Refined Enumerations of Totally Symmetric Self-Complementary Plane Partitions and Constant Term Identities

Masao Ishikawa[†]

[†]Department of Mathematics
Tottori University

The 19th International Conference on Formal Power Series and Algebraic Combinatorics 2007,
Nankai University, Tianjin, China

Introduction

Abstract

In this talk we give Pfaffian or determinant expressions, and constant term identities for the conjectures in the paper "Self-complementary totally symmetric plane partitions" (*J. Combin. Theory Ser. A* **42**, (1986), 277–292) by W.H. Mills, D.P. Robbins and H. Rumsey. We also settle a weak version of Conjecture 6 in the paper, i.e., the number of shifted plane partitions invariant under a certain involution is equal to the number of alternating sign matrices invariant under the vertical flip.

The conjectures on TSSCPPs

1. Conjecture 2 (The refined TSSCPP conjecture)
2. Conjecture 3 (The doubly refined TSSCPP conjecture)
3. Conjecture 7, 7' (Related to the monotone triangles)
4. Conjecture 4 (Related to half-turn symmetric ASMs)
5. Conjecture 6 (Related to vertical symmetric ASMs)

The conjectures on TSSCPPs

1. Conjecture 2 (The refined TSSCPP conjecture)
2. Conjecture 3 (The doubly refined TSSCPP conjecture)
3. Conjecture 7, 7' (Related to the monotone triangles)
4. Conjecture 4 (Related to half-turn symmetric ASMs)
5. Conjecture 6 (Related to vertical symmetric ASMs)

The conjectures on TSSCPPs

1. Conjecture 2 (The refined TSSCPP conjecture)
2. Conjecture 3 (The doubly refined TSSCPP conjecture)
3. Conjecture 7, 7’ (Related to the monotone triangles)
4. Conjecture 4 (Related to half-turn symmetric ASMs)
5. Conjecture 6 (Related to vertical symmetric ASMs)

The conjectures on TSSCPPs

1. Conjecture 2 (The refined TSSCPP conjecture)
2. Conjecture 3 (The doubly refined TSSCPP conjecture)
3. Conjecture 7, 7' (Related to the monotone triangles)
4. Conjecture 4 (Related to half-turn symmetric ASMs)
5. Conjecture 6 (Related to vertical symmetric ASMs)

The conjectures on TSSCPPs

1. Conjecture 2 (The refined TSSCPP conjecture)
2. Conjecture 3 (The doubly refined TSSCPP conjecture)
3. Conjecture 7, 7' (Related to the monotone triangles)
4. Conjecture 4 (Related to half-turn symmetric ASMs)
5. Conjecture 6 (Related to vertical symmetric ASMs)

Plane partitions

Definition

A *plane partition* is an array $\pi = (\pi_{ij})_{i,j\geq 1}$ of nonnegative integers such that π has finite support (i.e., finitely many nonzero entries) and is weakly decreasing in rows and columns. If $\sum_{i,j\geq 1} \pi_{ij} = n$, then we write $|\pi| = n$ and say that π is a plane partition of n, or π has the *weight* n.

Example

A plane partition of 14

$$\begin{array}{cccccc} 3 & 2 & 1 & 1 & 0 & \dots \\ 2 & 2 & 1 & 0 & \dots & \\ 1 & 1 & 0 & 0 & \dots & \\ 0 & 0 & 0 & \ddots & & \end{array}$$

Plane partitions

Definition

A *plane partition* is an array $\pi = (\pi_{ij})_{i,j \geq 1}$ of nonnegative integers such that π has finite support (i.e., finitely many nonzero entries) and is weakly decreasing in rows and columns. If $\sum_{i,j\geq 1} \pi_{ij} = n$, then we write $|\pi| = n$ and say that π is a plane partition of n, or π has the *weight* n.

Example

A plane partition of 14

$$\begin{matrix} 3 & 2 & 1 & 1 & 0 & \dots \\ 2 & 2 & 1 & 0 & \dots & \\ 1 & 1 & 0 & 0 & \dots & \\ & & & & & \\ 0 & 0 & 0 & \ddots & & \end{matrix}$$

Plane partitions

Definition

A *plane partition* is an array $\pi = (\pi_{ij})_{i,j \geq 1}$ of nonnegative integers such that π has finite support (i.e., finitely many nonzero entries) and is weakly decreasing in rows and columns. If $\sum_{i,j \geq 1} \pi_{ij} = n$, then we write $|\pi| = n$ and say that π is a plane partition of n, or π has the *weight* n.

Example

A plane partition of 14

$$\begin{array}{cccccc} 3 & 2 & 1 & 1 & 0 & \dots \\ 2 & 2 & 1 & 0 & \dots & \\ 1 & 1 & 0 & 0 & \dots & \\ 0 & 0 & 0 & \ddots & & \end{array}$$

Shape

Definition

Let $\pi = (\pi_{ij})_{i,j\geq 1}$ be a plane partition.

- A *part* is a positive entry $\pi_{ij} > 0$.
- The *shape* of π is the ordinary partition λ for which π has λ_i nonzero parts in the ith row.
- We say that π has r *rows* if $r = \ell(\lambda)$. Similarly, π has s *columns* if $s = \ell(\lambda')$.

Example

A plane partition of shape (432) with 3 rows and 4 columns:

3	2	1	1
2	2	1	
1	1		

Shape

Definition

Let $\pi = (\pi_{ij})_{i,j\geq 1}$ be a plane partition.

- A *part* is a positive entry $\pi_{ij} > 0$.
- The *shape* of π is the ordinary partition λ for which π has λ_i nonzero parts in the ith row.
- We say that π has r *rows* if $r = \ell(\lambda)$. Similarly, π has s *columns* if $s = \ell(\lambda')$.

Example

A plane partition of shape (432) with 3 rows and 4 columns:

3	2	1	1
2	2	1	
1	1		

Shape

Definition

Let $\pi = (\pi_{ij})_{i,j\geq 1}$ be a plane partition.

- A *part* is a positive entry $\pi_{ij} > 0$.
- The *shape* of π is the ordinary partition λ for which π has λ_i nonzero parts in the ith row.
- We say that π has r *rows* if $r = \ell(\lambda)$. Similarly, π has s *columns* if $s = \ell(\lambda')$.

Example

A plane partition of shape (432) with 3 rows and 4 columns:

3	2	1	1
2	2	1	
1	1		

Shape

Definition

Let $\pi = (\pi_{ij})_{i,j\geq 1}$ be a plane partition.

- A *part* is a positive entry $\pi_{ij} > 0$.
- The *shape* of π is the ordinary partition λ for which π has λ_i nonzero parts in the ith row.
- We say that π has r *rows* if $r = \ell(\lambda)$. Similarly, π has s *columns* if $s = \ell(\lambda')$.

Example

A plane partition of shape (432) with 3 rows and 4 columns:

3	2	1	1
2	2	1	
1	1		

Shape

Definition

Let $\pi = (\pi_{ij})_{i,j\geq 1}$ be a plane partition.

- A *part* is a positive entry $\pi_{ij} > 0$.
- The *shape* of π is the ordinary partition λ for which π has λ_i nonzero parts in the ith row.
- We say that π has r *rows* if $r = \ell(\lambda)$. Similarly, π has s *columns* if $s = \ell(\lambda')$.

Example

A plane partition of shape (432) with 3 rows and 4 columns:

3	2	1	1
2	2	1	
1	1		

Example of plane partitions

Example

- Plane partitions of 0: $\emptyset$
- Plane partitions of 1: $\begin{array}{|c|}\hline 1 \\ \hline\end{array}$
- Plane partitions of 2:

$$\begin{array}{|c|}\hline 2 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline 1 \\ \hline\end{array}$$

- Plane partitions of 3:

$$\begin{array}{|c|}\hline 3 \\ \hline\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline\end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline 1 \\ \hline 1 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline\end{array} \qquad \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array} \qquad \begin{array}{|c|c}\hline 1 & \multicolumn{1}{c|}{1} \\ \hline 1 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}$$

Example of plane partitions

Example

- Plane partitions of 0: $\emptyset$
- Plane partitions of 1: $\begin{array}{|c|}\hline 1\\\hline\end{array}$
- Plane partitions of 2:

$$\begin{array}{|c|}\hline 2\\\hline\end{array}\qquad\begin{array}{|c|c|}\hline 1&1\\\hline\end{array}\qquad\begin{array}{|c|}\hline 1\\\hline 1\\\hline\end{array}$$

- Plane partitions of 3:

$$\begin{array}{|c|}\hline 3\\\hline\end{array}\qquad\begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array}\qquad\begin{array}{|c|}\hline 1\\\hline 1\\\hline 1\\\hline\end{array}\qquad\begin{array}{|c|c|}\hline 2&1\\\hline\end{array}\qquad\begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}\qquad\begin{array}{|c|c|}\hline 1&1\\\hline 1&\\\cline{1-1}\end{array}$$

Example of plane partitions

Example

- Plane partitions of 0: $\emptyset$
- Plane partitions of 1:

1

- Plane partitions of 2:

2

1	1

1
1

- Plane partitions of 3:

3

1	1	1

1
1
1

2	1

2
1

1	1
1	

Example of plane partitions

Example

- Plane partitions of 0: $\emptyset$
- Plane partitions of 1: $\boxed{1}$
- Plane partitions of 2:

$$\begin{array}{|c|}\hline 2 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline 1 \\ \hline\end{array}$$

- Plane partitions of 3:

$$\begin{array}{|c|}\hline 3 \\ \hline\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline\end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline 1 \\ \hline 1 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline\end{array} \qquad \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

Ferrers graph

Definition

The *Ferrers graph* $D(\pi)$ of π is the subset of $\mathbb{P}^3$ defined by

$$D(\pi) = \left\{(i,j,k) : k \leq \pi_{ij}\right\}$$

Example

Ferrers graph

3	2	1	1
2	2	1	
1	1		

$\longleftrightarrow$

Ferrers graph

Definition

The *Ferrers graph* $D(\pi)$ of π is the subset of $\mathbb{P}^3$ defined by

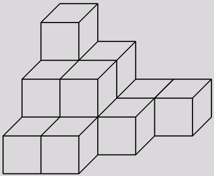
$$D(\pi) = \left\{(i,j,k) : k \leq \pi_{ij}\right\}$$

Example

Ferrers graph

3	2	1	1
2	2	1	
1	1		

$\longleftrightarrow$



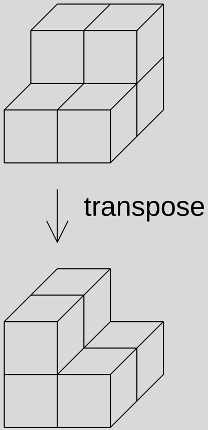
Symmetries of plane paritions

Definition

If $\pi = (\pi_{ij})$ is a plane partition, then the *transpose* π^* of π is defined by $\pi^* = (\pi_{ji})$.

- π is *symmetric* if $\pi = \pi^*$.
- π is *cyclically symmetric* if whenever $(i, j, k) \in \pi$ then $(j, k, i) \in \pi$.
- π is called *totally symmetric* if it is cyclically symmetric and symmetric.

Example



Symmetries of plane paritions

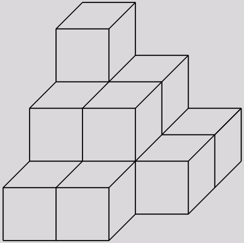
Definition

If $\pi = (\pi_{ij})$ is a plane partition, then the *transpose* π^* of π is defined by $\pi^* = (\pi_{ji})$.

- π is *symmetric* if $\pi = \pi^*$.
- π is *cyclically symmetric* if whenever $(i, j, k) \in \pi$ then $(j, k, i) \in \pi$.
- π is called *totally symmetric* if it is cyclically symmetric and symmetric.

Example

A symmetric PP



Symmetries of plane paritions

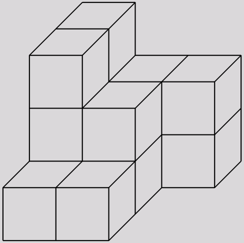
Definition

If $\pi = (\pi_{ij})$ is a plane partition, then the *transpose* π^* of π is defined by $\pi^* = (\pi_{ji})$.

- π is *symmetric* if $\pi = \pi^*$.
- π is *cyclically symmetric* if whenever $(i, j, k) \in \pi$ then $(j, k, i) \in \pi$.
- π is called *totally symmetric* if it is cyclically symmetric and symmetric.

Example

A cyclicaly symmetric PP



Symmetries of plane paritions

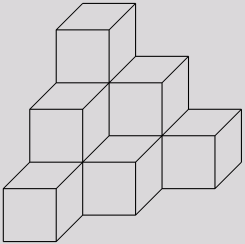
Definition

If $\pi = (\pi_{ij})$ is a plane partition, then the *transpose* π^* of π is defined by $\pi^* = (\pi_{ji})$.

- π is *symmetric* if $\pi = \pi^*$.
- π is *cyclically symmetric* if whenever $(i, j, k) \in \pi$ then $(j, k, i) \in \pi$.
- π is called *totally symmetric* if it is cyclically symmetric and symmetric.

Example

A totally symmetric PP



Complement

Definition

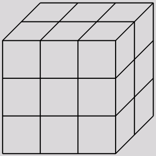
Let $\pi = (\pi_{ij})$ be a plane partition contained in the box $B(r, s, t) = [r] \times [s] \times [t]$.

Define the *complement* π^c of π by

$\pi^c = \{ (r+1-i, s+1-j, t+1-k) : (i, j, k) \notin \pi \}$.

- π is said to be *(r, s, t)-self-complementary* if $\pi = \pi^c$. i.e. $(i, j, k) \in \pi \Leftrightarrow (r+1-i, s+1-j, t+1-k) \notin \pi$.

Example



$B(2, 3, 3)$

Complement

Definition

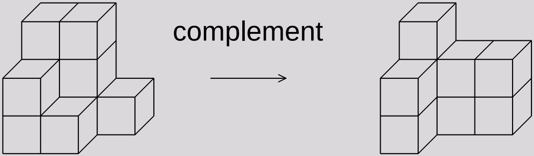
Let $\pi = (\pi_{ij})$ be a plane partition contained in the box $B(r,s,t) = [r] \times [s] \times [t]$.

Define the *complement* π^c of π by

$$\pi^c = \{ (r+1-i, s+1-j, t+1-k) \ : \ (i,j,k) \notin \pi \}.$$

- π is said to be *(r,s,t)-self-complementary* if $\pi = \pi^c$. i.e. $(i,j,k) \in \pi \Leftrightarrow (r+1-i, s+1-j, t+1-k) \notin \pi$.

Example



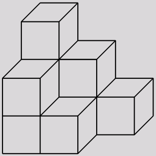
Complement

Definition

Let $\pi = (\pi_{ij})$ be a plane partition contained in the box $B(r,s,t) = [r] \times [s] \times [t]$.
Define the *complement* π^c of π by
$\pi^c = \{(r+1-i, s+1-j, t+1-k) : (i,j,k) \notin \pi\}$.

- π is said to be (r,s,t)*-self-complementary* if $\pi = \pi^c$. i.e. $(i,j,k) \in \pi \Leftrightarrow (r+1-i, s+1-j, t+1-k) \notin \pi$.

Example



A $(2,3,3)$-self-complementary PP

Symmetry classes of plane partitions

Symmetry classes (Stanley)

The transformation ^{c} and the group S_3 generate a group T of order 12. The group T has ten conjugacy classes of subgroups, giving rise to ten enumeration problems.

Table

1	[illegible]	[illegible]
2	$B(r,r,t)$	*Symmetric*
[illegible]	[illegible]	*Cyclically symmetric*
4	$B(r,r,r)$	*Totally symmetric*
[illegible]	[illegible]	*Self-complementary*
6	$B(r,r,t)$	*Complement = transpose*
[illegible]	[illegible]	*Symmetric and self-complementary*
8	$B(r,r,r)$	*Cyclically symmetric and complement = transpose*
[illegible]	[illegible]	*Cyclically symmetric and self-complementary*
10	$B(r,r,r)$	*Totally symmetric and self-complementary*

Symmetry classes of plane partitions

Symmetry classes (Stanley)

The transformation ^{c} and the group S_3 generate a group T of order 12. The group T has ten conjugacy classes of subgroups, giving rise to ten enumeration problems.

Table

1	[illegible]	[illegible]
2	$B(r,r,t)$	*Symmetric*
[illegible]	[illegible]	*Cyclically symmetric*
4	$B(r,r,r)$	*Totally symmetric*
[illegible]	[illegible]	*Self-complementary*
6	$B(r,r,t)$	*Complement = transpose*
[illegible]	[illegible]	*Symmetric and self-complementary*
8	$B(r,r,r)$	*Cyclically symmetric and complement = transpose*
[illegible]	[illegible]	*Cyclically symmetric and self-complementary*
10	$B(r,r,r)$	*Totally symmetric and self-complementary*

Symmetry classes of plane partitions

Symmetry classes (Stanley)

The transformation ^{c} and the group S_3 generate a group T of order 12. The group T has ten conjugacy classes of subgroups, giving rise to ten enumeration problems.

Table (R. P. Stanley, "Symmetries of Plane Partitions", *J. Combin. Theory Ser.* **A 43**, 103-113 (1986))

1	$B(r,s,t)$	Any
2	$B(r,r,t)$	*Symmetric*
3	$B(r,r,r)$	*Cyclically symmetric*
4	$B(r,r,r)$	*Totally symmetric*
5	$B(r,s,t)$	*Self-complementary*
6	$B(r,r,t)$	*Complement = transpose*
7	$B(r,r,t)$	*Symmetric and self-complementary*
8	$B(r,r,r)$	*Cyclically symmetric and complement = transpose*
9	$B(r,r,r)$	*Cyclically symmetric and self-complementary*
10	$B(r,r,r)$	*Totally symmetric and self-complementary*

Totally symmetric self-complementary plane partitions

Definition

A plane partition is said to be *totally symmetric self-complementary plane parition of size* $2n$ if it is totally symmetric and $(2n, 2n, 2n)$-self-complementary.

