Classifying walks in the quarter plane

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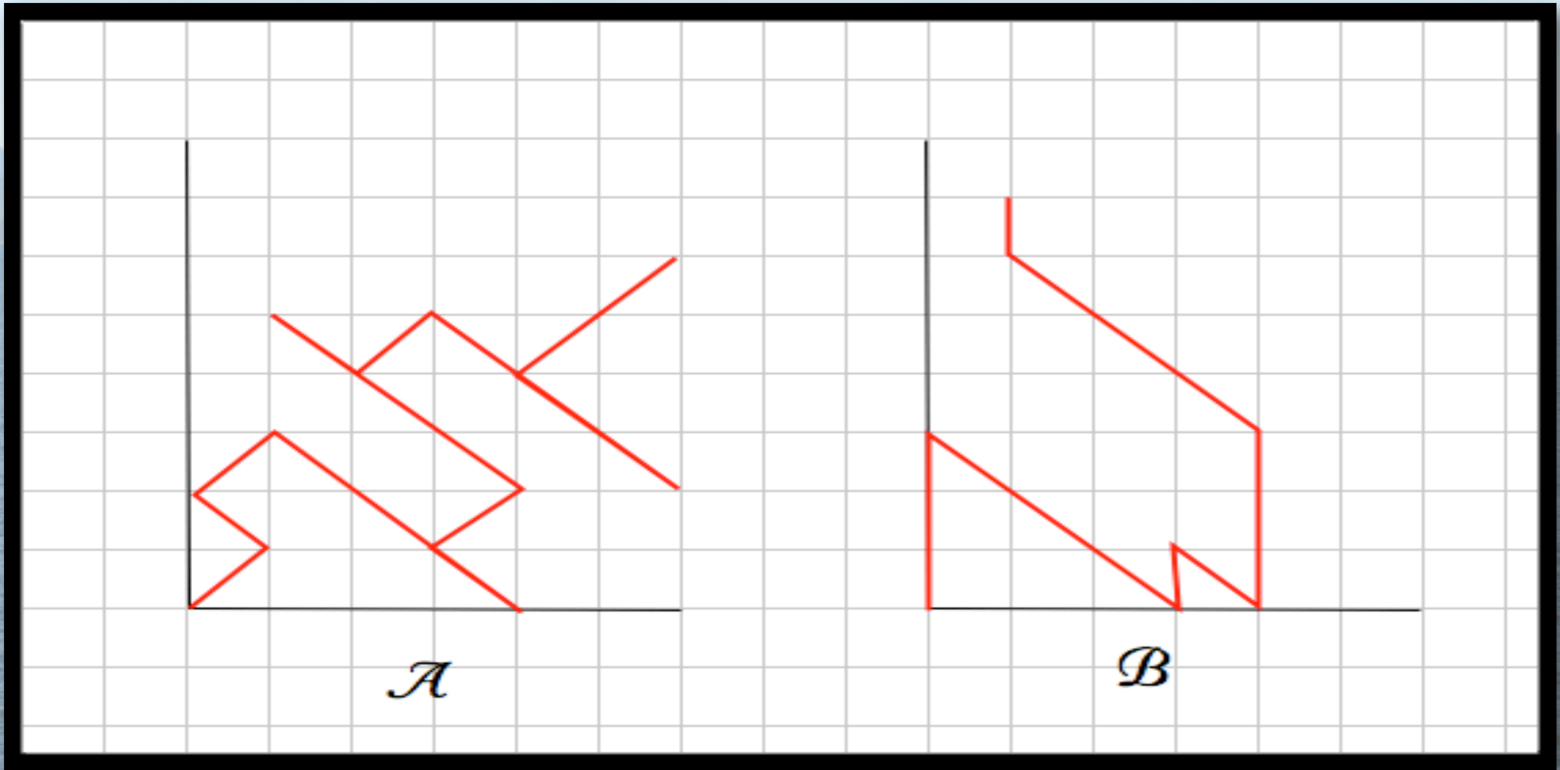
Two walks in the quarter plane

Two step sets: A={↖↗↘} B={↖↑↘}



Two theorems

Theorem 1. [M.-Rechnitzer 07]

The generating function $W_A(t) = \sum_w t^{\text{length}(w)}$

for walks w in the quarter plane with steps from

A = {↖↗↘} is not holonomic.

