 x_1^{n_1} \cdots x_r^{n_r}$$

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Hence: No such algorithm exists.

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Such an algorithm exists if and only if there exists an algorithm for deciding whether a given univariate rational function has only positive coefficients.

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- For determining useful recurrence equations satisfied by the coefficients
- For deriving and deciding (proving or disproving) sufficient conditions that, if true, imply deciding certain sufficient conditions

Computer Algebra Tools

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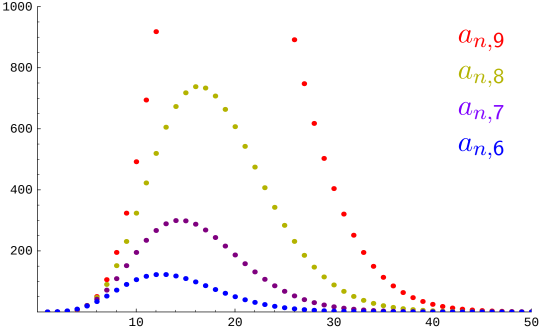

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The coefficients $a_{n,m}$ satisfy

$$\begin{aligned}
&\tfrac{3}{16}(m+n+2)(m+n+3)a_{n,m} \\
&- \tfrac{1}{8}(m+n+3)(m+7n+13)a_{n+1,m} \\
&+ \tfrac{1}{2}(n+2)(2n+5)a_{n+2,m} = 0
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For positivity of $a_{n,m}$ it would be sufficient to show

$$\begin{aligned}
&\forall N, M, A_0, A_1, A_2 \in \mathbb{R} : \\
&N \geq M \geq 0 \wedge A_1 > 0 \wedge A_0 > 0 \\
&\wedge \tfrac{3}{16}(M + N + 2)(M + N + 3)A_0 \\
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Collins's algorithm tells us that this is *wrong.*

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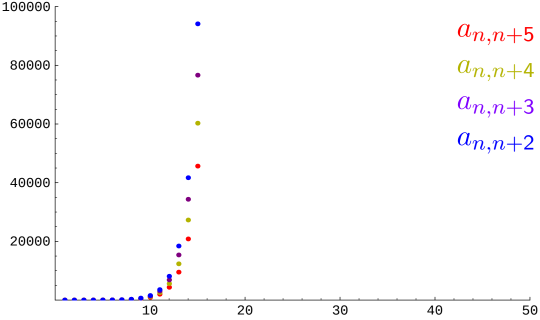

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The coefficients $a_{n,m}$ also satisfy

$$\begin{aligned}
&- \tfrac{9}{32}(m+n+2)(m+n+3) \\
&\times (m+n+4)(m+n+5)(2m+2n+9)a_{n,m} \\
&+ \tfrac{1}{4}(m+n+4)(m+n+5)(2m+2n+7) \\
&\times \left(2m^2 + 16nm + 35m + 2n^2 + 35n + 57\right) a_{n+1,m+1} \\
&- 2(m+2)(2m+5)(n+2) \\
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&+ \tfrac{1}{4}(M + N + 4) \cdots \left(2M^2 + \cdots + 57\right) A_1 \\
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Collin's algorithm tells us that this is *also wrong.*

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But it can be shown that the $a_{n,m}$ are diagonally *increasing:*

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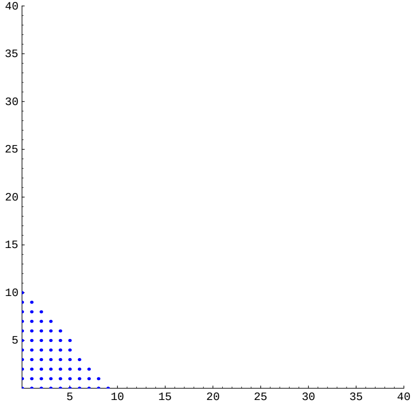
is *true*. This proves the positivity result.

Increasing Evidence

- Easy: Compute some large (but finite) number of coefficients.