We denote the set of all self-complementary totally symmetric plane partitions of size $2n$ by $\mathscr{S}_n$.

Example

$\mathscr{S}_1$ consists of the single partition

Totally symmetric self-complementary plane partitions

Definition

A plane partition is said to be *totally symmetric self-complementary plane parition of size 2n* if it is totally symmetric and $(2n, 2n, 2n)$-self-complementary.
We denote the set of all self-complementary totally symmetric plane partitions of size $2n$ by $\mathscr{S}_n$.

Example

$\mathscr{S}_1$ consists of the single partition

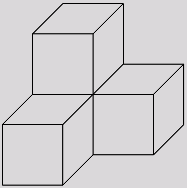
Totally symmetric self-complementary plane partitions

Definition

A plane partition is said to be *totally symmetric self-complementary plane parition of size 2n* if it is totally symmetric and $(2n, 2n, 2n)$-self-complementary.
We denote the set of all self-complementary totally symmetric plane partitions of size $2n$ by $\mathscr{S}_n$.

Example

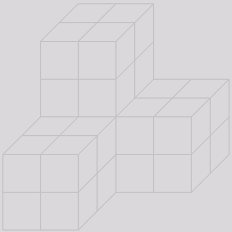
$\mathscr{S}_1$ consists of the single partition

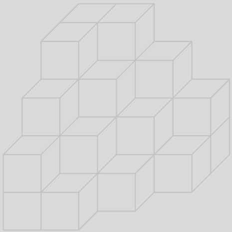


TSSCPPs of size 4

Example

$\mathscr{S}_2$ consists of the following two partitions:

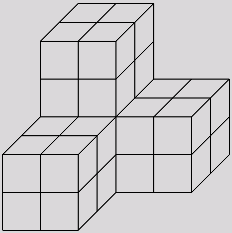


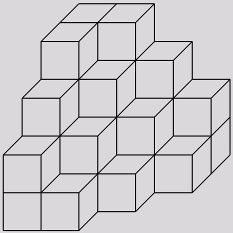


TSSCPPs of size 4

Example

$\mathscr{S}_2$ consists of the following two partitions:

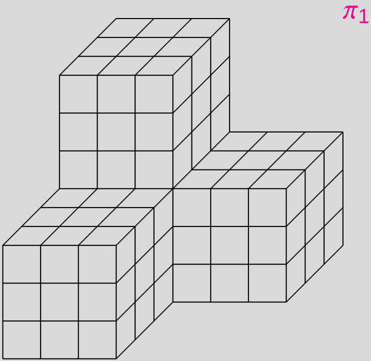


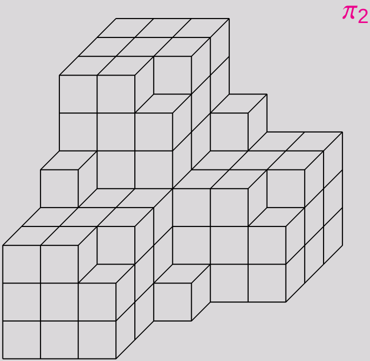


TSSCPPs of size 6

Example

$\mathscr{S}_3$ consists of the following seven partitions:

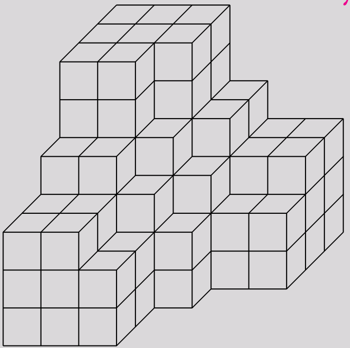




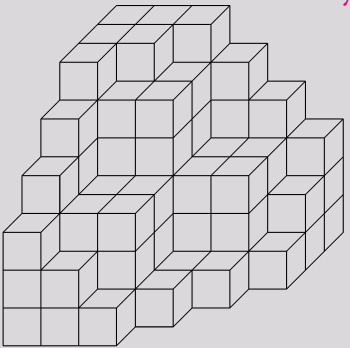
TSSCPPs of size 6

Example

$\mathscr{S}_3$ consists of the following seven partitions:



π_3

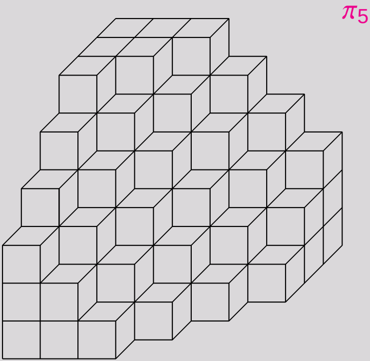


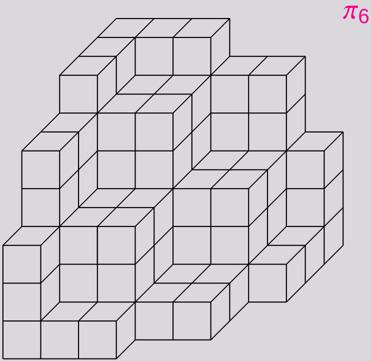
π_4

TSSCPPs of size 6

Example

$\mathscr{S}_3$ consists of the following seven partitions:

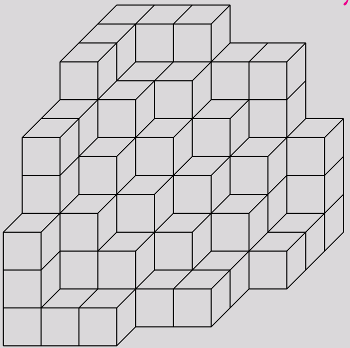




TSSCPPs of size 6

Example

$\mathscr{S}_3$ consists of the following seven partitions:



π_7

Triangular shifted plane partitions

Definition (Mills, Robbins and Rumsey)

Let $\mathscr{B}_n$ denote the set of shifted plane partitions $b = (b_{ij})_{1 \le i \le j}$ subject to the constraints that

(B1) the shifted shape of b is $(n-1, n-2, \ldots, 1)$;

(B2) $n - i \le b_{ij} \le n$ for $1 \le i \le j \le n-1$.

We call an element of $\mathscr{B}_n$ a *triangular shifted plane partition*.

Triangular shifted plane partitions

Definition (Mills, Robbins and Rumsey)

Let $\mathscr{B}_n$ denote the set of shifted plane partitions $b = (b_{ij})_{1 \leq i \leq j}$ subject to the constraints that

(B1) the shifted shape of b is $(n-1, n-2, \ldots, 1)$;

(B2) $n - i \leq b_{ij} \leq n$ for $1 \leq i \leq j \leq n-1$.

We call an element of $\mathscr{B}_n$ a *triangular shifted plane partition*.

Triangular shifted plane partitions

Definition (Mills, Robbins and Rumsey)

Let $\mathscr{B}_n$ denote the set of shifted plane partitions $b = (b_{ij})_{1 \le i \le j}$ subject to the constraints that

(B1) the shifted shape of b is $(n-1, n-2, \dots, 1)$;

(B2) $n - i \le b_{ij} \le n$ for $1 \le i \le j \le n-1$.

We call an element of $\mathscr{B}_n$ a *triangular shifted plane partition*.

Example

Triangular shifted plane partitions

Definition (Mills, Robbins and Rumsey)

Let $\mathscr{B}_n$ denote the set of shifted plane partitions $b = (b_{ij})_{1 \le i \le j}$ subject to the constraints that

(B1) the shifted shape of b is $(n-1, n-2, \dots, 1)$;

(B2) $n - i \le b_{ij} \le n$ for $1 \le i \le j \le n-1$.

We call an element of $\mathscr{B}_n$ a *triangular shifted plane partition*.

Triangular shifted plane partitions

Definition (Mills, Robbins and Rumsey)

Let $\mathscr{B}_n$ denote the set of shifted plane partitions $b = (b_{ij})_{1\le i\le j}$ subject to the constraints that

(B1) the shifted shape of b is $(n-1, n-2, \ldots, 1)$;

(B2) $n - i \le b_{ij} \le n$ for $1 \le i \le j \le n-1$.

We call an element of $\mathscr{B}_n$ a *triangular shifted plane partition*.

Example

$\mathscr{B}_1$ consists of the single PP $\emptyset$.

Triangular shifted plane partitions

Definition (Mills, Robbins and Rumsey)

Let $\mathscr{B}_n$ denote the set of shifted plane partitions $b = (b_{ij})_{1 \le i \le j}$ subject to the constraints that

(B1) the shifted shape of b is $(n-1, n-2, \ldots, 1)$;

(B2) $n - i \le b_{ij} \le n$ for $1 \le i \le j \le n-1$.

We call an element of $\mathscr{B}_n$ a *triangular shifted plane partition*.

Example

$\mathscr{B}_2$ consists of the following 2 PPs:

$\boxed{2} \qquad \boxed{1}$

Triangular shifted plane partitions

Definition (Mills, Robbins and Rumsey)

Let $\mathscr{B}_n$ denote the set of shifted plane partitions $b = (b_{ij})_{1 \le i \le j}$ subject to the constraints that

(B1) the shifted shape of b is $(n-1, n-2, \dots, 1)$;

(B2) $n - i \le b_{ij} \le n$ for $1 \le i \le j \le n-1$.

We call an element of $\mathscr{B}_n$ a *triangular shifted plane partition*.

Example

$\mathscr{B}_3$ consists of the followng 7 PPs

$$\begin{array}{|c|c|}\hline 3 & 3 \\ \hline \multicolumn{1}{c|}{} & 3 \\ \cline{2-2}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 3 \\ \hline \multicolumn{1}{c|}{} & 2 \\ \cline{2-2}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 3 \\ \hline \multicolumn{1}{c|}{} & 1 \\ \cline{2-2}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 2 \\ \hline \multicolumn{1}{c|}{} & 2 \\ \cline{2-2}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 2 \\ \hline \multicolumn{1}{c|}{} & 1 \\ \cline{2-2}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline \multicolumn{1}{c|}{} & 2 \\ \cline{2-2}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline \multicolumn{1}{c|}{} & 1 \\ \cline{2-2}\end{array}$$

A bijection

Theorem (Mills, Robbins and Rumsey)

Let n be a positive integer.
Then there is a bijection from $\mathscr{S}_n$ to $\mathscr{B}_n$.

Example

A bijection

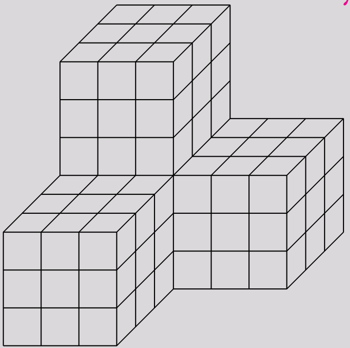
Theorem (Mills, Robbins and Rumsey)

Let n be a positive integer.
Then there is a bijection from $\mathscr{S}_n$ to $\mathscr{B}_n$.

Example

π_1

$n = 3$



$\longrightarrow$

3	3
	3

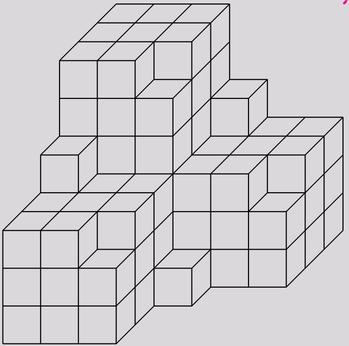
A bijection

Theorem (Mills, Robbins and Rumsey)

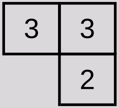
Let n be a positive integer.
Then there is a bijection from $\mathscr{S}_n$ to $\mathscr{B}_n$.

Example

π_2 $n = 3$



$\longrightarrow$



A bijection

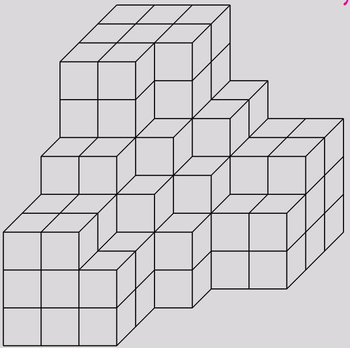
Theorem (Mills, Robbins and Rumsey)

Let n be a positive integer.
Then there is a bijection from $\mathscr{S}_n$ to $\mathscr{B}_n$.

Example

π_3

$n = 3$



$\longrightarrow$

3	3
	1

A bijection

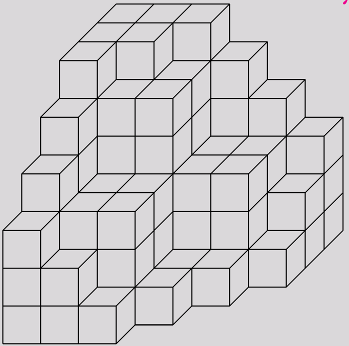
Theorem (Mills, Robbins and Rumsey)

Let n be a positive integer.
Then there is a bijection from $\mathscr{S}_n$ to $\mathscr{B}_n$.

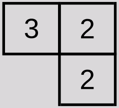
Example

$n = 3$

π_4



$\longrightarrow$



A bijection

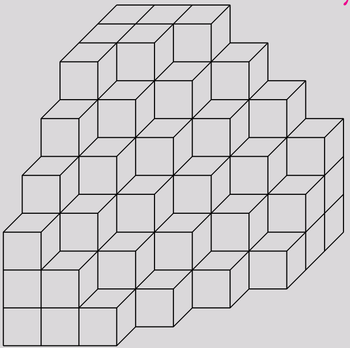
Theorem (Mills, Robbins and Rumsey)

Let n be a positive integer.
Then there is a bijection from $\mathscr{S}_n$ to $\mathscr{B}_n$.

Example

π_5

$n = 3$



$\longrightarrow$

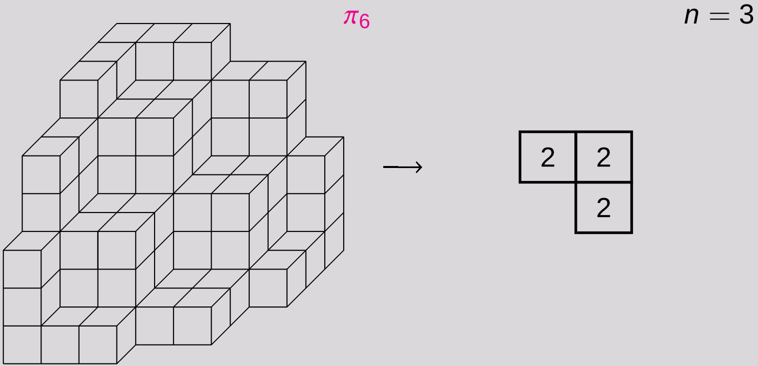
3	2
	1

A bijection

Theorem (Mills, Robbins and Rumsey)

Let n be a positive integer.
Then there is a bijection from $\mathscr{S}_n$ to $\mathscr{B}_n$.

Example

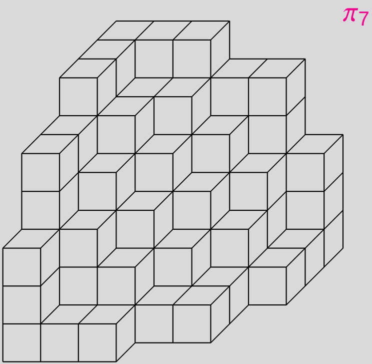


A bijection

Theorem (Mills, Robbins and Rumsey)

Let n be a positive integer.
Then there is a bijection from $\mathscr{S}_n$ to $\mathscr{B}_n$.

Example



$n = 3$

$\longrightarrow$

2	2
	1

Statistics

Definition (Mills, Robbins and Rumsey)

Let $b = (b_{ij})_{1 \le i \le j \le n-1}$ be in $\mathscr{B}_n$ and $k = 1, \dots, n$,

Let

$$U_k(b) = \sum_{t=1}^{n-k} (b_{t,t+k-1} - b_{t,t+k}) + \sum_{t=n-k+1}^{n-1} \chi\{b_{t,n-1} > n-t\}.$$

Here We set $b_{tn} = n - t$ for all $t = l, ..., n-1$ by convention, and $\chi\{\dots\}$ has value 1 when the statement "…" is true and 0 otherwise.

Statistics

Definition (Mills, Robbins and Rumsey)

Let $b = (b_{ij})_{1\leq i\leq j\leq n-1}$ be in $\mathscr{B}_n$ and $k = 1, \dots, n$,

Let

$$U_k(b) = \sum_{t=1}^{n-k}(b_{t,t+k-1} - b_{t,t+k}) + \sum_{t=n-k+1}^{n-1} \chi\{b_{t,n-1} > n-t\}.$$

Here We set $b_{tn} = n - t$ for all $t = l, \dots, n-1$ by convention, and $\chi\{\dots\}$ has value 1 when the statement "$\dots$" is true and 0 otherwise.

Statistics

Definition (Mills, Robbins and Rumsey)

Let $b = (b_{ij})_{1 \le i \le j \le n-1}$ be in $\mathscr{B}_n$ and $k = 1, \dots, n$,
Let

$$U_k(b) = \sum_{t=1}^{n-k} (b_{t,t+k-1} - b_{t,t+k}) + \sum_{t=n-k+1}^{n-1} \chi\{b_{t,n-1} > n-t\}.$$

Here We set $b_{tn} = n - t$ for all $t = l, ..., n-1$ by convention,
and $\chi\{\dots\}$ has value 1 when the statement "…" is true and 0 otherwise.

Statistics

Definition (Mills, Robbins and Rumsey)

Let $b = (b_{ij})_{1 \le i \le j \le n-1}$ be in $\mathscr{B}_n$ and $k = 1, \dots, n$,
Let

$$U_k(b) = \sum_{t=1}^{n-k} (b_{t,t+k-1} - b_{t,t+k}) + \sum_{t=n-k+1}^{n-1} \chi\{b_{t,n-1} > n-t\}.$$

Here We set $b_{tn} = n - t$ for all $t = l, ..., n-1$ by convention,
and $\chi\{\dots\}$ has value 1 when the statement "..." is true
and 0 otherwise.