Theorem 2. [M.-Rechnitzer 07]

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Can we characterize "holonomic walks"?

Main Problem

Characterize (combinatorially) lattice walks with a holonomic generating function.

- By step set $S \subseteq \{\uparrow \nearrow \rightarrow \searrow \downarrow \swarrow \leftarrow \nwarrow\}$

- By region (quarter plane, general wedge, ...)

Develop general techniques and algorithms to automate this decision.

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  - Many are algebraic
  - Several standard techniques
- Proving non-holonomy is still hard
  - So far: very few examples proved, many suspected.

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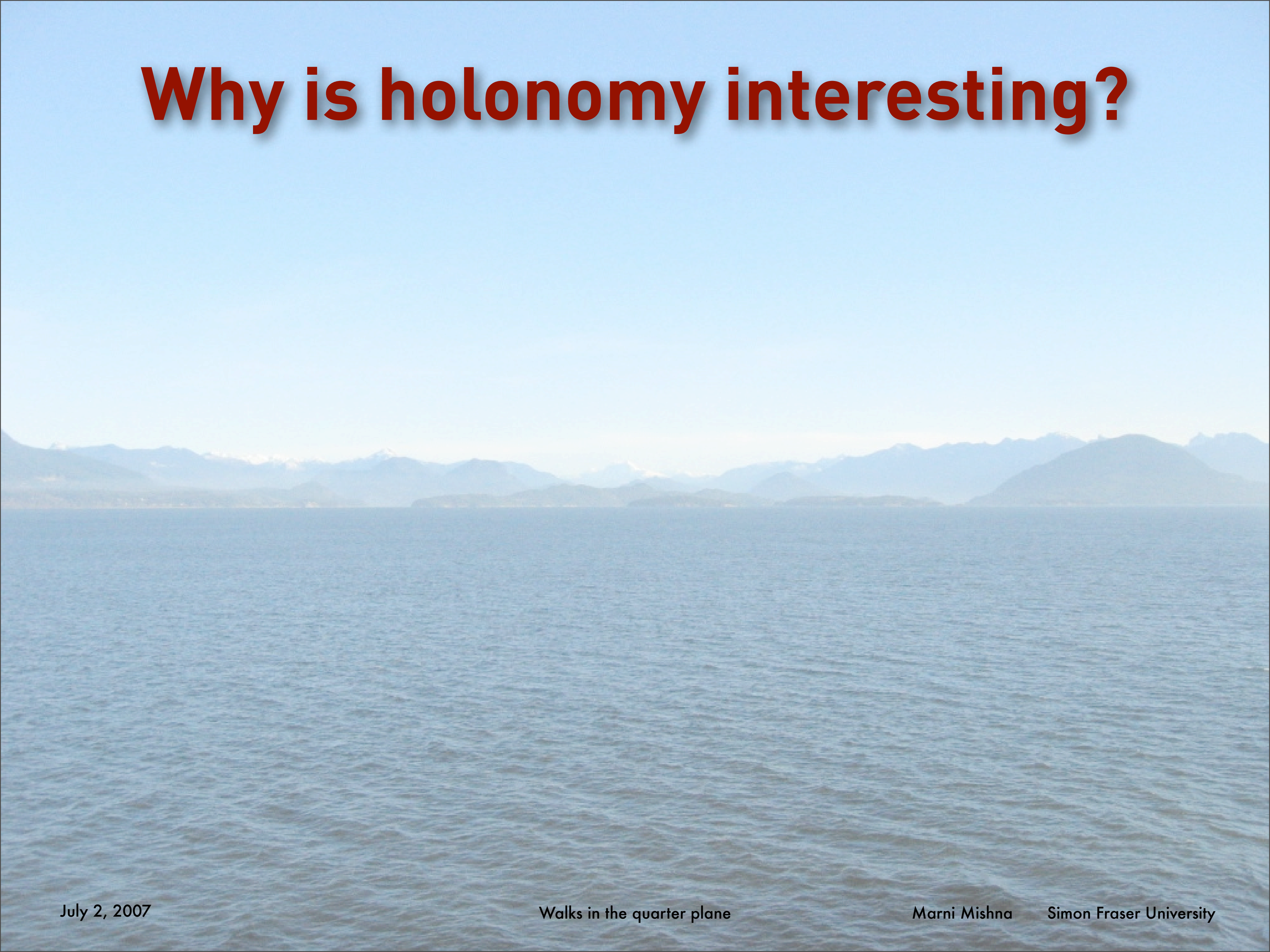
- |S|=3: classified (M.)