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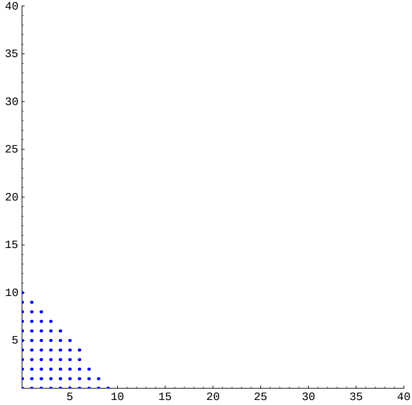

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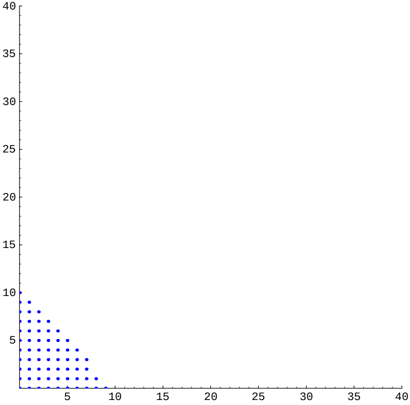

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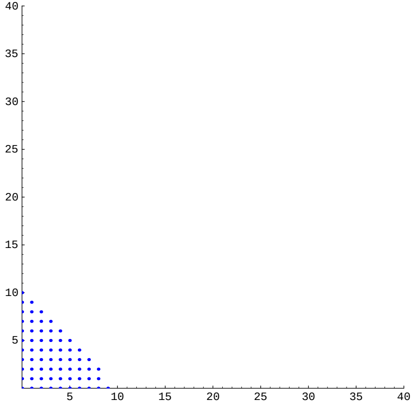

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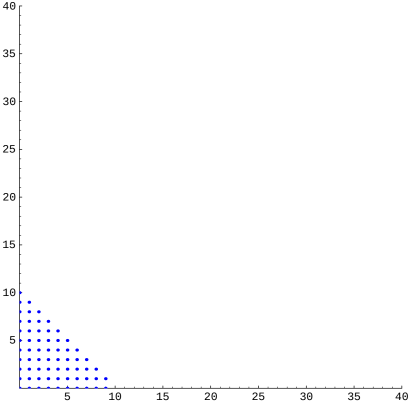

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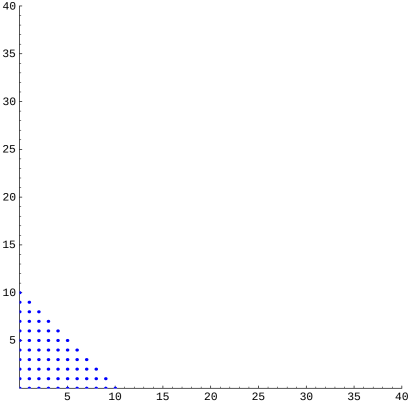

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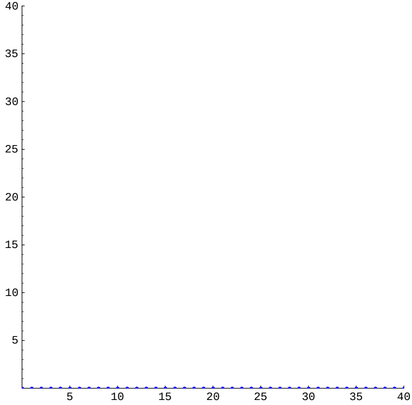

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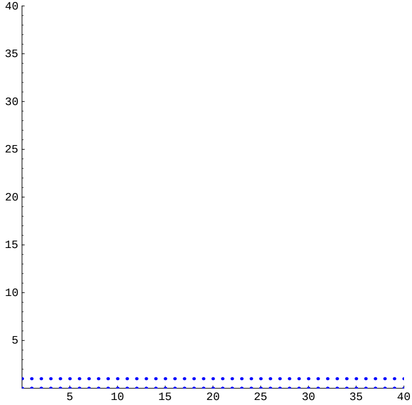

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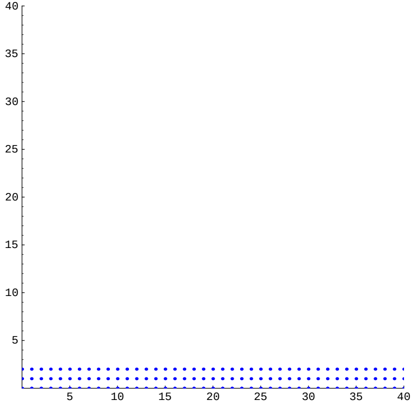