Statistics

$$U_k(b) = \sum_{t=1}^{n-k} (b_{t,t+k-1} - b_{t,t+k}) + \sum_{t=n-k+1}^{n-1} \chi\{b_{t,n-1} > n-t\}.$$

Example

$n = 7, \quad k = 1, \quad U_1(b) = 3$

7	7	7	7	7	7	6
	6	6	6	5	5	5
		5	4	4	4	4
			4	4	4	3
				3	2	2
					2	1

Statistics

$$U_k(b) = \sum_{t=1}^{n-k} (b_{t,t+k-1} - b_{t,t+k}) + \sum_{t=n-k+1}^{n-1} \chi\{b_{t,n-1} > n - t\}.$$

Example

$n = 7, \quad k = 1, \quad U_1(b) = 3$

7	7	7	7	7	7	6
	6	6	6	5	5	5
		5	4	4	4	4
			4	4	4	3
				3	2	2
					2	1

Statistics

$$U_k(b) = \sum_{t=1}^{n-k} (b_{t,t+k-1} - b_{t,t+k}) + \sum_{t=n-k+1}^{n-1} \chi\{b_{t,n-1} > n - t\}.$$

Example

$n = 7, \quad k = 2, \quad U_2(b) = 1$

7	7	7	7	7	7	6
	6	6	6	5	5	5
		5	4	4	4	4
			4	4	4	3
				3	2	2
					2	1

Statistics

$$U_k(b) = \sum_{t=1}^{n-k} (b_{t,t+k-1} - b_{t,t+k}) + \sum_{t=n-k+1}^{n-1} \chi\{b_{t,n-1} > n-t\}.$$

Example

$n = 7, \quad k = 3, \quad U_3(b) = 3$

7	7	7	7	7	7	6
	6	6	6	5	5	5
		5	4	4	4	4
			4	4	4	3
				3	2	2
					2	1

Statistics

$$U_k(b) = \sum_{t=1}^{n-k} (b_{t,t+k-1} - b_{t,t+k}) + \sum_{t=n-k+1}^{n-1} \chi\{b_{t,n-1} > n-t\}.$$

Example

$n = 7, \quad k = 4, \quad U_4(b) = 2$

7	7	7	7	7	7	6
	6	6	6	5	5	5
		5	4	4	4	4
			4	4	4	3
				3	2	2
					2	1

Statistics

$$U_k(b) = \sum_{t=1}^{n-k} (b_{t,t+k-1} - b_{t,t+k}) + \sum_{t=n-k+1}^{n-1} \chi\{b_{t,n-1} > n-t\}.$$

Example

$n = 7, \quad k = 5, \quad U_5(b) = 2$

7	7	7	7	7	7	6
	6	6	6	5	5	5
		5	4	4	4	4
			4	4	4	3
				3	2	2
					2	1

Statistics

$$U_k(b) = \sum_{t=1}^{n-k} (b_{t,t+k-1} - b_{t,t+k}) + \sum_{t=n-k+1}^{n-1} \chi\{b_{t,n-1} > n-t\}.$$

Example

$n = 7, \quad k = 6, \quad U_6(b) = 3$

7	7	7	7	7	7	6
	6	6	6	5	5	5
		5	4	4	4	4
			4	4	4	3
				3	2	2
					2	1

Statistics

$$U_k(b) = \sum_{t=1}^{n-k} (b_{t,t+k-1} - b_{t,t+k}) + \sum_{t=n-k+1}^{n-1} \chi\{b_{t,n-1} > n - t\}.$$

Example

$n = 7, \quad k = 7, \quad U_7(b) = 3$

7	7	7	7	7	7	6
	6	6	6	5	5	5
		5	4	4	4	4
			4	4	4	3
				3	2	2
					2	1

The refined TSSCPP conjecture

Conjecture (Conjecture 2 of Mills, Robbins and Rumsey, "Self-complementary totally symmetric plane partitions", *J. Combin. Theory Ser.* **A 42**, (1986).)

Let $0 \leq r \leq n-1$ and $1 \leq k \leq n$. Then the number of elements b of $\mathscr{B}_n$ such that $U_k(b) = r$ is the same as the number of n by n alternating sign matrices $a = (a_{ij})$ such that $a_{1,r+1} = 1$.

The refined TSSCPP conjecture

Conjecture (Conjecture 2 of Mills, Robbins and Rumsey, “Self-complementary totally symmetric plane partitions”, *J. Combin. Theory Ser.* **A 42**, (1986).)

Let $0 \leq r \leq n-1$ and $1 \leq k \leq n$. Then the number of elements b of $\mathscr{B}_n$ such that $U_k(b) = r$ is the same as the number of n by n alternating sign matrices $a = (a_{ij})$ such that $a_{1,r+1} = 1$.

Example

$n = 3$, $b \in \mathscr{B}_3$

b	3 3 / 3	3 3 / 2	3 3 / 1	3 2 / 2	3 2 / 1	2 2 / 2	2 2 / 1
$U_1(b)$	2	1	0	2	1	1	0
$U_2(b)$	2	2	1	1	0	1	0
$U_3(b)$	2	2	1	1	0	1	0

The refined TSSCPP conjecture

Conjecture (Conjecture 2 of Mills, Robbins and Rumsey, “Self-complementary totally symmetric plane partitions”, *J. Combin. Theory Ser.* **A 42**, (1986).)

Let $0 \leq r \leq n-1$ and $1 \leq k \leq n$. Then the number of elements b of $\mathscr{B}_n$ such that $U_k(b) = r$ is the same as the number of n by n alternating sign matrices $a = (a_{ij})$ such that $a_{1,r+1} = 1$.

Example

For $k = 1, 2, 3$, we have

$$\sum_{b \in \mathscr{B}_3} t^{U_k(b)} = 2 + 3t + 2t^2.$$

The refined enumeration of ASM

Zeilberger (1996), Kuperberg (1996)

The number of n by n alternating sign matrices $a = (a_{ij})$ such that $a_{1,r+1} = 1$ is equal to

$$\frac{\binom{n+r-2}{n-1}\binom{2n-r-1}{n-1}}{\binom{2n-2}{n-1}} A_{n-1} = \frac{\binom{n+r-2}{n-1}\binom{2n-1-r}{n-1}}{\binom{3n-2}{n-1}} A_n.$$

Here A_n is

$$\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}.$$

The doubly refined TSSCPP conjecture

Conjecture (Conjecture 3 of Mills, Robbins and Rumsey, “Self-complementary totally symmetric plane partitions”, *J. Combin. Theory Ser.* **A 42**, (1986).)

Let $n \geq 2$ and r, s with $0 \leq r, s \leq n-1$ be integers. Then the number of partitions in $\mathscr{B}_n$ with $U_1(b) = r$ and $U_2(b) = s$ is the same as the number of n by n alternating sign matrices $a = (a_{ij})$ with

$$a_{1,r+l} = a_{n,n-s} = 1.$$

The doubly refined TSSCPP conjecture

Conjecture (Conjecture 3 of Mills, Robbins and Rumsey, “Self-complementary totally symmetric plane partitions”, *J. Combin. Theory Ser.* **A 42**, (1986).)

Let $n \geq 2$ and r, s with $0 \leq r, s \leq n-1$ be integers. Then the number of partitions in $\mathscr{B}_n$ with $U_1(b) = r$ and $U_2(b) = s$ is the same as the number of n by n alternating sign matrices $a = (a_{ij})$ with

$$a_{1,r+l} = a_{n,n-s} = 1.$$

Example

$b \in \mathscr{B}_3$	$\begin{array}{cc} 3 & 3 \\ & 3 \end{array}$	$\begin{array}{cc} 3 & 3 \\ & 2 \end{array}$	$\begin{array}{cc} 3 & 3 \\ & 1 \end{array}$	$\begin{array}{cc} 3 & 2 \\ & 2 \end{array}$	$\begin{array}{cc} 3 & 2 \\ & 1 \end{array}$	$\begin{array}{cc} 2 & 2 \\ & 2 \end{array}$	$\begin{array}{cc} 2 & 2 \\ & 1 \end{array}$
$U_1(b)$	2	1	0	2	1	1	0
$U_2(b)$	2	2	1	1	0	1	0
$U_3(b)$	2	2	1	1	0	1	0

The doubly refined TSSCPP conjecture

Conjecture (Conjecture 3 of Mills, Robbins and Rumsey, "Self-complementary totally symmetric plane partitions", *J. Combin. Theory Ser.* **A 42**, (1986).)

Let $n \geq 2$ and r, s with $0 \leq r, s \leq n-1$ be integers. Then the number of partitions in $\mathscr{B}_n$ with $U_1(b) = r$ and $U_2(b) = s$ is the same as the number of n by n alternating sign matrices $a = (a_{ij})$ with

$$a_{1,r+l} = a_{n,n-s} = 1.$$

Example

Thus we have

$$\sum_{b \in \mathscr{B}_3} t^{U_1(b)} u^{U_2(b)} = 1 + t + u + tu + t^2u + tu^2 + t^2u^2.$$

The doubly refined enumeration of ASM

Di Francesco and Zinn-Justin (2004)

The doubly-refined ASM number generating function is given by

$$A_n(t,u) = \frac{\{\omega^2(\omega+t)(\omega+u)\}^{n-1}}{3^{n(n-1)/2}} \times s^{(2n)}_{\delta(n-1,n-1)}\left(\frac{1+\omega t}{\omega+t}, \frac{1+\omega u}{\omega+u}, 1, \dots, 1\right)$$

Here $s^{(n)}_{\lambda}(x_1,\dots,x_n)$ stands for the Schur function in the n variables $x_1, \dots, x_n$, corresponding to the partition λ, and $\delta(n-1,n-1) = (n-1,n-1,n-2,n-2,\dots,1,1)$ and $\omega = e^{2i\pi/3}$. (The coefficient of $t^{j-1}s^{k-1}$ is the number of $n \times n$ ASM with a 1 in position r on the top row (counted from left to right) and k on the bottom row (counted from right to left).)

TSSCPP and monotone triangles

Conjecture (Conjecture 7 of Mills, Robbins and Rumsey, "Self-complementary totally symmetric plane partitions", *J. Combin. Theory Ser.* **A 42**, (1986).)

For $n \geq 2$ and $k = 0, \ldots, n-1$, let $\mathscr{B}_{nk}$ be the subset of those $b = (b_{ij})_{1 \leq i \leq j}$ in $\mathscr{B}_n$ such that all b_{ij} in the first $n-1-k$ columns are equal to their maximum values n. Then the cardinality of $\mathscr{B}_{nk}$ is equal to the cardinality of the set of the monotone triangles with all entries m_{ij} in the first $n-1-k$ columns equal to their minimum values $j-i+1$.

TSSCPP and monotone triangles

Conjecture (Conjecture 7 of Mills, Robbins and Rumsey, “Self-complementary totally symmetric plane partitions”, *J. Combin. Theory Ser.* **A 42**, (1986).)

For $n \geq 2$ and $k = 0, \dots, n-1$, let $\mathscr{B}_{nk}$ be the subset of those $b = (b_{ij})_{1 \leq i \leq j}$ in $\mathscr{B}_n$ such that all b_{ij} in the first $n-1-k$ columns are equal to their maximum values n.

Example

$n = 3$, $k = 0$: The first 2 columns are equal to the maximum values 3.

$b \in \mathscr{B}_{3,0}$	$\begin{array}{cc} 3 & 3 \\ & 3 \end{array}$
$U_1(b)$	2
$U_2(b)$	2
$U_3(b)$	2

TSSCPP and monotone triangles

Conjecture (Conjecture 7 of Mills, Robbins and Rumsey, "Self-complementary totally symmetric plane partitions", *J. Combin. Theory Ser.* **A 42**, (1986).)

For $n \geq 2$ and $k = 0, \ldots, n-1$, let $\mathscr{B}_{nk}$ be the subset of those $b = (b_{ij})_{1 \leq i \leq j}$ in $\mathscr{B}_n$ such that all b_{ij} in the first $n-1-k$ columns are equal to their maximum values n.

Example

For $k = 1, 2, 3$, we have

$$\sum_{b \in \mathscr{B}_{3,0}} t^{U_k(b)} = t^2.$$

TSSCPP and monotone triangles

Conjecture (Conjecture 7 of Mills, Robbins and Rumsey, “Self-complementary totally symmetric plane partitions”, *J. Combin. Theory Ser.* **A 42**, (1986).)

For $n \geq 2$ and $k = 0, \ldots, n-1$, let $\mathscr{B}_{nk}$ be the subset of those $b = (b_{ij})_{1 \leq i \leq j}$ in $\mathscr{B}_n$ such that all b_{ij} in the first $n-1-k$ columns are equal to their maximum values n.

Example

$n = 3$, $k = 1$: The first column equals the maximum values 3.

$b \in \mathscr{B}_{3,1}$	$\begin{matrix} 3 & 3 \\ & 3 \end{matrix}$	$\begin{matrix} 3 & 3 \\ & 2 \end{matrix}$	$\begin{matrix} 3 & 3 \\ & 1 \end{matrix}$	$\begin{matrix} 3 & 2 \\ & 2 \end{matrix}$	$\begin{matrix} 3 & 2 \\ & 1 \end{matrix}$
$U_1(b)$	2	1	0	2	1
$U_2(b)$	2	2	1	1	0
$U_3(b)$	2	2	1	1	0

TSSCPP and monotone triangles

Conjecture (Conjecture 7 of Mills, Robbins and Rumsey, “Self-complementary totally symmetric plane partitions”, *J. Combin. Theory Ser.* **A 42**, (1986).)

For $n \geq 2$ and $k = 0, \ldots, n-1$, let $\mathscr{B}_{nk}$ be the subset of those $b = (b_{ij})_{1 \leq i \leq j}$ in $\mathscr{B}_n$ such that all b_{ij} in the first $n-1-k$ columns are equal to their maximum values n.

Example

For $k = 1, 2, 3$, we have

$$\sum_{b \in \mathscr{B}_{3,1}} t^{U_k(b)} = 1 + 2t + 2t^2.$$

TSSCPP and monotone triangles

Conjecture (Conjecture 7 of Mills, Robbins and Rumsey, "Self-complementary totally symmetric plane partitions", *J. Combin. Theory Ser.* **A 42**, (1986).)

For $n \geq 2$ and $k = 0, \ldots, n-1$, let $\mathscr{B}_{nk}$ be the subset of those $b = (b_{ij})_{1 \leq i \leq j}$ in $\mathscr{B}_n$ such that all b_{ij} in the first $n-1-k$ columns are equal to their maximum values n.

Example

$n = 3$, $k = 2$: No restriction.

$b \in \mathscr{B}_{3,2}$	$\begin{smallmatrix} 3 & 3 \\ & 3 \end{smallmatrix}$	$\begin{smallmatrix} 3 & 3 \\ & 2 \end{smallmatrix}$	$\begin{smallmatrix} 3 & 3 \\ & 1 \end{smallmatrix}$	$\begin{smallmatrix} 3 & 2 \\ & 2 \end{smallmatrix}$	$\begin{smallmatrix} 3 & 2 \\ & 1 \end{smallmatrix}$	$\begin{smallmatrix} 2 & 2 \\ & 2 \end{smallmatrix}$	$\begin{smallmatrix} 2 & 2 \\ & 1 \end{smallmatrix}$
$U_1(b)$	2	1	0	2	1	1	0
$U_2(b)$	2	2	1	1	0	1	0
$U_3(b)$	2	2	1	1	0	1	0

TSSCPP and monotone triangles

Conjecture (Conjecture 7 of Mills, Robbins and Rumsey, "Self-complementary totally symmetric plane partitions", *J. Combin. Theory Ser.* **A 42**, (1986).)

For $n \geq 2$ and $k = 0, \ldots, n-1$, let $\mathscr{B}_{nk}$ be the subset of those $b = (b_{ij})_{1 \leq i \leq j}$ in $\mathscr{B}_n$ such that all b_{ij} in the first $n-1-k$ columns are equal to their maximum values n.

Example

For $k = 1, 2, 3$, we have

$$\sum_{b \in \mathscr{B}_{3,2}} t^{U_k(b)} = 2 + 3t + 2t^2.$$

Flip

Definition (Mills, Robbins and Rumsey)

Let b be an element of $\mathscr{B}_n$.

- If b_{ij} is a part of b off the main diagonal, then by the *flip* of b_{ij} we mean the operation of replacing b_{ij} by b'_{ij} where b_{ij} and b'_{ij} are related by

$$b'_{ij} + b_{ij} = \min(b_{i-1,j}, b_{i,j-1}) + \max(b_{i,j+1}, b_{i+1,j}).$$

- Similarly, the *flip* of a part b_{ii} is the operation of replacing b_{ii} by b'_{ii} where

$$b'_{ii} + b_{ii} = b_{i-1,i} + b_{i,i+1}.$$

In the above expression we take $b_{0,j} = n$ for all j and $b_{i,n} = n - i$ for all i.

Flip

Definition (Mills, Robbins and Rumsey)

Let b be an element of $\mathscr{B}_n$.

- If b_{ij} is a part of b off the main diagonal, then by the *flip* of b_{ij} we mean the operation of replacing b_{ij} by b'_{ij} where b_{ij} and b'_{ij} are related by
$$b'_{ij} + b_{ij} = \min(b_{i-1,j}, b_{i,j-1}) + \max(b_{i,j+1}, b_{i+1,j}).$$
- Similarly, the *flip* of a part b_{ii} is the operation of replacing b_{ii} by b'_{ii} where
$$b'_{ii} + b_{ii} = b_{i-1,i} + b_{i,i+1}.$$

In the above expression we take $b_{O,j} = n$ for all j and $b_{i,n} = n - i$ for all i.