Why is holonomy interesting?



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- "Guessability" (gfun, rate, ...)

Examples and Non-Examples

- ✓ Algebraic: Dyck paths, Motzkin paths, ...
- ✓ Shuffles of Dyck paths
- ✓ Walks in 1/4 plane with steps {↑→↓←}
- ✓ k-regular graphs (Gessel)
- x Regular graphs
- x Partitions
- x Knights walks {(2, -1), (-1, 2)} (B.-M.+P)
- x $\sum \log(n) z^n$ (Flajolet+Gerhold+Salvy)
- ? Pattern avoiding permutations
- ? Walks in 1/4 plane with steps {↑↓↗↙}

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- x $Q(0,0;t)$ not holonomic $\Rightarrow$ $Q(x,y;t)$ not holonomic

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  - **Example:**
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  - **Example:**
    ...W(t) is holonomic if S has small height variations and is symmetric across the y-axis.
    (Bousquet-Mélou+ Petkovšek)
- A coherent, complete, combinatorial theory of holonomy, ideally, similar to the theory of algebraic functions.

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Conjecture (M.) The generating function W(t) for walks in the quarter plane with steps from $S \subseteq \{0, \pm 1\} \times \{0, \pm 1\}$

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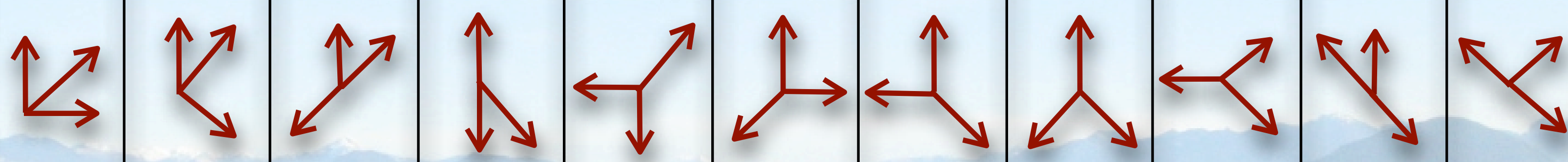
Rotational symmetry in the triangular lattice.

Evidence (part 1)

- All known + conjectured "nice" cases
  - square lattice
  - diamond lattice
  - triangular lattice
  - step sets of cardinality 3
  - ...

Evidence (part 2)

Isomorphism classes for cardinality 3 step sets in the quarter plane



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Recall our two theorems

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Is it difficult to prove these theorems?

