CHARACTERS OF SYMMETRIC GROUPS

FREE CUMULANTS

AND KEROV POLYNOMIALS

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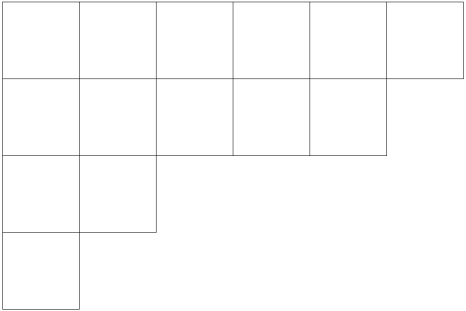
PARTITIONS

A partition is a nonincreasing finite sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$.
Partitions label *irreducible representations of symmetric group* on $\lambda_1 + \lambda_2 + \ldots + \lambda_n$ letters.



$$6 + 5 + 2 + 1 = 14$$

$$4 + 3 + 2 + 2 + 2 + 1 = 14$$

FRENCH CONVENTION

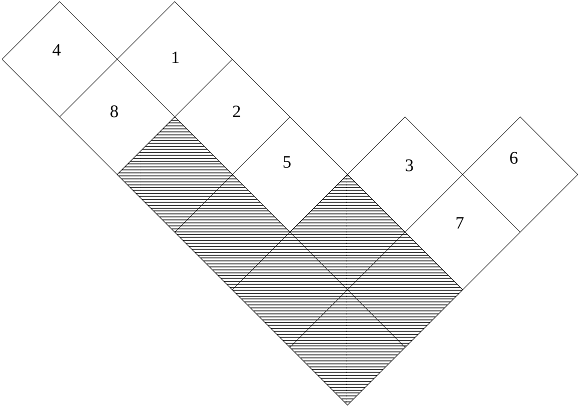
14			
12	13		
4	11		
3	8		
2	6	10	
1	5	7	9

1			
3	1		
4	2		
5	3		
7	5	1	
9	7	3	1

Dimension of a representation=number of Young tableaux.
Hook formula

$$\frac{n!}{\prod_{i,j} h_{ij}}$$

RUSSIAN CONVENTION



Restriction of a representation $S_{14} \downarrow S_6$.
The multiplicity is the number of ways to erase boxes.
cf experiment.

LARGE SYMMETRIC GROUPS

Normalized characters $\chi_\lambda(\mu) = \frac{Tr(\rho_\lambda(\mu))}{\dim(\lambda)}$

$\mu =$ fixed conjugacy class of $S_\infty = \cup_n S_n$

$N = \sum_i \lambda_i \qquad \lambda_i/N \to \alpha_i \qquad \lambda'_i/N \to \beta_i$
$\chi_\lambda(\mu) \to \chi^\infty_{\alpha,\beta}(\mu)$ for a factor representation of S_∞.

Thoma/Vershik/Kerov theory $\to$ representation theory of S_∞ in terms of S_N for $N \to \infty$.

For "most" Young diagrams $\lambda_i = o(N)$ and $\chi_\lambda(\mu) \to 0$.
In this regime representation theory of symmetric groups is governed by *free probability*.

FREE COMPRESSION

$$X = UDU^*$$

D=diagonal $N \times N$ matrix, eigenvalues $D_1, ..., D_N$.
U=random Haar unitary $N \times N$ matrix.

$$\frac{1}{N} Tr(X^k) = \frac{1}{N} \sum_j D_j^k \to_{N \to \infty} \int x^k \mu(dx)$$

$0 < p < 1$, $X^{(p)}$=$pN \times pN$ upper corner of X.

$$\frac{1}{pN} Tr((X^{(p)})^k) \to_{N \to \infty} \int x^k \mu^{(p)}(dx)$$

$\mu^{(p)}$=free compression of μ, depends only on μ and p.

FREE CUMULANTS

$$G_\mu(z) = \int \frac{1}{z-x}\mu(dx) \quad = \frac{1}{z} + \sum_{n=1}^{\infty} z^{-n-1} \int x^n \mu(dx)$$
$$= \frac{1}{z} + \sum_{n=1}^{\infty} z^{-n-1} M_n$$

$$K_{\mu_i}(G_\mu(z)) = G_\mu(K_\mu(z)) = z; \qquad K_\mu(z) = \frac{1}{z} + \sum_{n=0}^{\infty} R_n(\mu) z^n$$

$R_n(\mu) =$ **free cumulants** (D. Voiculescu, R. Speicher) of μ.
Free cumulants are polynomial functions of moments

$$M_n = \int x^n \mu(dx)$$

Conversely moments are polynomial functions of free cumulants.