Flip

Definition (Mills, Robbins and Rumsey)

Let b be an element of $\mathscr{B}_n$.

- If b_{ij} is a part of b off the main diagonal, then by the *flip* of b_{ij} we mean the operation of replacing b_{ij} by b'_{ij} where b_{ij} and b'_{ij} are related by
$$b'_{ij} + b_{ij} = \min(b_{i-1,j}, b_{i,j-1}) + \max(b_{i,j+1}, b_{i+1,j}).$$
- Similarly, the *flip* of a part b_{ii} is the operation of replacing b_{ii} by b'_{ii} where
$$b'_{ii} + b_{ii} = b_{i-1,i} + b_{i,i+1}.$$

In the above expression we take $b_{0,j} = n$ for all j and $b_{i,n} = n - i$ for all i.

Flip

Definition (Mills, Robbins and Rumsey)

Let b be an element of $\mathscr{B}_n$.

- If b_{ij} is a part of b off the main diagonal, then by the *flip* of b_{ij} we mean the operation of replacing b_{ij} by b'_{ij} where b_{ij} and b'_{ij} are related by
  $$b'_{ij} + b_{ij} = \min(b_{i-1,j}, b_{i,j-1}) + \max(b_{i,j+1}, b_{i+1,j}).$$
- Similarly, the *flip* of a part b_{ii} is the operation of replacing b_{ii} by b'_{ii} where
  $$b'_{ii} + b_{ii} = b_{i-1,i} + b_{i,i+1}.$$

In the above expression we take $b_{O,j} = n$ for all j and $b_{i,n} = n - i$ for all i.

Flips

Example

$n = 7$, Flip on the off-diagonal part $b_{2,4} = 5$

7	7	7	7	7	7	
7	7	7	7	7	7	6
	6	6	6	5	5	5
		5	4	4	4	4
			4	4	4	3
				3	2	2
					2	1

Flips

Example

$n = 7, \quad 5 + b'_{2,4} = \min(7, 6) + \max(5, 4)$

7	7	7	7	7	7	
7	7	7	7	7	7	6
	6	6	6	5	5	5
		5	4	4	4	4
			4	4	4	3
				3	2	2
					2	1

Flips

Example

$n = 7, \quad 5 + b'_{2,4} = 6 + 5$

7	7	7	7	7	7	
7	7	7	7	7	7	6
	6	6	6	5	5	5
		5	4	4	4	4
			4	4	4	3
				3	2	2
					2	1

Flips

Example

$n = 7$, Change $b_{2,4} = 5$ to $b'_{2,4} = 6$.

7	7	7	7	7	7	
7	7	7	7	7	7	6
	6	6	6	6	5	5
		5	4	4	4	4
			4	4	4	3
				3	2	2
					2	1

Flips

Example

$n = 7$, Flip on the diagonal part $b_{2,1} = 6$

7	7	7	7	7	7	
7	7	7	7	7	7	6
	6	6	6	5	5	5
		5	4	4	4	4
			4	4	4	3
				3	2	2
					2	1

Flips

Example

$n = 7, \quad 6 + b'_{2,1} = 7 + 6$

7	7	7	7	7	7	
7	7	7	7	7	7	6
	6	6	6	5	5	5
		5	4	4	4	4
			4	4	4	3
				3	2	2
					2	1

Flips

Example

$n = 7$, Change $b_{2,1} = 6$ to $b'_{2,1} = 7$.

7	7	7	7	7	7	
7	7	7	7	7	7	6
	7	6	6	5	5	5
		5	4	4	4	4
			4	4	4	3
				3	2	2
					2	1

An involution

Definition

For each $k = l, \ldots, n-1$, we define an operation π_k from $\mathscr{B}_n$ to itself. Let b be an element of $\mathscr{B}_n$. Then $\pi_k(b)$ is the result of flipping all the $b_{i,i+k-1}$, $1 \leq i \leq n-k$.

An involution

Definition

For each $k = l, \ldots, n-1$, we define an operation π_k from $\mathscr{B}_n$ to itself. Let b be an element of $\mathscr{B}_n$. Then $\pi_k(b)$ is the result of flipping all the $b_{i,i+k-1}$, $1 \le i \le n-k$.

Example $n = 7$, $k = 1$, Apply π_1 to the following $b \in \mathscr{B}_3$.

7	7	7	7	7	7
	7	6	6	5	5
		5	4	4	4
			4	4	4
				3	2
					2

An involution

Definition

For each $k = l, \ldots, n-1$, we define an operation π_k from $\mathscr{B}_n$ to itself. Let b be an element of $\mathscr{B}_n$. Then $\pi_k(b)$ is the result of flipping all the $b_{i,i+k-1}$, $1 \le i \le n-k$.

Example $n = 7$, $k = 1$, Then we obtain the following $\pi_1(b) \in \mathscr{B}_3$.

7	7	7	7	7	7
	6	6	6	5	5
		5	4	4	4
			4	4	4
				3	2
					1

An involution

Definition

For each $k = l, \dots, n - 1$, we define an operation π_k from $\mathscr{B}_n$ to itself. Let b be an element of $\mathscr{B}_n$. Then $\pi_k(b)$ is the result of flipping all the $b_{i,i+k-1}$, $1 \le i \le n - k$.

Example $n = 7$, $k = 2$, Apply π_2 to the following $b \in \mathscr{B}_3$.

7	7	7	7	7	7
	7	6	6	5	5
		5	4	4	4
			4	4	4
				3	2
					2

An involution

Definition

For each $k = l, \ldots, n-1$, we define an operation π_k from $\mathscr{B}_n$ to itself. Let b be an element of $\mathscr{B}_n$. Then $\pi_k(b)$ is the result of flipping all the $b_{i,i+k-1}$, $1 \le i \le n-k$.

Example $n = 7$, $k = 2$, Then we obtain the following $\pi_2(b) \in \mathscr{B}_3$.

7	7	7	7	7	7
	7	7	6	5	5
		5	5	4	4
			4	4	4
				3	3
					2

An involution

Definition

For each $k = l, \ldots, n-1$, we define an operation π_k from $\mathscr{B}_n$ to itself. Let b be an element of $\mathscr{B}_n$. Then $\pi_k(b)$ is the result of flipping all the $b_{i,i+k-1}$, $1 \le i \le n-k$.

Example $n = 7$, $k = 3$, Apply π_3 to the following $b \in \mathscr{B}_3$.

7	7	7	7	7	7
	7	6	6	5	5
		5	4	4	4
			4	4	4
				3	2
					2

An involution

Definition

For each $k = l, \ldots, n-1$, we define an operation π_k from $\mathscr{B}_n$ to itself. Let b be an element of $\mathscr{B}_n$. Then $\pi_k(b)$ is the result of flipping all the $b_{i,i+k-1}$, $1 \le i \le n-k$.

Example $n = 7$, $k = 3$, Then we obtain the following $\pi_3(b) \in \mathscr{B}_3$.

7	7	7	7	7	7
	7	6	5	5	5
		5	4	4	4
			4	4	3
				3	2
					2

An involution

Definition

For each $k = l, \dots, n-1$, we define an operation π_k from $\mathscr{B}_n$ to itself. Let b be an element of $\mathscr{B}_n$. Then $\pi_k(b)$ is the result of flipping all the $b_{i,i+k-1}$, $1 \le i \le n-k$.

Example $n = 7$, $k = 4$, Apply π_4 to the following $b \in \mathscr{B}_3$.

7	7	7	7	7	7
	7	6	6	5	5
		5	4	4	4
			4	4	4
				3	2
					2

An involution

Definition

For each $k = l, \dots, n-1$, we define an operation π_k from $\mathscr{B}_n$ to itself. Let b be an element of $\mathscr{B}_n$. Then $\pi_k(b)$ is the result of flipping all the $b_{i,i+k-1}$, $1 \le i \le n-k$.

Example $n = 7$, $k = 4$, Then we obtain the following $\pi_4(b) \in \mathscr{B}_3$.

7	7	7	7	7	7
	7	6	6	6	5
		5	4	4	4
			4	4	4
				3	2
					2

An involution

Definition

For each $k = l, \ldots, n-1$, we define an operation π_k from $\mathscr{B}_n$ to itself. Let b be an element of $\mathscr{B}_n$. Then $\pi_k(b)$ is the result of flipping all the $b_{i,i+k-1}$, $1 \le i \le n-k$.

Example $n = 7$, $k = 5$, Apply π_5 to the following $b \in \mathscr{B}_3$.

7	7	7	7	7	7
	7	6	6	5	5
		5	4	4	4
			4	4	4
				3	2
					2

An involution

Definition

For each $k = l, \ldots, n-1$, we define an operation π_k from $\mathscr{B}_n$ to itself. Let b be an element of $\mathscr{B}_n$. Then $\pi_k(b)$ is the result of flipping all the $b_{i,i+k-1}$, $1 \le i \le n-k$.

Example $n = 7$, $k = 5$, Then we obtain the following $\pi_5(b) \in \mathscr{B}_3$.

7	7	7	7	7	7
	7	6	6	5	5
		5	4	4	4
			4	4	4
				3	2
					2

An involution

Definition

For each $k = l, \dots, n-1$, we define an operation π_k from $\mathscr{B}_n$ to itself. Let b be an element of $\mathscr{B}_n$. Then $\pi_k(b)$ is the result of flipping all the $b_{i,i+k-1}$, $1 \le i \le n-k$.

<u>Example</u> $n = 7$, $k = 6$, Apply π_6 to the following $b \in \mathscr{B}_3$.

7	7	7	7	7	7
	7	6	6	5	5
		5	4	4	4
			4	4	4
				3	2
					2

An involution

Definition

For each $k = l, \ldots, n-1$, we define an operation π_k from $\mathscr{B}_n$ to itself. Let b be an element of $\mathscr{B}_n$. Then $\pi_k(b)$ is the result of flipping all the $b_{i,i+k-1}$, $1 \le i \le n-k$.

Example $n = 7$, $k = 6$, Then we obtain the following $\pi_6(b) \in \mathscr{B}_6$.

7	7	7	7	7	6
	7	6	6	5	5
		5	4	4	4
			4	4	4
				3	2
					2

Conjecture 4

Definition

Define the involution $\rho : \mathscr{B}_n \to \mathscr{B}_n$ by

$$\rho = \pi_2 \pi_4 \pi_6 \cdots .$$

Conjecture (Conjecture 4 of Mills, Robbins and Rumsey, "Self-complementary totally symmetric plane partitions", *J. Combin. Theory Ser.* **A 42**, (1986).)

Let $n \geq 2$ and r, $0 \leq r \leq n$ be integers. Then the number of elements of $\mathscr{B}_n$ with $\rho(b) = b$ and $U_1(b) = r$ is the same as the number of n by n alternating sign matrices a invariant under the half turn in their own planes (that is $a_{ij} = a_{n+1-i,n+1-j}$ for $1 < i, j < n$) and satisfying $a_{1,r} = 1$.

Conjecture 4

Definition

Define the involution $\rho : \mathscr{B}_n \to \mathscr{B}_n$ by

$$\rho = \pi_2 \pi_4 \pi_6 \cdots .$$

Conjecture (Conjecture 4 of Mills, Robbins and Rumsey, “Self-complementary totally symmetric plane partitions”, *J. Combin. Theory Ser.* **A 42**, (1986).)

Let $n \geq 2$ and r, $0 \leq r \leq n$ be integers. Then the number of elements of $\mathscr{B}_n$ with $p(b) = b$ and $U_1(b) = r$ is the same as the number of n by n alternating sign matrices a invariant under the half turn in their own planes (that is $a_{ij} = a_{n+1-i,n+1-i}$ for $1 < i, j < n$) and satisfying $a_{1,r} = 1$.

Conjecture 6

Definition

Define the involution $\gamma : \mathscr{B}_n \to \mathscr{B}_n$ by

$$\gamma = \pi_1 \pi_3 \pi_5 \cdots .$$

Conjecture

(Conjecture 6 of Mills, Robbins and Rumsey, "Self-complementary totally symmetric plane partitions", *J. Combin. Theory Ser.* **A 42**, (1986).)

Let $n \geq 3$ an odd integer and i, $0 \leq i \leq n-1$ be an integer. Then the number of b in $\mathscr{B}_n$ with $\gamma(b) = b$ and $U_2(b) = i$ is the same as the number of n by n alternating sign matrices with $a_{i1} = 1$ and which are invariant under the vertical flip (that is $a_{ij} = a_{i,n+1-j}$ for $1 \leq i, j \leq n$).

Conjecture 6

Definition

Define the involution $\gamma : \mathscr{B}_n \to \mathscr{B}_n$ by

$$\gamma = \pi_1\pi_3\pi_5\cdots.$$

Conjecture (Conjecture 6 of Mills, Robbins and Rumsey, “Self-complementary totally symmetric plane partitions”, *J. Combin. Theory Ser.* **A 42**, (1986).)

Let $n \geq 3$ an odd integer and i, $0 \leq i \leq n-1$ be an integer. Then the number of b in $\mathscr{B}_n$ with $\gamma(b) = b$ and $U_2(b) = i$ is the same as the number of n by n alternating sign matrices with $a_{i1} = 1$ and which are invariant under the vertical flip (that is $a_{ij} = a_{i,n+1-j}$ for $1 \leq i, j \leq n$).

Restricted column-strict plane partitions

Definition

Let $\mathscr{P}_n$ denote the set of (ordinary) plane partitions $c = (c_{ij})_{1 \le i,j}$ subject to the constraints that

(C1) c is column-strict;

(C2) jth column is less than or equal to $n - j$.

We call an element of $\mathscr{P}_n$ a *restricted column-strict plane partition*.

A part c_{ij} of c is said to be *saturated* if $c_{ij} = n - j$.

Example

Restricted column-strict plane partitions

Definition

Let $\mathscr{P}_n$ denote the set of (ordinary) plane partitions $c = (c_{ij})_{1 \le i,j}$ subject to the constraints that

(C1) c is column-strict;

(C2) jth column is less than or equal to $n - j$.

We call an element of $\mathscr{P}_n$ a *restricted column-strict plane partition*.

A part c_{ij} of c is said to be *saturated* if $c_{ij} = n - j$.

Example

Restricted column-strict plane partitions

Definition

Let $\mathscr{P}_n$ denote the set of (ordinary) plane partitions $c = (c_{ij})_{1 \le i,j}$ subject to the constraints that

(C1) c is column-strict;

(C2) jth column is less than or equal to $n - j$.

We call an element of $\mathscr{P}_n$ a *restricted column-strict plane partition*.

A part c_{ij} of c is said to be *saturated* if $c_{ij} = n - j$.

Example

Restricted column-strict plane partitions

Definition

Let $\mathscr{P}_n$ denote the set of (ordinary) plane partitions $c = (c_{ij})_{1 \le i,j}$ subject to the constraints that

(C1) c is column-strict;

(C2) jth column is less than or equal to $n - j$.

We call an element of $\mathscr{P}_n$ a *restricted column-strict plane partition*.

A part c_{ij} of c is said to be *saturated* if $c_{ij} = n - j$.

Example

Restricted column-strict plane partitions

Definition

Let $\mathscr{P}_n$ denote the set of (ordinary) plane partitions $c = (c_{ij})_{1 \le i,j}$ subject to the constraints that

(C1) c is column-strict;

(C2) jth column is less than or equal to $n - j$.

We call an element of $\mathscr{P}_n$ a *restricted column-strict plane partition*.

A part c_{ij} of c is said to be *saturated* if $c_{ij} = n - j$.

Example

Restricted column-strict plane partitions

Definition

Let $\mathscr{P}_n$ denote the set of (ordinary) plane partitions $c = (c_{ij})_{1 \le i,j}$ subject to the constraints that

(C1) c is column-strict;

(C2) jth column is less than or equal to $n - j$.

We call an element of $\mathscr{P}_n$ a *restricted column-strict plane partition*.

A part c_{ij} of c is said to be *saturated* if $c_{ij} = n - j$.

Example

$\mathscr{P}_1$ consists of the single PP $\emptyset$.

Restricted column-strict plane partitions

Definition

Let $\mathscr{P}_n$ denote the set of (ordinary) plane partitions $c = (c_{ij})_{1\le i,j}$ subject to the constraints that

(C1) c is column-strict;

(C2) jth column is less than or equal to $n - j$.

We call an element of $\mathscr{P}_n$ a *restricted column-strict plane partition*. A part c_{ij} of c is said to be *saturated* if $c_{ij} = n - j$.

Example

$\mathscr{P}_2$ consists of the following 2 PPs:

$$\emptyset \qquad \boxed{1}$$

Restricted column-strict plane partitions

Definition

Let $\mathscr{P}_n$ denote the set of (ordinary) plane partitions $c = (c_{ij})_{1 \le i,j}$ subject to the constraints that

(C1) c is column-strict;

(C2) jth column is less than or equal to $n - j$.

We call an element of $\mathscr{P}_n$ a *restricted column-strict plane partition*.

A part c_{ij} of c is said to be *saturated* if $c_{ij} = n - j$.

Example

$\mathscr{P}_2$ consists of the following 2 PPs:

$$\emptyset \qquad \boxed{1}$$

Restricted column-strict plane partitions

Definition

Let $\mathscr{P}_n$ denote the set of (ordinary) plane partitions $c = (c_{ij})_{1 \le i,j}$ subject to the constraints that

(C1) c is column-strict;

(C2) jth column is less than or equal to $n - j$.

We call an element of $\mathscr{P}_n$ a *restricted column-strict plane partition*.

A part c_{ij} of c is said to be *saturated* if $c_{ij} = n - j$.