Proving Thm 1: $S = \{\nearrow \searrow \nwarrow\}$

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$$Q(x, y; t) = \sum a_{ij}(n) x^i y^j t^n$$

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A walk is a shorter walk and a step.

$$Q(x,y) = 1 + t\left(xy + \frac{x}{y} + \frac{y}{x}\right) Q(x,y) - t\frac{x}{y}Q(x,0) - t\frac{y}{x}Q(0,y)$$

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2. Forms an infinite group of power series

$$Y_1(Y_{-1}(x)) = x \qquad Y_n(x) = Y_1(Y_{n-1}(x)) = xt^n + \ldots$$

The iterated kernel method

$$K(x,y)Q(x,y) = xy - tx^2Q(x,0) - ty^2Q(0,y)$$

- Using $K(Y_n(x), Y_{n+1}(x)) = 0$ and the above equation, we show that

$$tQ(x,0) = \sum_n (-1)^n Y_n(x) Y_{n+1}(x)$$

- We substitute x=1, and rearrange (1) to get

$$W(t) = Q(1,1;t) = \frac{2\sum(-1)^n Y_n(1;t) Y_{n+1}(1;t)}{1-3t}$$

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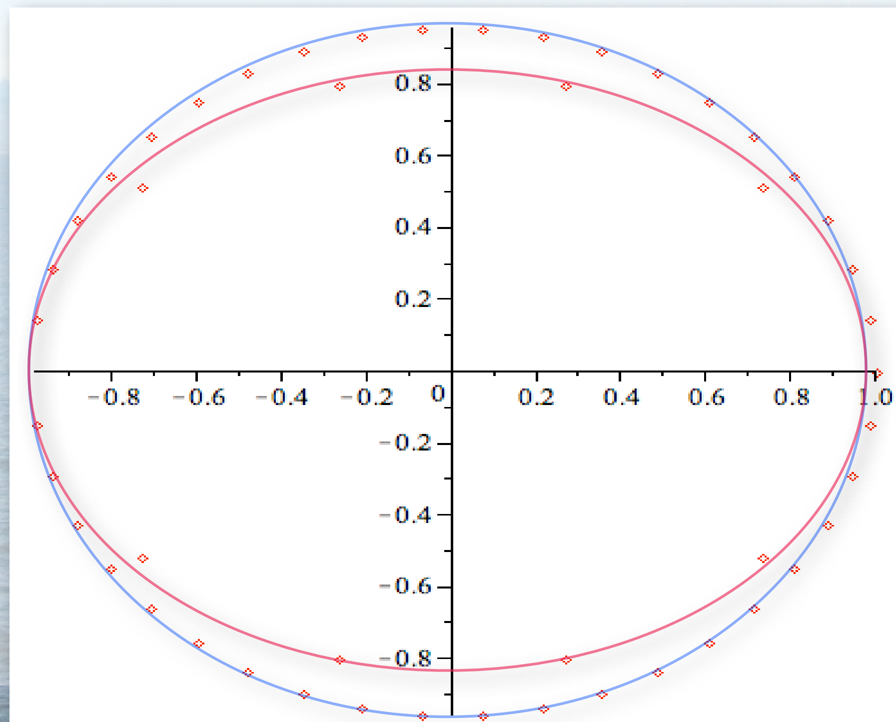
The singularities of W(t) are 1/3 and the singularities of the Y_n, hence they are infinite in number and **W(t) is not holonomic.**

Where are the singularities?

The singularities of Yn(1;t) are the solutions to

$$z^{2n} + z^{-2n} + z^2 + z^{-2} = 4$$

inside the unit circle.



Asymptotic result

Theorem (M.+Rechnitzer): The number of lattice paths of length n with steps from {↖↗↘} confined to the quarter plane is asymptotic to

$$\alpha 3^n + O(8^{n/2})$$

where

$$\alpha = 1 - 2\sum_{n\geq 0} \frac{(-1)^n}{F_{2n}F_{2n+2}} = 0.1731788...$$

Other non-holonomic class?

Conjecture (M.+ Laferrière) Walks with steps {→↑↓} in wedge bounded by $y = \pm mx$ centered on x-axis has non-holonomic generating function.