FREE COMPRESSION

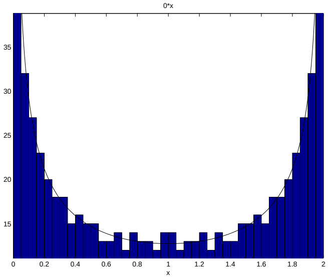
The free compression of a measure is obtained by the rule

$$R_n(\mu^{(p)}) = p^{n-1} R_n(\mu)$$

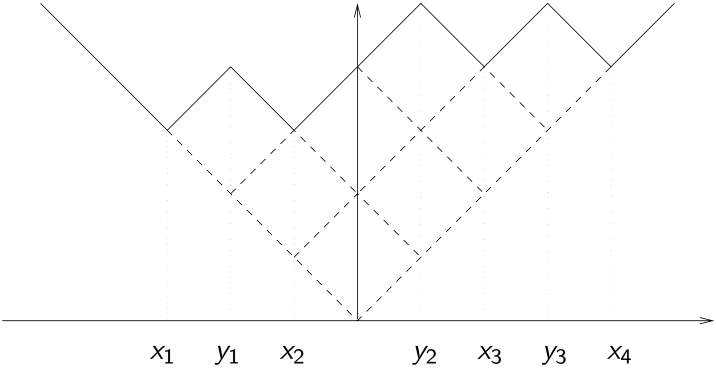
Since free cumulants determine the measure, this determines $\mu^{(p)}$.

Example: $\mu = \frac{1}{2}(\delta_0 + \delta_1)$
Random matrix model: compute the spectrum of $\Pi_1\Pi_2\Pi_1$ where $\Pi_1, \Pi_2 =$ orthogonal projections on random subspaces of dimensions $N/2$.

$$\mu^{(1/2)} = \frac{dx}{\pi\sqrt{x(1-x)}} \qquad \text{arcsine distribution}$$



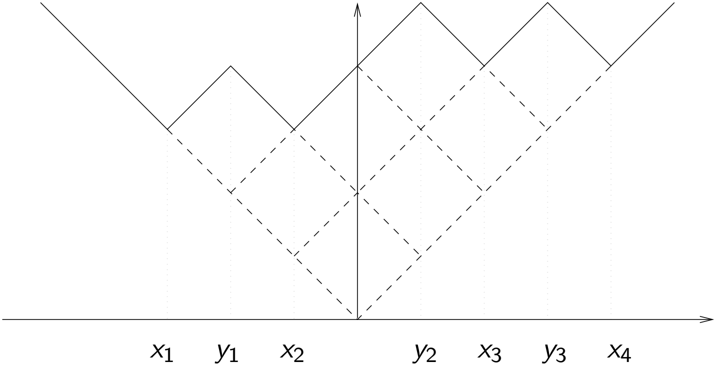
Histogram with a 400×400 random matrix.



A diagram may be identified with a function $\omega(x)$ such that

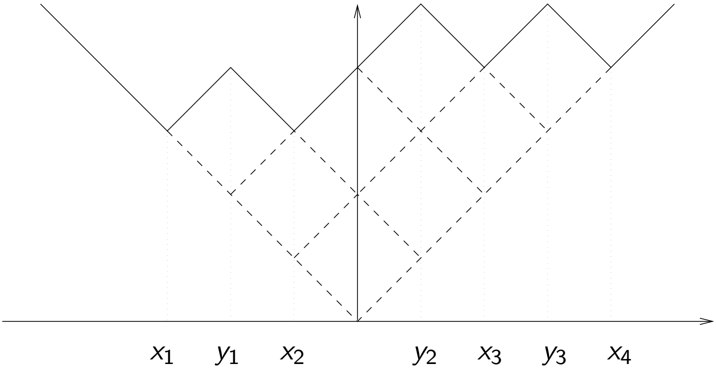
$$|\omega(x)| = |x| \text{ for } x >> 1 \qquad |\omega(x) - \omega(y)| \leq |x - y|.$$