Example

$\mathscr{P}_3$ consists of the followng 7 PPs

$$\emptyset \qquad \begin{array}{|c|}\hline 1 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array} \qquad \begin{array}{|c|}\hline 2 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline\end{array} \qquad \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 & \\ \cline{1-1}\end{array}$$

Restricted column-strict plane partitions

Definition

Let $\mathscr{P}_n$ denote the set of (ordinary) plane partitions $c = (c_{ij})_{1 \le i,j}$ subject to the constraints that

(C1) c is column-strict;

(C2) jth column is less than or equal to $n - j$.

We call an element of $\mathscr{P}_n$ a *restricted column-strict plane partition*. A part c_{ij} of c is said to be *saturated* if $c_{ij} = n - j$.

Example

$\mathscr{P}_3$ consists of the followng 7 PPs

$$\emptyset \qquad \begin{array}{|c|}\hline 1 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array} \qquad \begin{array}{|c|}\hline 2 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline\end{array} \qquad \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 & \\ \cline{1-1}\end{array}$$

Another bijection

Theorem

Let n be a positive integer.
Then there is a bijection from $\mathscr{S}_n$ to $\mathscr{P}_n$.

Example

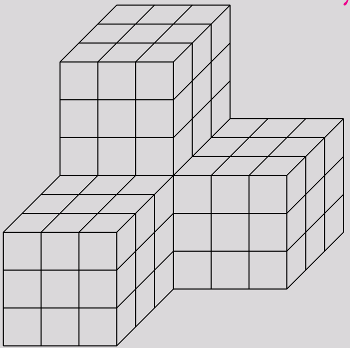
Another bijection

Theorem

Let n be a positive integer.
Then there is a bijection from $\mathscr{S}_n$ to $\mathscr{P}_n$.

Example

$n = 3$



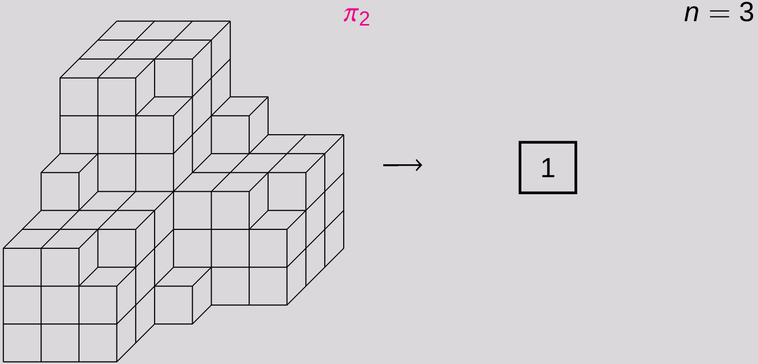
$\longrightarrow \quad \emptyset$

Another bijection

Theorem

Let n be a positive integer.
Then there is a bijection from $\mathscr{S}_n$ to $\mathscr{P}_n$.

Example



Another bijection

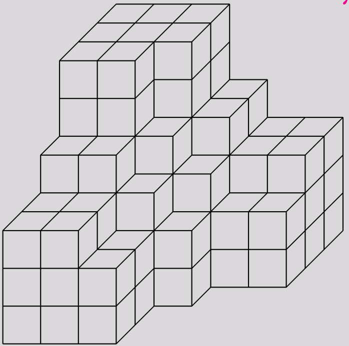
Theorem

Let n be a positive integer.
Then there is a bijection from $\mathscr{S}_n$ to $\mathscr{P}_n$.

Example

π_3

$n = 3$



$\longrightarrow$

1	1

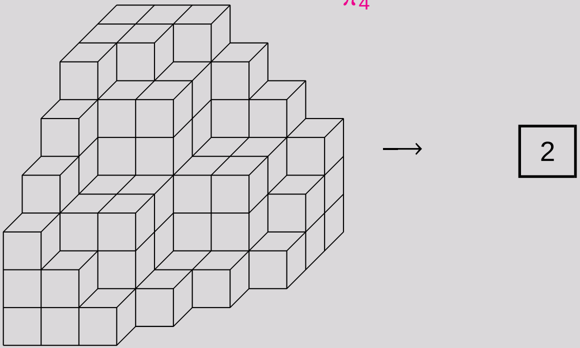
Another bijection

Theorem

Let n be a positive integer.
Then there is a bijection from $\mathscr{S}_n$ to $\mathscr{P}_n$.

Example

π_4 $\quad n = 3$



Another bijection

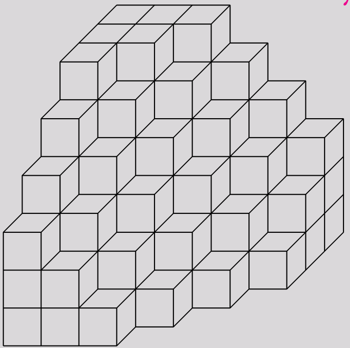
Theorem

Let n be a positive integer.
Then there is a bijection from $\mathscr{S}_n$ to $\mathscr{P}_n$.

Example

π_5

$n = 3$



$\longrightarrow$

2	1

Another bijection

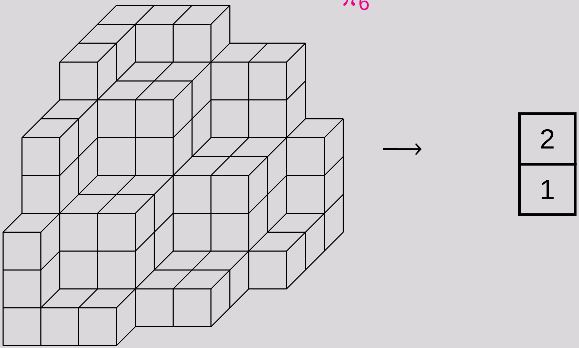
Theorem

Let n be a positive integer.
Then there is a bijection from $\mathscr{S}_n$ to $\mathscr{P}_n$.

Example

π_6

$n = 3$



Another bijection

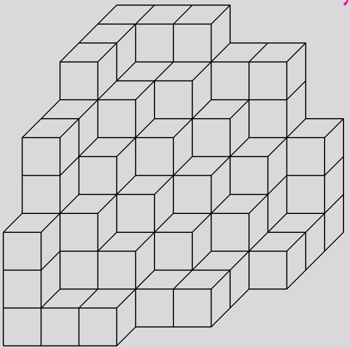
Theorem

Let n be a positive integer.
Then there is a bijection from $\mathscr{S}_n$ to $\mathscr{P}_n$.

Example

π_7

$n = 3$



$\longrightarrow$

2	1
1	

Composition of the bijectons

Corollary

Let n be a positive integer.
Then there is a bijection φ_n from $\mathscr{B}_n$ to $\mathscr{P}_n$.

Example

The case of $n = 3$

$b \in \mathscr{B}_3$	3 3 / 3	3 3 / 2	3 3 / 1	3 2 / 2	3 2 / 1	2 2 / 2	2 2 / 1
$c \in \mathscr{P}_3$	$\emptyset$	1	1 1	2	2 1	2 / 1	2 1 / 1

Composition of the bijectons

Corollary

Let n be a positive integer.
Then there is a bijection φ_n from $\mathscr{B}_n$ to $\mathscr{P}_n$.

Example

The case of $n = 3$

$b \in \mathscr{B}_3$	$\begin{array}{cc} 3 & 3 \\ & 3 \end{array}$	$\begin{array}{cc} 3 & 3 \\ & 2 \end{array}$	$\begin{array}{cc} 3 & 3 \\ & 1 \end{array}$	$\begin{array}{cc} 3 & 2 \\ & 2 \end{array}$	$\begin{array}{cc} 3 & 2 \\ & 1 \end{array}$	$\begin{array}{cc} 2 & 2 \\ & 2 \end{array}$	$\begin{array}{cc} 2 & 2 \\ & 1 \end{array}$
$c \in \mathscr{P}_3$	$\emptyset$	$\begin{array}{c} 1 \end{array}$	$\begin{array}{cc} 1 & 1 \end{array}$	$\begin{array}{c} 2 \end{array}$	$\begin{array}{cc} 2 & 1 \end{array}$	$\begin{array}{c} 2 \\ 1 \end{array}$	$\begin{array}{cc} 2 & 1 \\ 1 & \end{array}$

The statistics in words of RCSPP

Definition

Let $c = (c_{ij})_{1 \leq i,j} \in \mathscr{P}_n$ and $k = 1, \ldots, n$,

Let $\overline{U}_k(c)$ denote the number parts equal to k plus the number of saturated parts less than k.

Example

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

The statistics in words of RCSPP

Definition

Let $c = (c_{ij})_{1 \le i,j} \in \mathscr{P}_n$ and $k = 1, \dots, n$,
Let $\overline{U}_k(c)$ denote the number parts equal to k plus the number of saturated parts less than k.

Example

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

The statistics in words of RCSPP

Definition

Let $c = (c_{ij})_{1 \le i,j} \in \mathscr{P}_n$ and $k = 1, \dots, n$,
Let $\overline{U}_k(c)$ denote the number parts equal to k plus the number of saturated parts less than k.

Example

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

The statistics in words of RCSPP

Definition

Let $c = (c_{ij})_{1 \le i,j} \in \mathscr{P}_n$ and $k = 1, \ldots, n$,
Let $\overline{U}_k(c)$ denote the number parts equal to k plus the number of saturated parts less than k.

Example

$n = 7$, $c \in \mathscr{P}_3$, Saturated parts

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

The statistics in words of RCSPP

Definition

Let $c = (c_{ij})_{1\leq i,j} \in \mathscr{P}_n$ and $k = 1, \ldots, n$,
Let $\overline{U}_k(c)$ denote the number parts equal to k plus the number of saturated parts less than k.

Example

$n = 7$, $c \in \mathscr{P}_3$, $k = 1$, $\overline{U}_1(c) = 3$

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

The statistics in words of RCSPP

Definition

Let $c = (c_{ij})_{1 \le i,j} \in \mathscr{P}_n$ and $k = 1, \dots, n$,
Let $\overline{U}_k(c)$ denote the number parts equal to k plus the number of saturated parts less than k.

Example

$n = 7$, $c \in \mathscr{P}_3$, $k = 2$, $\overline{U}_2(c) = 5$

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

The statistics in words of RCSPP

Definition

Let $c = (c_{ij})_{1 \le i,j} \in \mathscr{P}_n$ and $k = 1, \dots, n$,
Let $\overline{U}_k(c)$ denote the number parts equal to k plus the number of saturated parts less than k.

Example

$n = 7$, $c \in \mathscr{P}_3$, $k = 3$, $\overline{U}_3(c) = 3$

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

The statistics in words of RCSPP

Definition

Let $c = (c_{ij})_{1 \le i,j} \in \mathscr{P}_n$ and $k = 1, \dots, n$,
Let $\overline{U}_k(c)$ denote the number parts equal to k plus the number of saturated parts less than k.

Example

$n = 7$, $c \in \mathscr{P}_3$, $k = 4$, $\overline{U}_4(c) = 4$

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

The statistics in words of RCSPP

Definition

Let $c = (c_{ij})_{1 \le i,j} \in \mathscr{P}_n$ and $k = 1, \ldots, n$,
Let $\overline{U}_k(c)$ denote the number parts equal to k plus the number of saturated parts less than k.

Example

$n = 7$, $c \in \mathscr{P}_3$, $k = 5$, $\overline{U}_5(c) = 4$

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

The statistics in words of RCSPP

Definition

Let $c = (c_{ij})_{1 \le i,j} \in \mathscr{P}_n$ and $k = 1, \dots, n$,
Let $\overline{U}_k(c)$ denote the number parts equal to k plus the number of saturated parts less than k.

Example

$n = 7$, $c \in \mathscr{P}_3$, $k = 6$, $\overline{U}_6(c) = 3$

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

The statistics in words of RCSPP

Definition

Let $c = (c_{ij})_{1 \le i,j} \in \mathscr{P}_n$ and $k = 1, \ldots, n$,
Let $\overline{U}_k(c)$ denote the number parts equal to k plus the number of saturated parts less than k.

Example

$n = 7$, $c \in \mathscr{P}_3$, $k = 7$, $\overline{U}_7(c) = 3$

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

Relation between $U_k(b)$ and $\overline{U}_k(c)$

Theorem

For $n \geq 1$ and $k = 1, \ldots, n$, assume that the bijection φ_n maps $b \in \mathscr{B}_n$ to $c = \varphi(b) \in \mathscr{P}_n$. Then

$$\overline{U}_k(c) = n - 1 - U_k(b).$$

Relation between $U_k(b)$ and $\overline{U}_k(c)$

Theorem

For $n \geq 1$ and $k = 1, \dots, n$, assume that the bijection φ_n maps $b \in \mathscr{B}_n$ to $c = \varphi(b) \in \mathscr{P}_n$. Then

$$\overline{U}_k(c) = n - 1 - U_k(b).$$

Example

$n = 3$, $b \in \mathscr{B}_3$

b	3 3 / 3	3 3 / 2	3 3 / 1	3 2 / 2	3 2 / 1	2 2 / 2	2 2 / 1
$U_1(b)$	2	1	0	2	1	1	0
$U_2(b)$	2	2	1	1	0	1	0
$U_3(b)$	2	2	1	1	0	1	0

Relation between $U_k(b)$ and $\overline{U}_k(c)$

Theorem

For $n \geq 1$ and $k = 1, \dots, n$, assume that the bijection φ_n maps $b \in \mathscr{B}_n$ to $c = \varphi(b) \in \mathscr{P}_n$. Then

$$\overline{U}_k(c) = n - 1 - U_k(b).$$

Example

$n = 3$, $c \in \mathscr{P}_3$

c	$\emptyset$	1	1 1	2	2 1	2 / 1	2 1 / 1
$\overline{U}_1(c)$	0	1	2	0	1	1	2
$\overline{U}_2(c)$	0	0	1	1	2	1	2
$\overline{U}_3(c)$	0	0	1	1	2	1	2

From RCSPPs to lattce paths

Theorem

Let $V = \{(x, y) \in \mathbb{N}^2 : 0 \leq y \leq x\}$ *be the vertex set, and direct an edge from u to v whenever* $v - u = (1, -1)$ *or* $(0, -1)$.
Let $u_j = (n - j, n - j)$ *and* $v_j = (\lambda_j + n - j, 0)$ *for* $j = 1, \ldots, n$*, and let* $\boldsymbol{u} = (u_1, \ldots, u_n)$ *and* $\boldsymbol{v} = (v_1, \ldots, v_n)$*. We claim that the* $c \in \mathscr{P}_n$ *of shape* λ' *can be identified with n-tuples of nonintersecting D-paths in* $\mathscr{P}(\boldsymbol{u}, \boldsymbol{v})$.

From RCSPPs to lattce paths

Theorem

Let $V = \{(x, y) \in \mathbb{N}^2 : 0 \le y \le x\}$ *be the vertex set, and direct an edge from u to v whenever* $v - u = (1, -1)$ *or* $(0, -1)$.
Let $u_j = (n-j, n-j)$ *and* $v_j = (\lambda_j + n - j, 0)$ *for* $j = 1, \dots, n$*, and let* $\boldsymbol{u} = (u_1, \dots, u_n)$ *and* $\boldsymbol{v} = (v_1, \dots, v_n)$*. We claim that the* $c \in \mathscr{P}_n$ *of shape* λ' *can be identified with n-tuples of nonintersecting D-paths in* $\mathscr{P}(\boldsymbol{u}, \boldsymbol{v})$*.*

From RCSPPs to lattce paths

Theorem

Let $V = \{(x, y) \in \mathbb{N}^2 : 0 \le y \le x\}$ *be the vertex set, and direct an edge from u to v whenever* $v - u = (1, -1)$ *or* $(0, -1)$.
Let $u_j = (n - j, n - j)$ *and* $v_j = (\lambda_j + n - j, 0)$ *for* $j = 1, \ldots, n$, *and let* $\mathbf{u} = (u_1, \ldots, u_n)$ *and* $\mathbf{v} = (v_1, \ldots, v_n)$. *We claim that the* $c \in \mathscr{P}_n$ *of shape* λ' *can be identified with n-tuples of nonintersecting D-paths in* $\mathscr{P}(\mathbf{u}, \mathbf{v})$.

From RCSPPs to lattce paths

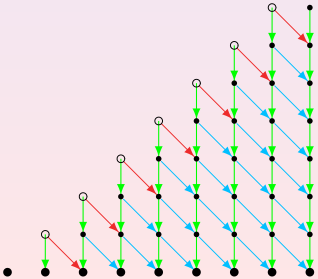
Theorem

Let $V = \{(x, y) \in \mathbb{N}^2 : 0 \le y \le x\}$ *be the vertex set, and direct an edge from* u *to* v *whenever* $v - u = (1, -1)$ *or* $(0, -1)$.
Let $u_j = (n - j, n - j)$ *and* $v_j = (\lambda_j + n - j, 0)$ *for* $j = 1, \ldots, n$*, and let* $\boldsymbol{u} = (u_1, \ldots, u_n)$ *and* $\boldsymbol{v} = (v_1, \ldots, v_n)$. *We claim that the* $c \in \mathscr{P}_n$ *of shape* λ' *can be identified with* n*-tuples of nonintersecting* D*-paths in* $\mathscr{P}(\boldsymbol{u}, \boldsymbol{v})$.

From RCSPPs to lattce paths

Theorem

Let $V = \{(x,y) \in \mathbb{N}^2 : 0 \leq y \leq x\}$ *be the vertex set, and direct an edge from* u *to* v *whenever* $v - u = (1,-1)$ *or* $(0,-1)$*.*
Let $u_j = (n-j, n-j)$ *and* $v_j = (\lambda_j + n - j, 0)$ *for* $j = 1, \dots, n$*, and let* $\boldsymbol{u} = (u_1, \dots, u_n)$ *and* $\boldsymbol{v} = (v_1, \dots, v_n)$*. We claim that the* $c \in \mathscr{P}_n$ *of shape* λ' *can be identified with* n*-tuples of nonintersecting* D*-paths in* $\mathscr{P}(\boldsymbol{u}, \boldsymbol{v})$*.*



Example of lattice paths

Example

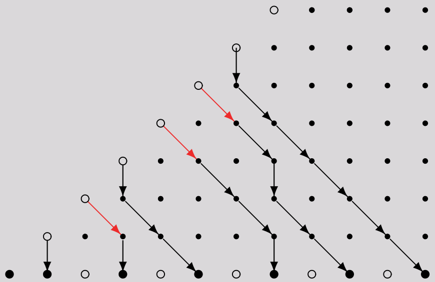
$n = 7$, $c \in \mathscr{P}_7$: RCSPP

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

Example of lattice paths

Example

Lattice paths



Weight of each edge

Definition

Let $u \to v$ be an edge in from u to v.