Evidence

Combinatorial similarity to {↖↗↘} in ¼ plane.

Bi-variate GF counting walks ending on x=k:

- $B(t,u) = \sum_k B_k(t)u^k$
- $B_k(t)$ is rational and as k increases, poles fill unit circle.

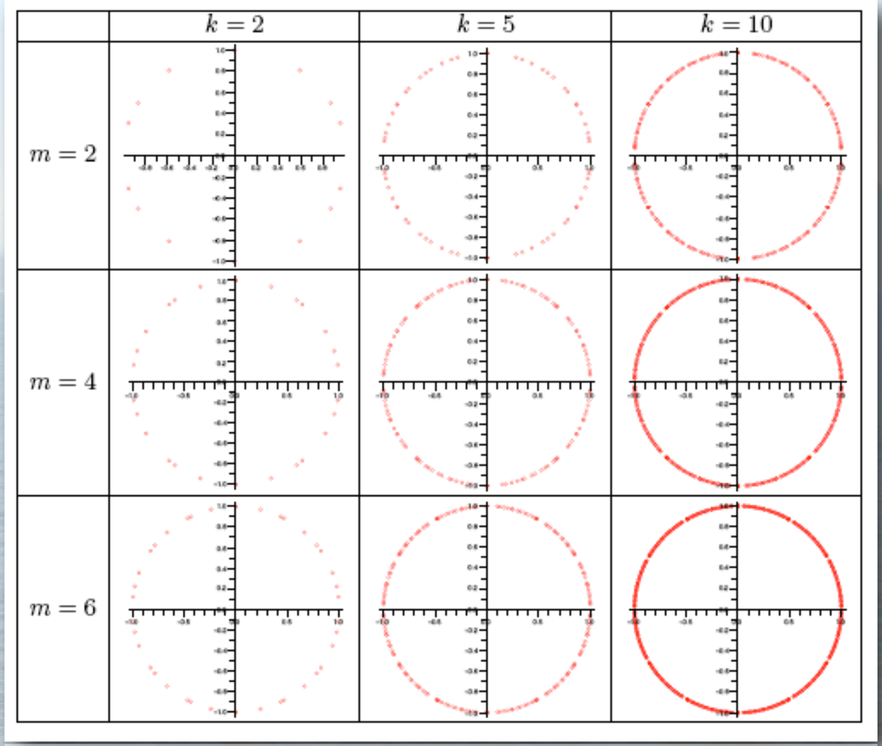
Hints of non-holonomicity

$B_k^{(m)}(t)$ counts walks in the wedge bounded by $y = \pm mx$.

$B^{(m)}(t,u) = \sum_k B_k^{(m)}(t) u^k$

shows evidence of being non-holonomic, as does

$B^{(m)}(t) = B^{(m)}(t,1)$



Combinatorial criteria

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Any indications on how to prove this?

The group of the walk

To each step set S we define G(S), a group of transformations which fix the polynomial

$$K(x,y;t) = 1 - t\sum_{(i,j)\in S} x^i y^j$$

Motivation: Fayolle+Iasnogordski+Malyshev

e.g. S={←→↑↓} = {(-1,0), (1,0), (0,1), (0,-1)}

$$K(x,y;t) = 1 - t(x + y + 1/x + 1/y)$$

$$G(S) = \left\langle \underbrace{(x,y) \to (1/y, x)}_{\tau_x}, \underbrace{(x,y) \to (y, 1/x)}_{\tau_y} \right\rangle$$

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Can they be interpreted in generating function language or combinatorial terms?

Projects and goals

- Prove the holonomy of the Gessel walks
- Prove the conjecture and "combinatorialize" F+I+M approach to quarter plane
- Classify walks in other wedges (with Laferrière):
  - Strategies for "OR" constraints
- Consider other infinite Cayley graphs
- Prove that miracles don't exist- rather understand precisely when they do.


merci beaucoup!