TRANSITION MEASURE OF A DIAGRAM



(S.Kerov) there exists a unique probability measure m_ω such that

$$m_\omega = \sum_{k=1}^{n} \mu_k \delta_{x_k} \qquad \mu_k = \frac{\prod_{i=1}^{n-1}(x_k - y_i)}{\prod_{i \neq k}(x_k - x_i)}$$



m_ω gives the decomposition of $\omega \uparrow S_{n+1}$.

$$\sigma(u) = (\omega(u) - |u|)/2$$

$$
\begin{aligned}
G_{m_\omega}(z) &= \tfrac{1}{z}\exp\int \tfrac{1}{x-z}\sigma'(x)\,dx \\
&= \int \tfrac{1}{z-x} m_\omega(dx) \\
&= \frac{\prod_{i=1}^{n-1}(z-y_k)}{\prod_{i=1}^{n}(z-x_k)}
\end{aligned}
$$

$$K_\omega = G_\omega^{\langle -1\rangle}$$

$$K_\omega(z) = \frac{1}{z} + \sum_{n=1}^{\infty} R_n(\omega) z^{n-1}$$

$R_n(\omega)$= the free cumulants of the diagram.

Remark $\omega \mapsto m_\omega$ can be extended to 1-Lipschitz maps.

ASYMPTOTIC EVALUATION OF CHARACTERS

$\lambda =$ Young diagram with q boxes, $\lambda \sim \sqrt{q}\omega$.

Number of rows and columns $= O(\sqrt{q})$.

$\chi_\lambda =$ normalized character of λ.

$$\chi_\lambda(\sigma) = q^{-|\sigma|/2}(\prod_{c|\sigma} R_{|c|+2}(\omega) + O(q^{-1}))$$

$|\sigma|=$ length of σ w.r.t generating set of all transpositions, the product is over cycles of σ.

ASYMPTOTIC OF RESTRICTION

$\omega=$ continuous diagram, $0 < t < 1$,
define ω_t by

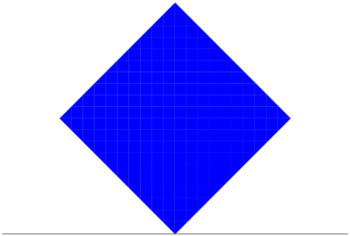
$$R_n(\omega_t) = t^{n-1} R_n(\omega)$$

The restriction of λ to $S_p \times S_{q-p} \subset S_q$ splits into irreducible

$$\bigoplus c^{\lambda}_{\mu\nu} \, [\mu] \otimes [\nu] \quad \text{(Littlewood-Richarson rule)}.$$

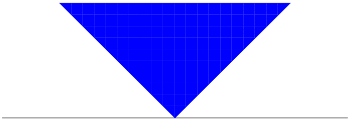
Give a weight $c^{\lambda}_{\mu\nu} \, dim(\mu) \, dim(\nu)$ to the pair (μ, ν).
Then as $q \to \infty$ and $p/q \to t$, almost all pairs (μ, ν) (rescaled by $\sqrt{q}$),
become close to (ω_t, ω_{1-t}).

EXAMPLE



Square diagram

$$m_\omega = \frac{1}{2}(\delta_{-1} + \delta_1)$$



1/2 Compression of the square diagram

$$m_\omega^{(1/2)} = \frac{dx}{\pi\sqrt{x(1-x)}}$$

Asymptotic of induction of representations

$$S_p \times S_q \uparrow S_{p+q}$$

can be interpreted in terms of *sums* of independent random matrices and *free convolution*.

FROBENIUS FORMULA FOR CHARACTERS OF CYCLES

$\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots)$=partition of n,

$$\varphi(z) = \prod_i (z - \lambda_i - n + i)$$

$$z\varphi(z-1)/\varphi(z) = 1/G_\lambda(z+n-1) = H_\lambda(z+n-1)$$

Frobenius' formula is
(c_k=cycle of order k, $\chi_\lambda(\sigma) = \frac{Tr(\rho_\lambda(\sigma))}{Tr(\rho_\lambda(e))}$)

$$(n)_k \chi_\lambda(c_k) = -\frac{1}{k}[z^{-1}]\, z(z-1)\ldots(z-k+1)\varphi(z-k)/\varphi(z).$$

$$(n)_k \chi_\lambda(c_k) = -\frac{1}{k}[z^{-1}]\, H_\lambda(z+n-1)\ldots H_\lambda(z+n-k)$$