1. We assign the weight

$$\begin{cases} \prod_{k=j}^{n} t_k \cdot x_j & \text{if } j = i, \\ t_j x_j & \text{if } j < i, \end{cases}$$

to the horizontal edge from $u = (i, j)$ to $v = (i+1, j-1)$.

2. We assign the weight 1 to the vertical edge from $u = (i, j)$ to $v = (i, j-1)$.

Weight of each edge

Definition

Let $u \to v$ be an edge in from u to v.

1. We assign the weight

$$\begin{cases} \prod_{k=j}^{n} t_k \cdot x_j & \text{if } j = i, \\ t_j x_j & \text{if } j < i, \end{cases}$$

   to the horizontal edge from $u = (i, j)$ to $v = (i + 1, j - 1)$.
2. We assign the weight 1 to the vertical edge from $u = (i, j)$ to $v = (i, j - 1)$.

Weight of each edge

Definition

Let $u \to v$ be an edge in from u to v.

1. We assign the weight

$$\begin{cases} \prod_{k=j}^{n} t_k \cdot x_j & \text{if } j = i, \\ t_j x_j & \text{if } j < i, \end{cases}$$

to the horizontal edge from $u = (i, j)$ to $v = (i + 1, j - 1)$.

2. We assign the weight 1 to the vertical edge from $u = (i, j)$ to $v = (i, j - 1)$.

Generating function

Theorem

Let n be a positive integer. Let λ be a partition such that $\ell(\lambda) \le n$. Then the generating function of all plane partitions $c \in \mathscr{P}_n$ of shape λ' with the weight $\boldsymbol{t}^{\overline{U}(c)}\boldsymbol{x}^c$ is given by

$$\sum_{\substack{c\in\mathscr{P}_n \\ \mathrm{sh}\,c=\lambda'}} \boldsymbol{t}^{\overline{U}(c)}\boldsymbol{x}^c = \det\left(e^{(n-i)}_{\lambda_j-j+i}(t_1x_1,\dots,t_{n-i-1}x_{n-i-1},T_{n-i}x_{n-i})\right)_{1\le i,j\le n},$$

where $T_i = \prod_{k=i}^n t_k$.

$\emptyset$	1	1 1	2	2 1	2 / 1	2 1 / 1
1	t_1x_1	$t_1^2t_2t_3x_1^2$	$t_2t_3x_1x_2$	$t_1t_2t_3x_1x_2$	$t_1t_2t_3x_1x_2$	$t_1^2t_2^2t_3^2x_1^2x_2$

Generating function

Theorem

Let n be a positive integer. Let λ be a partition such that $\ell(\lambda) \leq n$. Then the generating function of all plane partitions $c \in \mathscr{P}_n$ of shape λ' with the weight $t^{\overline{U}(c)} x^c$ is given by

$$\sum_{\substack{c \in \mathscr{P}_n \\ \mathrm{sh}\, c = \lambda'}} t^{\overline{U}(c)} x^c = \det\left(e^{(n-i)}_{\lambda_j - j + i}(t_1 x_1, \dots, t_{n-i-1} x_{n-i-1}, T_{n-i} x_{n-i}) \right)_{1 \leq i,j \leq n},$$

where $T_i = \prod_{k=i}^{n} t_k$.

$\emptyset$	$\begin{array}{\|c\|}\hline 1 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 1 & 1 \\ \hline\end{array}$	$\begin{array}{\|c\|}\hline 2 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 2 & 1 \\ \hline\end{array}$	$\begin{array}{\|c\|}\hline 2 \\ \hline 1 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 2 & 1 \\ \hline 1 \\ \cline{1-1}\end{array}$
1	$t_1 x_1$	$t_1^2 t_2 t_3 x_1^2$	$t_2 t_3 x_1 x_2$	$t_1 t_2 t_3 x_1 x_2$	$t_1 t_2 t_3 x_1 x_2$	$t_1^2 t_2^2 t_3^2 x_1^2 x_2$

Generating function

Theorem

Let n be a positive integer. Let λ be a partition such that $\ell(\lambda) \leq n$. Then the generating function of all plane partitions $c \in \mathscr{P}_n$ of shape λ' with the weight $\boldsymbol{t}^{\overline{U}(c)}\boldsymbol{x}^c$ is given by

$$\sum_{\substack{c \in \mathscr{P}_n \\ \operatorname{sh} c = \lambda'}} \boldsymbol{t}^{\overline{U}(c)}\boldsymbol{x}^c = \det\left(e^{(n-i)}_{\lambda_j - j + i}(t_1 x_1, \dots, t_{n-i-1} x_{n-i-1}, T_{n-i} x_{n-i})\right)_{1 \leq i,j \leq n},$$

where $T_i = \prod_{k=i}^{n} t_k$.

$\emptyset$	1	1 1	2	2 1	2 / 1	2 1 / 1
1	t_1x_1	$t_1^2t_2t_3x_1^2$	$t_2t_3x_1x_2$	$t_1t_2t_3x_1x_2$	$t_1t_2t_3x_1x_2$	$t_1^2t_2^2t_3^2x_1^2x_2$

Generating function

Theorem

Let n be a positive integer. Let λ be a partition such that $\ell(\lambda) \leq n$. Then the generating function of all plane partitions $c \in \mathscr{P}_n$ of shape λ' with the weight $\boldsymbol{t}^{\overline{U}(c)}\boldsymbol{x}^c$ is given by

$$\sum_{\substack{c \in \mathscr{P}_n \\ \operatorname{sh} c = \lambda'}} \boldsymbol{t}^{\overline{U}(c)}\boldsymbol{x}^c = \det\left(e^{(n-i)}_{\lambda_j - j + i}(t_1 x_1, \dots, t_{n-i-1} x_{n-i-1}, T_{n-i} x_{n-i})\right)_{1 \leq i,j \leq n},$$

where $T_i = \prod_{k=i}^{n} t_k$.

$\emptyset$	1	1 1	2	2 1	2 1	2 1 1
1	$t_1 x_1$	$t_1^2 t_2 t_3 x_1^2$	$t_2 t_3 x_1 x_2$	$t_1 t_2 t_3 x_1 x_2$	$t_1 t_2 t_3 x_1 x_2$	$t_1^2 t_2^2 t_3^2 x_1^2 x_2$

A Pfaffian expression for the refined TSSCPP conj.

Definition

For positive integers n and N, let $B_n^N(t) = (b_{ij}(t))_{0\le i\le n-1,\ 0\le j\le n+N-1}$ be the $n \times (n+N)$ matrix whose (i,j)th entry is

$$b_{ij}(t) = \begin{cases} \delta_{0,j} & \text{if } i = 0, \\ \binom{i-1}{j-i} + \binom{i-1}{j-i-1} t & \text{otherwise.} \end{cases}$$

A Pfaffian expression for the refined TSSCPP conj.

Definition

For positive integers n and N, let $B_n^N(t) = (b_{ij}(t))_{0\le i\le n-1,\ 0\le j\le n+N-1}$ be the $n \times (n+N)$ matrix whose (i,j)th entry is

$$b_{ij}(t) = \begin{cases} \delta_{0,j} & \text{if } i = 0, \\ \binom{i-1}{j-i} + \binom{i-1}{j-i-1}t & \text{otherwise.} \end{cases}$$

Example

If $n = 3$ and $N = 2$, then

$$B_3^2(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & t & 0 & 0 \\ 0 & 0 & 1 & 1+t & t \end{pmatrix}$$

A Pfaffian expression for the refined TSSCPP conj.

Definition

For positive integers n, let $J_n = (\delta_{i,n+1-j})_{1 \le i,j \le n}$ be the $n \times n$ anti-diagonal matrix.

A Pfaffian expression for the refined TSSCPP conj.

Definition

For positive integers n, let $J_n = (\delta_{i,n+1-j})_{1 \le i,j \le n}$ be the $n \times n$ anti-diagonal matrix.

Example

If $n = 4$, then

$$J_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

A Pfaffian expression for the refined TSSCPP conj.

Definition

For positive integers n, let $\overline{S}_n = (\overline{s}_{i,j})_{1\leq i,j\leq n}$ be the $n \times n$ skew-symmetricl matrix whose (i,j)th entry is

$$\overline{s}_{i,j} = \begin{cases} (-1)^{j-i-1} & \text{if } i < j, \\ 0 & \text{if } i = j, \\ (-1)^{j-i} & \text{if } i > j. \end{cases}$$

A Pfaffian expression for the refined TSSCPP conj.

Definition

For positive integers n, let $\overline{S}_n = (\overline{s}_{i,j})_{1 \le i,j \le n}$ be the $n \times n$ skew-symmetricl matrix whose (i, j)th entry is

$$\overline{s}_{i,j} = \begin{cases} (-1)^{j-i-1} & \text{if } i < j, \\ 0 & \text{if } i = j, \\ (-1)^{j-i} & \text{if } i > j. \end{cases}$$

Example

If $n = 4$, then

$$\overline{S}_4 = \begin{pmatrix} 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ -1 & 1 & -1 & 0 \end{pmatrix}$$

A Pfaffian expression for the refined TSSCPP conj.

Theorem

Let n be a positive integer and let N be an even integer such that $N \geq n - 1$. If k is an integer such that $1 \leq k \leq n$, then

$$\sum_{c \in \mathscr{P}_n} t^{\overline{U}_k(c)} = \mathrm{Pf} \begin{pmatrix} O_n & J_n B_n^N(t) \\ -{}^t B_n^N(t) J_n & \overline{S}_{n+N} \end{pmatrix}.$$

A Pfaffian expression for the refined TSSCPP conj.

Example

If $n = 3$ and $N = 2$ then

$$\mathrm{Pf}\left(\begin{array}{ccc|ccccc} 0 & 0 & 0 & 0 & 0 & 1 & 1+t & t \\ 0 & 0 & 0 & 0 & 1 & t & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 & 1 & -1 & 1 & -1 \\ 0 & -1 & 0 & -1 & 0 & 1 & -1 & 1 \\ -1 & -t & 0 & 1 & -1 & 0 & 1 & -1 \\ -1-t & 0 & 0 & -1 & 1 & -1 & 0 & 1 \\ -t & 0 & 0 & 1 & -1 & 1 & -1 & 0 \end{array}\right).$$

A constant term identity for the refined TSSCPP conj.

Theorem

Let n be a positive integer. If k is an integer such that $1 \le k \le n$, then $\sum_{c \in \mathscr{P}_n} t^{\overline{U}_k(c)}$ is equal to

$$\mathrm{CT}_{\boldsymbol{x}} \prod_{1 \le i < j \le n} \left(1 - \frac{x_i}{x_j}\right) \prod_{i=2}^{n} \left(1 + \frac{1}{x_i}\right)^{i-2} \left(1 + \frac{t}{x_i}\right) \prod_{i=1}^{n} \frac{1}{1 - x_i} \prod_{1 \le i < j \le n} \frac{1}{1 - x_i x_j}.$$

Example

If $n = 3$, then the constant term of

$$\left(1 - \frac{x_1}{x_2}\right)\left(1 - \frac{x_1}{x_3}\right)\left(1 - \frac{x_2}{x_3}\right)\left(1 + \frac{t}{x_2}\right)\left(1 + \frac{1}{x_3}\right)\left(1 + \frac{t}{x_3}\right)$$
$$\times \frac{1}{(1 - x_1)(1 - x_2)(1 - x_3)(1 - x_1 x_2)(1 - x_1 x_3)(1 - x_2 x_3)}$$

is equal to $2 + 3t + 2t^2$.

A constant term identity for the refined TSSCPP conj.

Theorem

Let n be a positive integer. If k is an integer such that $1 \le k \le n$, then $\sum_{c\in\mathscr{P}_n} t^{\overline{U}_k(c)}$ is equal to

$$\mathrm{CT}_{\boldsymbol{x}} \prod_{1\le i<j\le n}\left(1-\frac{x_i}{x_j}\right)\prod_{i=2}^{n}\left(1+\frac{1}{x_i}\right)^{i-2}\left(1+\frac{t}{x_i}\right)\prod_{i=1}^{n}\frac{1}{1-x_i}\prod_{1\le i<j\le n}\frac{1}{1-x_ix_j}.$$

Example

If $n=3$, then the constant term of

$$\left(1-\frac{x_1}{x_2}\right)\left(1-\frac{x_1}{x_3}\right)\left(1-\frac{x_2}{x_3}\right)\left(1+\frac{t}{x_2}\right)\left(1+\frac{1}{x_3}\right)\left(1+\frac{t}{x_3}\right)$$
$$\times\frac{1}{(1-x_1)(1-x_2)(1-x_3)(1-x_1x_2)(1-x_1x_3)(1-x_2x_3)}$$

is equal to $2+3t+2t^2$.

A Pfaffian expression for the doubly refined TSSCPP enumeration

Definition

For positive integers n and N, let $B_n^N(t,u) = (b_{ij}(t,u))_{0 \le i \le n-1,\ 0 \le j \le n+N-1}$ be the $n \times (n+N)$ matrix whose (i,j)th entry is

$$b_{ij}(t,u) = \begin{cases} \delta_{0,j} & \text{if } i = 0, \\ \delta_{0,j-i} + \delta_{0,j-i-1} tu & \text{if } i = 1, \\ \binom{i-2}{j-i} + \binom{i-2}{j-i-1}(t+u) + \binom{i-2}{j-i-2} tu & \text{otherwise.} \end{cases}$$

A Pfaffian expression for the doubly refined TSSCPP enumeration

Example

If $n = 3$ and $N = 2$, then

$$B_3^2(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & tu & 0 & 0 \\ 0 & 0 & 1 & t+u & tu \end{pmatrix}$$

A Pfaffian expression for the doubly refined TSSCPP enumeration

Theorem

Let n be a positive integer and let N be an even integer such that $N \geq n-1$. If k is an integer such that $2 \leq k \leq n$, then

$$\sum_{c \in \mathscr{P}_n} t^{\overline{U}_1(c)} u^{\overline{U}_k(c)} = \operatorname{Pf} \begin{pmatrix} O_n & J_n B_n^N(t,u) \\ -{}^t B_n^N(t,u) J_n & \overline{S}_{n+N} \end{pmatrix}.$$

A Pfaffian expression for the doubly refined TSSCPP enumeration

Example

If $n = 3$ and $N = 2$ then

$$\operatorname{Pf}\left(\begin{array}{ccc|ccccc} 0 & 0 & 0 & 0 & 0 & 1 & t+u & tu \\ 0 & 0 & 0 & 0 & 1 & tu & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 & 1 & -1 & 1 & -1 \\ 0 & -1 & 0 & -1 & 0 & 1 & -1 & 1 \\ -1 & -tu & 0 & 1 & -1 & 0 & 1 & -1 \\ -t-u & 0 & 0 & -1 & 1 & -1 & 0 & 1 \\ -tu & 0 & 0 & 1 & -1 & 1 & -1 & 0 \end{array}\right).$$

A constant term identity for the doubly refined TSSCPP enumeration

Definition

Let $h_i(t,u;x)$ denote the function defined by

$$h_i(t,u;x) = \begin{cases} 1 & \text{if } i = 0, \\ 1 + tux & \text{if } i = 1, \\ (1+x)^{i-2}(1+tx)(1+ux) & \text{if } i \geq 2. \end{cases}$$

Theorem

Let n be a positive integer. If k is an integer such that $2 \leq k \leq n$, then $\sum_{c \in \mathscr{P}_n} t^{\overline{U}_1(c)} u^{\overline{U}_k(c)}$ is equal to

$$\mathrm{CT}_{\boldsymbol{x}} \prod_{1 \leq i < j \leq n} \left(1 - \frac{x_i}{x_j}\right) \prod_{i=1}^{n} h_{i-1}(t,u;x_i^{-1}) \prod_{i=1}^{n} \frac{1}{1-x_i} \prod_{1 \leq i < j \leq n} \frac{1}{1-x_i x_j}.$$

A constant term identity for the doubly refined TSSCPP enumeration

Definition

Let $h_i(t,u;x)$ denote the function defined by

$$h_i(t,u;x)=\begin{cases}1 & \text{if } i=0,\\ 1+tux & \text{if } i=1,\\ (1+x)^{i-2}(1+tx)(1+ux) & \text{if } i\geq 2.\end{cases}$$

Theorem

Let n be a positive integer. If k is an integer such that $2\leq k\leq n$, then $\sum_{c\in\mathscr{P}_n} t^{\overline{U}_1(c)}u^{\overline{U}_k(c)}$ is equal to

$$\mathrm{CT}_{\boldsymbol{x}} \prod_{1\leq i<j\leq n}\left(1-\frac{x_i}{x_j}\right)\prod_{i=1}^{n} h_{i-1}\left(t,u;x_i^{-1}\right)\prod_{i=1}^{n}\frac{1}{1-x_i}\prod_{1\leq i<j\leq n}\frac{1}{1-x_ix_j}.$$

A constant term identity for the doubly refined TSSCPP enumeration

Example

If $n = 3$, then the constant term of

$$\left(1-\frac{x_1}{x_2}\right)\left(1-\frac{x_1}{x_3}\right)\left(1-\frac{x_2}{x_3}\right)\left(1+\frac{tu}{x_2}\right)\left(1+\frac{t}{x_3}\right)\left(1+\frac{u}{x_3}\right)$$
$$\times\frac{1}{(1-x_1)(1-x_2)(1-x_3)(1-x_1x_2)(1-x_1x_3)(1-x_2x_3)}$$

is equal to $1+t+tu+t^2u+tu^2+ut^2u^2$.