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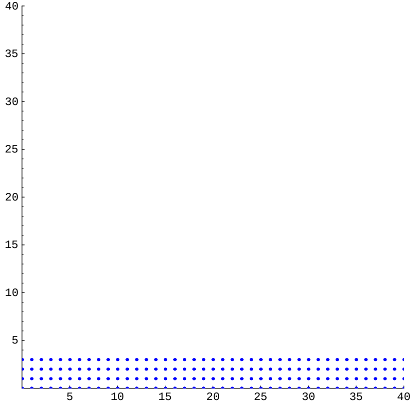

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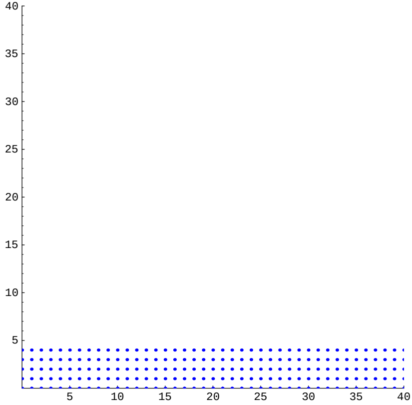

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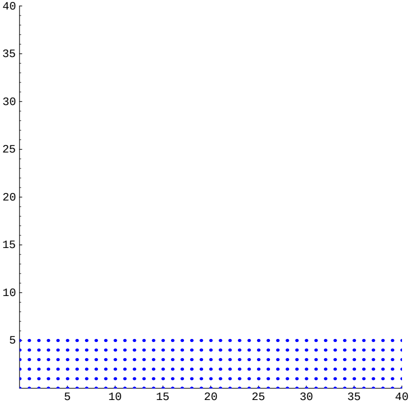

Increasing Evidence

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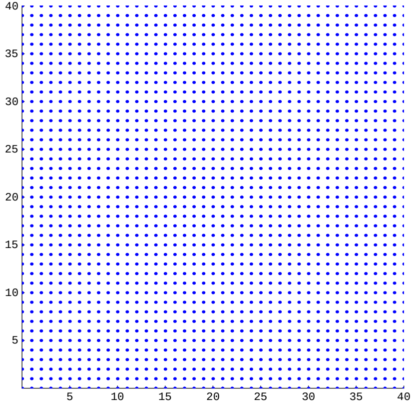
- Better: Prove positivity for some infinite set of coefficients.
- Best case: Prove positivity for all coefficients.



Szegö's Example

$$\frac{1}{1 - x - y - z + \frac{2}{3}(xy + xz + yz)} = \sum_{n,m,k} a_{n,m,k} x^n y^m z^k$$

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- The best we obtained were positivity implying recurrence equations in n and m when $k = 1, 2, \ldots, 16$ is fixed.
- Because for each fixed k, a shorter recurrence equations are available.

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Such results can probably be obtained for any fixed k, l, u, v, w.

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$$\frac{1}{1 - x - y - z + 4xyz} = \sum_{n,m,k} a_{n,m,k} x^n y^k z^m$$

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- Automated guessing delivers 10 linearly independent multivariate recurrence equations with linear coefficients.
- A positivity-asserting linear-combination can be found by making an ansatz and solving a quantifier elimination problem for the undetermined coefficients.

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- This way, it can also be shown that Szegö's result *cannot be shown* by a first-order linear positivity-asserting recurrence with linear coefficients.

Askey-Gasper's Example

The conjectured generalization

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The monotonicity-by-induction reasoning is applicable to this sum.

New Challenges & Conclusion

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We can give partial proofs for both, but no full proofs.

(The first one is easily proven, as pointed out by Armin Straub a few days ago. The second remains open so far.)

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Conclusion

- It is difficult to decide whether all the Taylor coefficients of a rational function are positive.
- Standard tools from Computer Algebra (Recurrence Guessing and Cylindrical Decomposition) can contribute to this topic.
- For nontrivial examples, we could obtain partial proofs in this way.
- This extends the computational evidence in support of these conjectures far beyond what was available so far.