Using the invariance of the residue under translation of the variable one gets

$$(n)_k \chi_\lambda(c_k) = -\frac{1}{k}[z^{-1}]\, H_\lambda(z)\ldots H_\lambda(z-k+1).$$

KEROV POLYNOMIALS

Consider the formal power series

$$H(z) = z - \sum_{j=2}^{\infty} B_j z^{1-j}.$$

Define

$$\Sigma_k = -\frac{1}{k}[z^{-1}]H(z)\dots H(z-k+1)$$

$$R_{k+1} = -\frac{1}{k}[z^{-1}]\, H(z)^k$$

The expression of Σ_k in terms of the R_j's is given by Kerov's polynomials.
Kerov's polynomials express normalized characters of cycles in terms of free cumulants of Young diagrams.

ANOTHER FORM OF THE FORMULA (Goulden, Rattan, 2005)

Use invariance of residue under change of variables $z = G(\zeta) = \zeta + \sum_{i=2}^{\infty} R_j \zeta^{1-j}$ and

$$[z^{-1}]f(z) = [\zeta^{-1}]u'(\zeta)f(u(\zeta))$$

to get

$$\Sigma_k = -\frac{1}{k}[\zeta^{-1}]G'(\zeta)\prod_{j=1}^{k-1}\left(\zeta + \sum_{r=1}^{\infty}\frac{(-j)^r}{r!}\left(\frac{1}{G'(\zeta)}\frac{d}{d\zeta}\right)^{r-1}\frac{1}{G'(\zeta)}\right)$$

$$\Sigma_1 = R_2$$

$$\Sigma_2 = R_3$$

$$\Sigma_3 = R_4 + R_2$$

$$\Sigma_4 = R_5 + 5R_3$$

$$\Sigma_5 = R_6 + 15R_4 + 5R_2^2 + 8R_2$$

$$\Sigma_6 = R_7 + 35R_5 + 35R_3R_2 + 84R_3$$

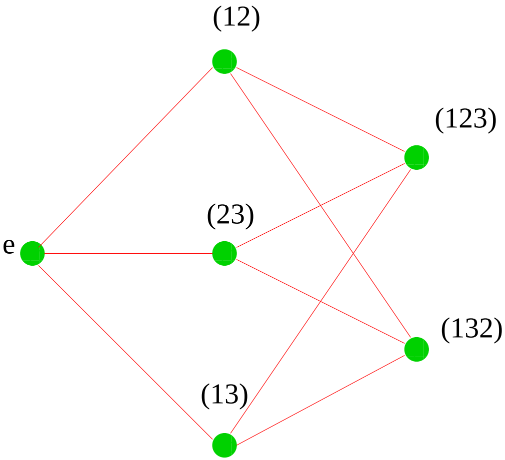
$$\Sigma_7 = R_8 + 70R_6 + 84R_4R_2 + 56R_3^2 + 14R_2^3 + 469R_4 + 224R_2^2 + 180R_2$$

$$\Sigma_8 = R_9 + 126R_7 + 169R_5R_2 + 252R_4R_3 + 30R_3R_2^2 + 1869R_5 + 3392R_3R_2 + 3044R_3$$

GEOMETRY OF SYMMETRIC GROUPS

Cayley graph of S_n: (π_1, π_2) edge if and only if $\pi_1\pi_2^{-1} =$ transposition.

$$d(\sigma_1, \sigma_2) = |\sigma_1\sigma_2^{-1}| = n - |\{\text{cycles of } \sigma_1\sigma_2^{-1}\}|$$

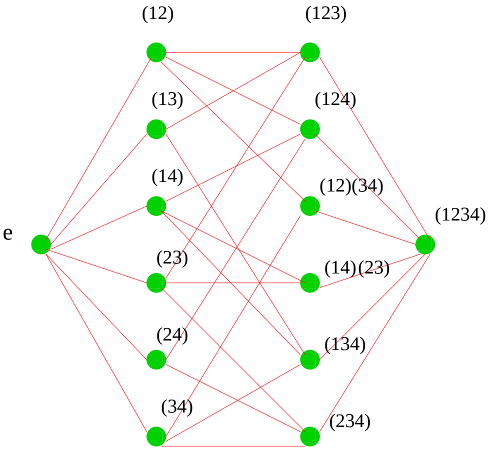


Cayley Graph of S_3

INTERVALS IN THE SYMMETRIC GROUPS

An interval in the Cayley graph

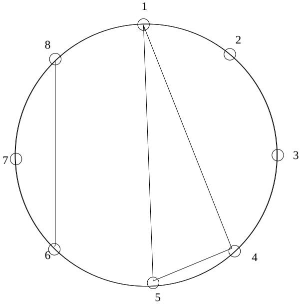
$$[\pi_1, \pi_2] = \{\sigma \mid d(\pi_1, \sigma) + d(\sigma, \pi_2) = d(\pi_1, \pi_2)\}$$