A constant term identity

Definition

Let $\mathscr{P}_{nk}$ denote the set of RCSPPs $c \in \mathscr{P}_n$ such that

- c has at most k rows.

Example

A constant term identity

Definition

Let $\mathscr{P}_{nk}$ denote the set of RCSPPs $c \in \mathscr{P}_n$ such that

- c has at most k rows.

Example

A constant term identity

Definition

Let $\mathscr{P}_{nk}$ denote the set of RCSPPs $c \in \mathscr{P}_n$ such that

- c has at most k rows.

Example

If $n = 3$ and $k = 0$, $\mathscr{P}_{3,0}$ consists of the single PP:

$$\emptyset.$$

A constant term identity

Definition

Let $\mathscr{P}_{nk}$ denote the set of RCSPPs $c \in \mathscr{P}_n$ such that

- c has at most k rows.

Example

If $n = 3$ and $k = 1$, $\mathscr{P}_{3,1}$ consists of the following 5 PPs:

$\emptyset$ | 1 | 1 1 | 2 | 2 1

A constant term identity

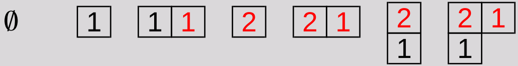
Definition

Let $\mathscr{P}_{nk}$ denote the set of RCSPPs $c \in \mathscr{P}_n$ such that

- c has at most k rows.

Example

If $n = 3$ and $k = 2$, $\mathscr{B}_{3,2}$ consists of the followng 7 PPs



A constant term identity

Theorem

Let n be a positive integer. The restriction of φ_n to $\mathscr{B}_{nk}$ gives a bijection from $\mathscr{B}_{nk}$ to $\mathscr{P}_{nk}$.

Theorem

Let n be a positive integer. If $0 \leq k \leq n-1$ and $1 \leq r \leq n$, then $\sum_{c \in \mathscr{P}_{nk}} t^{\overline{U}_r(c)}$ is equal to

$$\mathrm{CT}_{\boldsymbol{x}} \prod_{1 \leq i < j \leq n} \left(1 - \frac{x_i}{x_j}\right) \prod_{i=2}^{n} \left(1 + \frac{1}{x_i}\right)^{i-2} \left(1 + \frac{t}{x_i}\right)$$

$$\times \frac{\det\left(x_i^{j-1} - x_i^{k+2n-j}\right)_{1 \leq i,j \leq n}}{\prod_{i=1}^{n}(1 - x_i) \prod_{1 \leq i < j \leq n}(x_j - x_i)(1 - x_i x_j)}.$$

A constant term identity

Theorem

Let n be a positive integer. The restriction of φ_n to $\mathscr{B}_{nk}$ gives a bijection from $\mathscr{B}_{nk}$ to $\mathscr{P}_{nk}$.

Theorem

Let n be a positive integer. If $0 \le k \le n-1$ and $1 \le r \le n$, then $\sum_{c \in \mathscr{P}_{nk}} t^{\overline{U}_r(c)}$ is equal to

$$\mathrm{CT}_{\boldsymbol{x}} \prod_{1 \le i < j \le n} \left(1 - \frac{x_i}{x_j}\right) \prod_{i=2}^{n} \left(1 + \frac{1}{x_i}\right)^{i-2} \left(1 + \frac{t}{x_i}\right)$$
$$\times \frac{\det\left(x_i^{j-1} - x_i^{k+2n-j}\right)_{1 \le i,j \le n}}{\prod_{i=1}^{n}(1-x_i) \prod_{1 \le i < j \le n}(x_j - x_i)(1 - x_i x_j)}.$$

Example of $n = 3$

Example

If $n = 3$ and $k = 0$, then the constant term of

$$\left(1-\frac{x_1}{x_2}\right)\left(1-\frac{x_1}{x_3}\right)\left(1-\frac{x_2}{x_3}\right)\left(1+\frac{t}{x_2}\right)\left(1+\frac{1}{x_3}\right)\left(1+\frac{t}{x_3}\right)$$
$$\times\frac{1}{(1-x_1)(1-x_2)(1-x_3)}$$
$$\times\frac{\det\begin{pmatrix}1-x_1^{5} & x_1-x_1^{4} & x_1^2-x_1^{3}\\ 1-x_2^{5} & x_2-x_1^{4} & x_2^2-x_2^{3}\\ 1-x_3^{5} & x_3-x_1^{4} & x_3^2-x_3^{3}\end{pmatrix}}{(x_2-x_1)(x_3-x_1)(x_3-x_2)(1-x_1x_2)(1-x_1x_3)(1-x_2x_3)}$$

is equal to 1.

Example of $n = 3$

Example

If $n = 3$ and $k = 1$, then the constant term of

$$\left(1-\frac{x_1}{x_2}\right)\left(1-\frac{x_1}{x_3}\right)\left(1-\frac{x_2}{x_3}\right)\left(1+\frac{t}{x_2}\right)\left(1+\frac{1}{x_3}\right)\left(1+\frac{t}{x_3}\right)$$
$$\times \frac{1}{(1-x_1)(1-x_2)(1-x_3)}$$
$$\times \frac{\det\begin{pmatrix} 1-x_1^{6} & x_1-x_1^{5} & x_1^2-x_1^{5} \\ 1-x_2^{6} & x_2-x_1^{5} & x_2^2-x_2^{5} \\ 1-x_3^{6} & x_3-x_1^{5} & x_3^2-x_3^{5} \end{pmatrix}}{(x_2-x_1)(x_3-x_1)(x_3-x_2)(1-x_1x_2)(1-x_1x_3)(1-x_2x_3)}$$

is equal to $2+2t+t^2$.

Example of $n = 3$

Example

If $n = 3$ and $k = 2$, then the constant term of

$$\left(1-\frac{x_1}{x_2}\right)\left(1-\frac{x_1}{x_3}\right)\left(1-\frac{x_2}{x_3}\right)\left(1+\frac{t}{x_2}\right)\left(1+\frac{1}{x_3}\right)\left(1+\frac{t}{x_3}\right)$$
$$\times\frac{1}{(1-x_1)(1-x_2)(1-x_3)}$$
$$\times\frac{\det\begin{pmatrix}1-x_1^{7} & x_1-x_1^{6} & x_1^2-x_1^{5}\\ 1-x_2^{7} & x_2-x_1^{6} & x_2^2-x_2^{5}\\ 1-x_3^{7} & x_3-x_1^{6} & x_3^2-x_3^{5}\end{pmatrix}}{(x_2-x_1)(x_3-x_1)(x_3-x_2)(1-x_1x_2)(1-x_1x_3)(1-x_2x_3)}$$

is equal to $2+3t+2t^2$.

Twisted Bender-Knuth involution

The Bender-Knuth involution s_k on tableaux which swaps the number of k's and $(k-1)$'s, for each i.

Example

$n = 7$, $c \in \mathscr{P}_3$

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

Twisted Bender-Knuth involution

Definition

If $k \geq 2$, we define a Bender-Knuth-type involution $\widetilde{\pi}_k$ on $\mathscr{P}_n$ which swaps the number of k's and $(k-1)$'s while we ignore saturated $(k-1)$.

Example

$n = 7$, $c \in \mathscr{P}_3$

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

Twisted Bender-Knuth involution

Definition

If $k \geq 2$, we define a Bender-Knuth-type involution $\widetilde{\pi}_k$ on $\mathscr{P}_n$ which swaps the number of k's and $(k-1)$'s while we ignore saturated $(k-1)$.

Example

$n = 7$ Apply $\widetilde{\pi}_2$ to the following $c \in \mathscr{P}_3$.

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

Twisted Bender-Knuth involution

Definition

If $k \geq 2$, we define a Bender-Knuth-type involution $\widetilde{\pi}_k$ on $\mathscr{P}_n$ which swaps the number of k's and $(k-1)$'s while we ignore saturated $(k-1)$.

Example

$n = 7$ Apply $\widetilde{\pi}_2$ to the following $c \in \mathscr{P}_3$.

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

Twisted Bender-Knuth involution

Definition

If $k \geq 2$, we define a Bender-Knuth-type involution $\widetilde{\pi}_k$ on $\mathscr{P}_n$ which swaps the number of k's and $(k-1)$'s while we ignore saturated $(k-1)$.

Example

$n = 7$ Then we obtain the following $\widetilde{\pi}_2(c) \in \mathscr{P}_3$.

5	5	4	2	1
4	4	3	1	
3	2	1		
2	1			
1				

Twisted Bender-Knuth involution

Definition

If $k \geq 2$, we define a Bender-Knuth-type involution $\widetilde{\pi}_k$ on $\mathscr{P}_n$ which swaps the number of k's and $(k-1)$'s while we ignore saturated $(k-1)$.

Example

$n = 7$ Apply $\widetilde{\pi}_3$ to the following $c \in \mathscr{P}_3$.

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

Twisted Bender-Knuth involution

Definition

If $k \geq 2$, we define a Bender-Knuth-type involution $\widetilde{\pi}_k$ on $\mathscr{P}_n$ which swaps the number of k's and $(k-1)$'s while we ignore saturated $(k-1)$.

Example

$n = 7$ Then we obtain the following $\widetilde{\pi}_3(c) \in \mathscr{P}_3$.

5	5	4	3	2
4	4	3	1	
3	3	2		
2	1			
1				

Twisted Bender-Knuth involution

Definition

If $k \geq 2$, we define a Bender-Knuth-type involution $\widetilde{\pi}_k$ on $\mathscr{P}_n$ which swaps the number of k's and $(k-1)$'s while we ignore saturated $(k-1)$.

Example

$n = 7$ Apply $\widetilde{\pi}_4$ to the following $c \in \mathscr{P}_3$.

5	$\color{red}{5}$	$\color{red}{4}$	2	$\color{red}{2}$
$\color{teal}{4}$	$\color{teal}{4}$	$\color{teal}{3}$	1	
$\color{teal}{3}$	2	2		
2	1			
1				

Twisted Bender-Knuth involution

Definition

If $k \geq 2$, we define a Bender-Knuth-type involution $\widetilde{\pi}_k$ on $\mathscr{P}_n$ which swaps the number of k's and $(k-1)$'s while we ignore saturated $(k-1)$.

Example

$n = 7$ Then we obtain the following $\widetilde{\pi}_4(c) \in \mathscr{P}_3$.

5	5	4	2	2
4	3	3	1	
3	2	2		
2	1			
1				

Twisted Bender-Knuth involution

Definition

If $k \geq 2$, we define a Bender-Knuth-type involution $\widetilde{\pi}_k$ on $\mathscr{P}_n$ which swaps the number of k's and $(k-1)$'s while we ignore saturated $(k-1)$.

Example

$n = 7$ Apply $\widetilde{\pi}_5$ to the following $c \in \mathscr{P}_3$.

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

Twisted Bender-Knuth involution

Definition

If $k \geq 2$, we define a Bender-Knuth-type involution $\widetilde{\pi}_k$ on $\mathscr{P}_n$ which swaps the number of k's and $(k-1)$'s while we ignore saturated $(k-1)$.

Example

$n = 7$ Then we obtain the following $\widetilde{\pi}_5(c) \in \mathscr{P}_3$.

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

Twisted Bender-Knuth involution

Definition

If $k \geq 2$, we define a Bender-Knuth-type involution $\widetilde{\pi}_k$ on $\mathscr{P}_n$ which swaps the number of k's and $(k-1)$'s while we ignore saturated $(k-1)$.

Example

$n = 7$ Apply $\widetilde{\pi}_6$ to the following $c \in \mathscr{P}_3$.

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

Twisted Bender-Knuth involution

Definition

If $k \geq 2$, we define a Bender-Knuth-type involution $\widetilde{\pi}_k$ on $\mathscr{P}_n$ which swaps the number of k's and $(k-1)$'s while we ignore saturated $(k-1)$.

Example

$n = 7$ Then we obtain the following $\widetilde{\pi}_6(c) \in \mathscr{P}_3$.

6	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

Twisted Bender-Knuth involution

Definition

Let $c \in \mathscr{P}_n$. Set λ_i to be the number of parts ≥ 2 in the ith row of c. We set $\lambda_0 = n - 1$ by convention. Let k_i denote the number of 1's in the ith row. Let $\widetilde{\pi}_1$ be the involution on $\mathscr{P}_n$ that changes the number of 1's in the ith row from k_i to $\lambda_{i-1} - \lambda_i - k_i$.

Example

$n = 7$ Apply $\widetilde{\pi}_1$ to the following $c \in \mathscr{P}_3$.

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

Twisted Bender-Knuth involution

Definition

Let $c \in \mathscr{P}_n$. Set λ_i to be the number of parts ≥ 2 in the ith row of c. We set $\lambda_0 = n - 1$ by convention. Let k_i denote the number of 1's in the ith row. Let $\widetilde{\pi}_1$ be the involution on $\mathscr{P}_n$ that changes the number of 1's in the ith row from k_i to $\lambda_{i-1} - \lambda_i - k_i$.

Example

$n = 7$ Apply $\widetilde{\pi}_1$ to the following $c \in \mathscr{P}_3$.

5	5	4	2	2
4	4	3	1	
3	2	2		
2	1			
1				

Twisted Bender-Knuth involution

Definition

Let $c \in \mathscr{P}_n$. Set λ_i to be the number of parts ≥ 2 in the ith row of c. We set $\lambda_0 = n - 1$ by convention. Let k_i denote the number of 1's in the ith row. Let $\widetilde{\pi}_1$ be the involution on $\mathscr{P}_n$ that changes the number of 1's in the ith row from k_i to $\lambda_{i-1} - \lambda_i - k_i$.

Example

$n = 7$ Then we obtain the following $\widetilde{\pi}_1(c) \in \mathscr{P}_3$.

5	5	4	2	2	1
4	4	3	1		
3	2	2			
2	1				

Flips in words of RCSPP

Theorem

Let n be a positive integer and let $k = 1, \dots, n-1$. If $b \in \mathscr{B}_n$, then we have

$$\widetilde{\pi}_k\left(\varphi_n\left(b\right)\right) = \varphi_n\left(\pi_k\left(b\right)\right).$$

Definition

We define involutions on $\mathscr{P}_n$

$$\widetilde{\rho} = \widetilde{\pi}_2\widetilde{\pi}_4\widetilde{\pi}_6\cdots,$$

$$\widetilde{\gamma} = \widetilde{\pi}_1\widetilde{\pi}_3\widetilde{\pi}_5\cdots,$$

and we put $\mathscr{P}_n^{\widetilde{\rho}}$ (resp. $\mathscr{P}_n^{\widetilde{\gamma}}$) the set of elements $\mathscr{P}_n$ invariant under $\widetilde{\rho}$ (resp. $\widetilde{\gamma}$).

Flips in words of RCSPP

Theorem

Let n be a positive integer and let $k = 1, \dots, n-1$. If $b \in \mathscr{B}_n$, then we have

$$\widetilde{\pi}_k\left(\varphi_n\left(b\right)\right) = \varphi_n\left(\pi_k\left(b\right)\right).$$

Definition

We define involutions on $\mathscr{P}_n$

$$\widetilde{\rho} = \widetilde{\pi}_2\widetilde{\pi}_4\widetilde{\pi}_6\cdots,$$
$$\widetilde{\gamma} = \widetilde{\pi}_1\widetilde{\pi}_3\widetilde{\pi}_5\cdots,$$

and we put $\mathscr{P}_n^{\widetilde{\rho}}$ (resp. $\mathscr{P}_n^{\widetilde{\gamma}}$) the set of elements $\mathscr{P}_n$ invariant under $\widetilde{\rho}$ (resp. $\widetilde{\gamma}$).

Invariants under $\widetilde{\rho}$

Example

$\mathscr{P}_1^{\widetilde{\rho}} = \{\emptyset\}$

Invariants under $\widetilde{\rho}$

Example

$$\mathscr{P}_2^{\widetilde{\rho}} = \left\{ \emptyset, \boxed{1} \right\}$$

Invariants under $\widetilde{\rho}$

Example

$\mathscr{P}_3^{\widetilde{\rho}}$ is composed of the following 3 RCSPPs:

$$\emptyset \qquad \begin{array}{|c|}\hline \color{red}{2} \\ \hline 1 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline \color{red}{2} & \color{red}{1} \\ \hline 1 & \\ \cline{1-1} \end{array}$$

Invariants under $\widetilde{\rho}$

Example

$\mathscr{P}_4^{\widetilde{\rho}}$ is composed of the following 10 elements:

$$\emptyset \qquad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline\end{array} \qquad \begin{array}{|c|c|c|}\hline 2 & 1 & \color{red}{1} \\ \hline\end{array} \qquad \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 2 & \color{red}{2} \\ \hline 1 & 1 \\ \hline\end{array}$$

$$\begin{array}{|c|c|c|}\hline 2 & \color{red}{2} & \color{red}{1} \\ \hline 1 & 1 \\ \cline{1-2}\end{array} \qquad \begin{array}{|c|}\hline \color{red}{3} \\ \hline\end{array} \qquad \begin{array}{|c|}\hline \color{red}{3} \\ \hline 2 \\ \hline 1 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline \color{red}{3} & \color{red}{2} \\ \hline 2 & 1 \\ \hline 1 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline \color{red}{3} & \color{red}{2} & \color{red}{1} \\ \hline 2 & 1 \\ \cline{1-2} 1 \\ \cline{1-1}\end{array}$$

Invariants under $\widetilde{\rho}$

Example

$\mathscr{P}_5^{\widetilde{\rho}}$ has 25 elements, and $\mathscr{P}_6^{\widetilde{\rho}}$ has 140 elements.

Invariants under $\widetilde{\gamma}$

Proposition

If $c \in \mathscr{P}_n$ is invariant under $\widetilde{\gamma}$, then n must be an odd integer.

Example

Thus we have $\mathscr{P}_3^{\widetilde{\gamma}} = \left\{ \boxed{1} \right\}$,

$\mathscr{P}_5^{\widetilde{\gamma}}$ is composed of the following 3 RCSPPs:

1	1

3	2	1
1		

3	3	1
2	2	
1		

and $\mathscr{P}_5^{\widetilde{\gamma}}$ has 26 elements.

Invariants under $\widetilde{\gamma}$

Proposition

If $c \in \mathscr{P}_n$ is invariant under $\widetilde{\gamma}$, then n must be an odd integer.

Example

Thus we have $\mathscr{P}_3^{\widetilde{\gamma}} = \left\{ \boxed{1} \right\}$,

$\mathscr{P}_5^{\widetilde{\gamma}}$ is composed of the following 3 RCSPPs:

1	1

3	2	1
1		

3	3	1
2	2	
1		

and $\mathscr{P}_5^{\widetilde{\gamma}}$ has 26 elements.

Invariants under $\widetilde{\gamma}$

Theorem

If $c \in \mathscr{P}_{2n+1}$ is invariant under $\widetilde{\gamma}$, then c has no saturated parts.

Example

Invariants under $\widetilde{\gamma}$

Theorem

If $c \in \mathscr{P}_{2n+1}$ is invariant under $\widetilde{\gamma}$, then c has no saturated parts.

Example

The following $c \in \mathscr{P}_{11}$ is invariant under $\widetilde{\gamma}$:

$c =$

7	7	6	6	3	2	1	1
5	5	4	3	1			
4	3	2	2				
1	1						

Invariants under $\widetilde{\gamma}$

Theorem

If $c \in \mathscr{P}_{2n+1}$ is invariant under $\widetilde{\gamma}$, then c has no saturated parts.

Example

Remove all 1's from $c \in \mathscr{P}_{11}^{\widetilde{\gamma}}$.

$c =$

7	7	6	6	3	2	1	1
5	5	4	3	1			
4	3	2	2				
1	1						

Invariants under $\widetilde{\gamma}$

Theorem

If $c \in \mathscr{P}_{2n+1}$ is invariant under $\widetilde{\gamma}$, then c has no saturated parts.

Example

Then we obtain a PP in which each row has even length.

$$c = \begin{array}{|c|c|c|c|c|c|} \hline 7 & 7 & 6 & 6 & 3 & 2 \\ \hline 5 & 5 & 4 & 3 \\ \cline{1-4} 4 & 3 & 2 & 2 \\ \cline{1-4} \end{array}$$

Invariants under $\widetilde{\gamma}$

Theorem

If $c \in \mathscr{P}_{2n+1}$ is invariant under $\widetilde{\gamma}$, then c has no saturated parts.

Example

Identify 3 and 2, 5 and 4, 7 and 6.

$c =$

7	7	6	6	3	2
5	5	4	3		
4	3	2	2		

Invariants under $\widetilde{\gamma}$

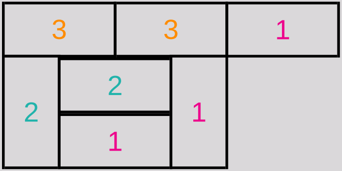
Theorem

If $c \in \mathscr{P}_{2n+1}$ is invariant under $\widetilde{\gamma}$, then c has no saturated parts.

Example

Repace 3 and 2 by dominos containing 1, 5 and 4 by dominos containing 2, 7 and 6 by dominos containing 3.

$d =$



Domino plane partitions

Definition

Let n be a positive integer. Let $\mathscr{D}_n^{\mathrm{R}}$ denote the set of column-strict domino plane partitions d such that

1. The jth column does not exceed $\lceil (n-j)/2 \rceil$.
2. Each row of d has even length.

Let [illegible] denote the number of 1's in $d \in \mathscr{D}_n^{\mathrm{R}}$.

Domino plane partitions

Definition

Let n be a positive integer. Let $\mathscr{D}_n^{\mathrm{R}}$ denote the set of column-strict domino plane partitions d such that

1. The jth column does not exceed $\lceil (n-j)/2 \rceil$,
2. Each row of d has even length.

Let $\overline{U}_1(d)$ denote the number of 1's in $d \in \mathscr{D}_n^{\mathrm{R}}$.

Example

Domino plane partitions

Definition

Let n be a positive integer. Let $\mathscr{D}_n^{\mathrm{R}}$ denote the set of column-strict domino plane partitions d such that

1. The jth column does not exceed $\lceil (n-j)/2 \rceil$,
2. Each row of d has even length.

Let $\overline{U}_1(d)$ denote the number of 1's in $d \in \mathscr{D}_n^{\mathrm{R}}$.

Example

Domino plane partitions

Definition

Let n be a positive integer. Let $\mathscr{D}_n^{\mathrm{R}}$ denote the set of column-strict domino plane partitions d such that

1. The jth column does not exceed $\lceil (n-j)/2 \rceil$,
2. Each row of d has even length.

Let $\overline{U}_1(d)$ denote the number of 1's in $d \in \mathscr{D}_n^{\mathrm{R}}$.

Example

Domino plane partitions

Definition

Let n be a positive integer. Let $\mathscr{D}_n^{\mathrm{R}}$ denote the set of column-strict domino plane partitions d such that

1. The jth column does not exceed $\lceil (n-j)/2 \rceil$,
2. Each row of d has even length.

Let $\overline{U}_1(d)$ denote the number of 1's in $d \in \mathscr{D}_n^{\mathrm{R}}$.

Example

$\mathscr{D}_1^{\mathrm{R}} = \mathscr{D}_2^{\mathrm{R}} = \{\emptyset\}$.

Domino plane partitions

Definition

Let n be a positive integer. Let $\mathscr{D}_n^{\mathrm{R}}$ denote the set of column-strict domino plane partitions d such that

1. The jth column does not exceed $\lceil (n-j)/2 \rceil$,
2. Each row of d has even length.

Let $\overline{U}_1(d)$ denote the number of 1's in $d \in \mathscr{D}_n^{\mathrm{R}}$.

Example

$\mathscr{D}_3^{\mathrm{R}}$ is composed of the following 3 elements:

$\emptyset$, [1], [1 | 1].

Domino plane partitions

Definition

Let n be a positive integer. Let $\mathscr{D}_n^{\mathrm{R}}$ denote the set of column-strict domino plane partitions d such that

1. The jth column does not exceed $\lceil (n-j)/2 \rceil$,
2. Each row of d has even length.

Let $\overline{U}_1(d)$ denote the number of 1's in $d \in \mathscr{D}_n^{\mathrm{R}}$.

Example

$\mathscr{D}_4^{\mathrm{R}}$ is composed of the following 4 elements:

$$\emptyset, \quad \boxed{\ 1\ \ }, \quad \begin{array}{|c|c|} \hline 1 & 1 \\ & \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline 2 & 1 \\ & \\ \hline \end{array}.$$

$\mathscr{D}_5^{\mathrm{R}}$ has 26 elements, $\mathscr{D}_6^{\mathrm{R}}$ has 50 elements, and $\mathscr{D}_7^{\mathrm{R}}$ has 646 elements.

A determinantal formula for Conjecture 6

Theorem

Let n be a positive integer. Then there is a bijection τ_{2n+1} from $\mathscr{P}^{\tilde{\gamma}}_{2n+1}$ to $\mathscr{D}^{\mathrm{R}}_{2n-1}$ such that $\overline{U}_1(\tau_{2n+1}(c)) = \overline{U}_2(c)$ for $c \in \mathscr{P}^{\tilde{\gamma}}_{2n+1}$.

Theorem

Let $n \geq 2$ be a positive integer. Let $R^o_n(t) = (R^o_{ij})_{0 \leq i,j \leq n-1}$ be the $n \times n$ matrix where

$$R^o_{ij} = \binom{i+j-1}{2i-j} + \left\{\binom{i+j-1}{2i-j-1} + \binom{i+j-1}{2i-j+1}\right\} t + \binom{i+j-1}{2i-j} t^2$$

with the convention that $R^o_{0,0} = R^o_{0,1} = 1$. Then we obtain

$$\sum_{c \in \mathscr{P}^{\tilde{\gamma}}_{2n+1}} t^{\overline{U}_2(c)} = \det R^o_n(t).$$

A determinantal formula for Conjecture 6

Theorem

Let n be a positive integer. Then there is a bijection τ_{2n+1} from $\mathscr{P}^{\widetilde{\gamma}}_{2n+1}$ to $\mathscr{D}^{\mathrm{R}}_{2n-1}$ such that $\overline{U}_1(\tau_{2n+1}(c)) = \overline{U}_2(c)$ for $c \in \mathscr{P}^{\widetilde{\gamma}}_{2n+1}$.

Theorem

Let $n \geq 2$ be a positive integer.

A determinantal formula for Conjecture 6

Theorem

Let n be a positive integer. Then there is a bijection τ_{2n+1} from $\mathscr{P}^{\widetilde{\gamma}}_{2n+1}$ to $\mathscr{D}^{\mathrm{R}}_{2n-1}$ such that $\overline{U}_1(\tau_{2n+1}(c)) = \overline{U}_2(c)$ for $c \in \mathscr{P}^{\widetilde{\gamma}}_{2n+1}$.

Theorem

Let $n \geq 2$ be a positive integer. Let $R^0_n(t) = (R^0_{i,j})_{0 \leq i,j \leq n-1}$ be the $n \times n$ matrix where

$$R^0_{i,j} = \binom{i+j-1}{2i-j} + \left\{ \binom{i+j-1}{2i-j-1} + \binom{i+j-1}{2i-j+1} \right\} t + \binom{i+j-1}{2i-j} t^2$$

with the convention that $R^0_{0,0} = R^0_{0,1} = 1$. Then we obtain

A determinantal formula for Conjecture 6

Theorem

Let n be a positive integer. Then there is a bijection τ_{2n+1} from $\mathscr{P}^{\widetilde{\gamma}}_{2n+1}$ to $\mathscr{D}^{R}_{2n-1}$ such that $\overline{U}_1(\tau_{2n+1}(c)) = \overline{U}_2(c)$ for $c \in \mathscr{P}^{\widetilde{\gamma}}_{2n+1}$.

Theorem

Let $n \geq 2$ be a positive integer. Let $R^{\mathrm{o}}_n(t) = (R^{\mathrm{o}}_{i,j})_{0 \leq i,j \leq n-1}$ be the $n \times n$ matrix where

$$R^{\mathrm{o}}_{i,j} = \binom{i+j-1}{2i-j} + \left\{ \binom{i+j-1}{2i-j-1} + \binom{i+j-1}{2i-j+1} \right\} t + \binom{i+j-1}{2i-j} t^2$$

with the convention that $R^{\mathrm{o}}_{0,0} = R^{\mathrm{o}}_{0,1} = 1$. Then we obtain

$$\sum_{c \in \mathscr{P}^{\gamma}_{2n+1}} t^{\overline{U}_2(c)} = \det R^{\mathrm{o}}_n(t).$$

A determinantal formula for Conjecture 6

Theorem

Let n be a positive integer. Then there is a bijection τ_{2n+1} from $\mathscr{P}^{\widetilde{\gamma}}_{2n+1}$ to $\mathscr{D}^{\mathrm{R}}_{2n-1}$ such that $\overline{U}_1(\tau_{2n+1}(c)) = \overline{U}_2(c)$ for $c \in \mathscr{P}^{\widetilde{\gamma}}_{2n+1}$.

Theorem

Let $n \geq 2$ be a positive integer. Let $R^{\mathrm{o}}_n(t) = (R^{\mathrm{o}}_{i,j})_{0 \leq i,j \leq n-1}$ be the $n \times n$ matrix where

$$R^{\mathrm{o}}_{i,j} = \binom{i+j-1}{2i-j} + \left\{ \binom{i+j-1}{2i-j-1} + \binom{i+j-1}{2i-j+1} \right\} t + \binom{i+j-1}{2i-j} t^2$$

with the convention that $R^{\mathrm{o}}_{0,0} = R^{\mathrm{o}}_{0,1} = 1$. Then we obtain

$$\sum_{c \in \mathscr{P}^{\gamma}_{2n+1}} t^{\overline{U}_2(c)} = \det R^{\mathrm{o}}_n(t).$$

A determinantal formula for Conjecture 6

Theorem

Let n be a positive integer. Then there is a bijection τ_{2n+1} from $\mathscr{P}^{\widetilde{\gamma}}_{2n+1}$ to $\mathscr{D}^{R}_{2n-1}$ such that $\overline{U}_1(\tau_{2n+1}(c)) = \overline{U}_2(c)$ for $c \in \mathscr{P}^{\widetilde{\gamma}}_{2n+1}$.

Theorem

Let $n \geq 2$ be a positive integer. Let $R^{o}_{n}(t) = (R^{o}_{i,j})_{0\leq i,j\leq n-1}$ be the $n \times n$ matrix where

$$R^{o}_{i,j} = \binom{i+j-1}{2i-j} + \left\{\binom{i+j-1}{2i-j-1} + \binom{i+j-1}{2i-j+1}\right\} t + \binom{i+j-1}{2i-j} t^{2}$$

with the convention that $R^{o}_{0,0} = R^{o}_{0,1} = 1$. Then we obtain

$$\sum_{c\in\mathscr{P}^{\gamma}_{2n+1}} t^{\overline{U}_2(c)} = \det R^{o}_{n}(t).$$

A determinantal formula for Conjecture 6

Theorem

Let n be a positive integer. Then there is a bijection τ_{2n+1} from $\mathscr{P}^{\widetilde{\gamma}}_{2n+1}$ to $\mathscr{D}^{R}_{2n-1}$ such that $\overline{U}_1(\tau_{2n+1}(c)) = \overline{U}_2(c)$ for $c \in \mathscr{P}^{\widetilde{\gamma}}_{2n+1}$.

Theorem

Let $n \geq 2$ be a positive integer. Let $R^{o}_{n}(t) = (R^{o}_{i,j})_{0 \leq i,j \leq n-1}$ be the $n \times n$ matrix where

$$R^{o}_{i,j} = \binom{i+j-1}{2i-j} + \left\{\binom{i+j-1}{2i-j-1} + \binom{i+j-1}{2i-j+1}\right\} t + \binom{i+j-1}{2i-j} t^2$$

with the convention that $R^{o}_{0,0} = R^{o}_{0,1} = 1$. Then we obtain

$$\sum_{c \in \mathscr{P}^{\gamma}_{2n+1}} t^{\overline{U}_2(c)} = \det R^{o}_{n}(t).$$

The determinants

Example

if $n = 2$, then $\sum_{c \in \mathscr{P}_5^{\tilde{\gamma}}} t^{\overline{U}_2(c)}$ is given by

$$\det \begin{pmatrix} 1 & 1 \\ 0 & 1 + t + t^2 \end{pmatrix}$$

which is equal to $1 + t + t^2$.

The determinants

Example

if $n = 3$, then $\sum_{c \in \mathscr{P}_7^{\tilde{\gamma}}} t^{\overline{U}_2(c)}$ is given by

$$\det \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 + t + t^2 & 1 + 2t + t^2 \\ 0 & t & 3 + 4t + 3t^2 \end{pmatrix}$$

which is equal to $3 + 6t + 8t^2 + 6t^3 + 3t^4$.

The determinants

Example

if $n = 4$, then $\sum_{c \in \mathscr{P}_7^{\tilde{\gamma}}} t^{\overline{U}_2(c)}$ is given by

$$\det \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1+t+t^2 & 1+2t+t^2 & t \\ 0 & t & 3+4t+3t^2 & 4+7t+4t^2 \\ 0 & 0 & 1+4t+t^2 & 10+15t+10t^2 \end{pmatrix}$$

which is equal to $26 + 78t + 138t^2 + 162t^3 + 138t^4 + 78t^5 + 26t^6$.

Determinant evaluation

Theorem (Andrews-Burge)

Let

$$M_n(x,y) = \det\left(\binom{i+j+x}{2i-j} + \binom{i+j+y}{2i-j}\right)_{0\le i,j\le n-1}.$$

Then

$$M_n(x,y) = \prod_{k=0}^{n-1} \Delta_{2k}(x+y),$$

where $\Delta_0(u) = 2$ and for $j > 0$

$$\Delta_{2j}(u) = \frac{(u+2j+2)_j(\frac{1}{2}u+2j+\frac{3}{2})_{j-1}}{(j)_j(\frac{1}{2}u+j+\frac{3}{2})_{j-1}}.$$

A weak version of Conjecture 6

Theorem

Let n be a positive integer. Then

$$\det R_n^o(1) = \frac{1}{2^n} \prod_{k=1}^{n} \frac{(6k-2)!(2k-1)!}{(4k-2)!(4k-1)!}.$$

This proves tha the number of $b \in \mathscr{B}_{2n+1}$ invariant under γ is equal to the number of vertically symmetric alternating sign matrices of size $2n+1$.

A weak version of Conjecture 6

Theorem

Let n be a positive integer. Then

$$\det R_n^o(1) = \frac{1}{2^n} \prod_{k=1}^{n} \frac{(6k-2)!(2k-1)!}{(4k-2)!(4k-1)!}.$$

This proves tha the number of $b \in \mathscr{B}_{2n+1}$ invariant under γ is equal to the number of vertically symmetric alternating sign matrices of size $2n+1$.

A weak version of Conjecture 6

Theorem

Let n be a positive integer. Then

$$\det R_n^{o}(1) = \frac{1}{2^n} \prod_{k=1}^{n} \frac{(6k-2)!(2k-1)!}{(4k-2)!(4k-1)!}.$$

This proves tha the number of $b \in \mathscr{B}_{2n+1}$ invariant under γ is equal to the number of vertically symmetric alternating sign matrices of size $2n+1$.

The end

Thank you!