The interval [e,(1234)]

NONCROSSING PARTITIONS AND FREE CUMULANTS

$[e, (1234...n)] \sim NC(n)$= lattice of noncrossing partitions of $\{1, 2, ..., n\}$.

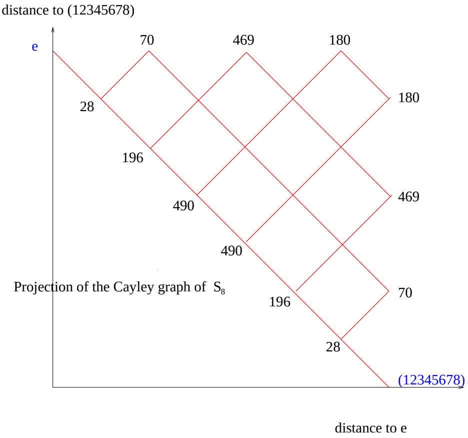


Moments and free cumulants are related by (Speicher)

$$M_n = \sum_{\pi \in NC(n)} R_\pi \qquad R_n = \sum_{\pi \in NC(n)} \mu([\pi, c_n]) M_\pi$$

$$R_\pi = \prod_{p \in \pi} R_{|p|} \qquad M_\pi = \prod_{p \in \pi} M_{|p|}$$

$$\Sigma_7 = R_8 + 70R_6 + 84R_4R_2 + 56R_3^2 + 14R_2^3 + 469R_4 + 224R_2^2 + 180R_2$$



The coefficient of R_{k+1-2l} in Σ_k is equal to the number of cycles $c \in S_k$, of length k, such that $(1\,2\ldots k)\,c^{-1}$ has $k-2l$ cycles (Stanley 2001, B. 2001).

$$\Sigma_8 = R_9 + 126R_7 + 169R_5R_2 + 252R_4R_3 + 30R_3R_2^2 + 1869R_5 + 3392R_3R_2 + 3044R_3$$

1. The coefficient of
$R_2^{l_2} \ldots R_s^{l_s}$ in Σ_k, with $k = 2l_2 + 3l_3 + \ldots + sl_s + 1$ is equal to

$$\frac{(k+1)k(k-1)}{24} \frac{(l_2 + \ldots + l_s)!}{l_2! \ldots l_s!} \prod_{j=2}^{s} (j-1)^{l_j}$$

(Sniady 2004; Goulden and Rattan 2005).

2. (Goulden and Rattan 2005) The coefficient of R_2^i in Σ_{2i+3} is

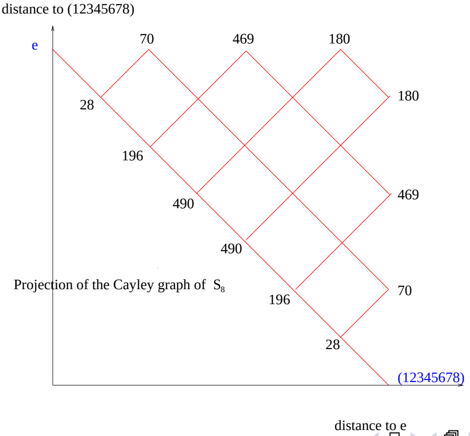
$$\frac{1}{540} i(i+1)^3(i+2)^3(i+3)(2i+3)$$

3. (V. Féray, 2007) The coefficient of R_jR_l in Σ_k is the number of $\sigma = c_1c_2$ such that $(12...k)\sigma^{-1}$ has $j + l - 2$ cycles, and among these cycles at least j have an element in common with c_1 and l with c_2.

Positivity conjecture (Kerov 2000): all coefficients are nonnegative.

Recent proof (still under checking) by V. Féray (May 2007), using Stanley's polynomials, and a signed covering of the symmetric group.

$$\Sigma_7 = R_8 + 70R_6 + 84R_4R_2 + 56R_3^2 + 14R_2^3 + 469R_4 + 224R_2^2 + 180R_2$$



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