A bijection between 2-triangulations and pairs of non-crossing Dyck paths

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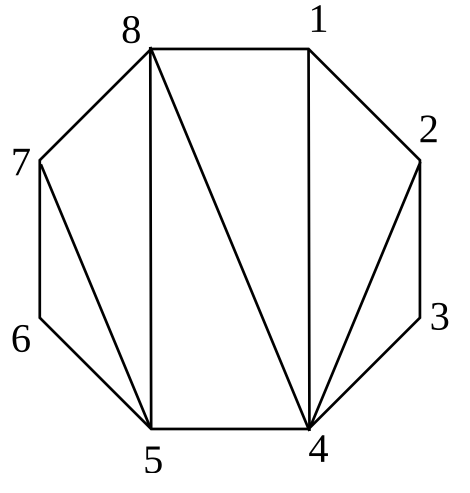
Let $\mathfrak{C}_n$ be a regular n-gon.
A triangulation of $\mathfrak{C}_n$ is a subdivision of $\mathfrak{C}_n$ into triangles, using diagonals that do not cross.

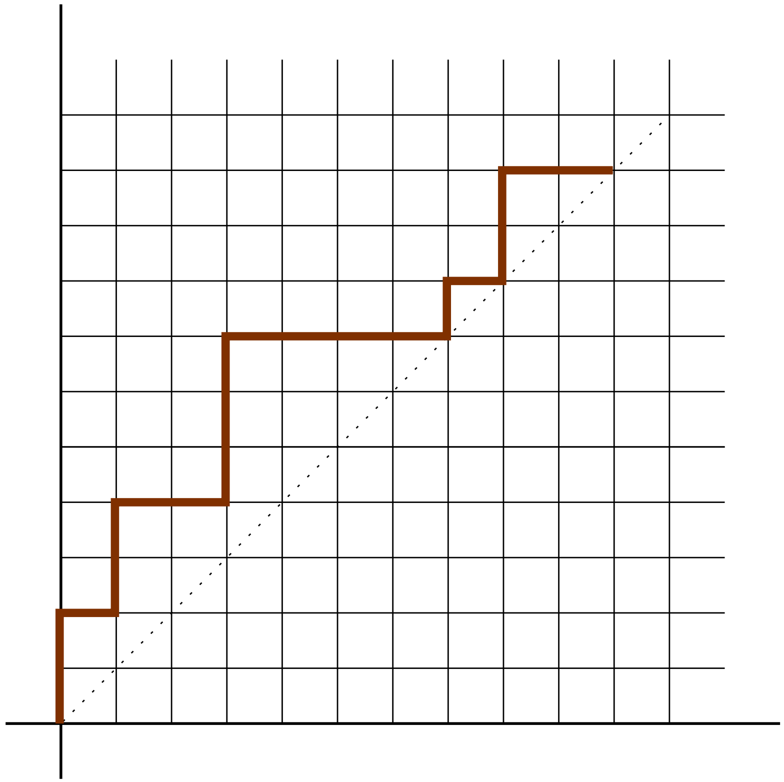


- Every triangulation of $\mathfrak{C}_n$ has exactly $n-3$ diagonals.
- The number of triangulations of $\mathfrak{C}_n$ is

$$C_{n-2} = \frac{1}{n-1}\binom{2n-4}{n-2},$$

where C_m is the m-th *Catalan number*.

A *Dyck path* is a lattice path from $(0, 0)$ to a point on the diagonal $y = x$ with steps $N = (0, 1)$ and $E = (1, 0)$ that never goes below this diagonal.



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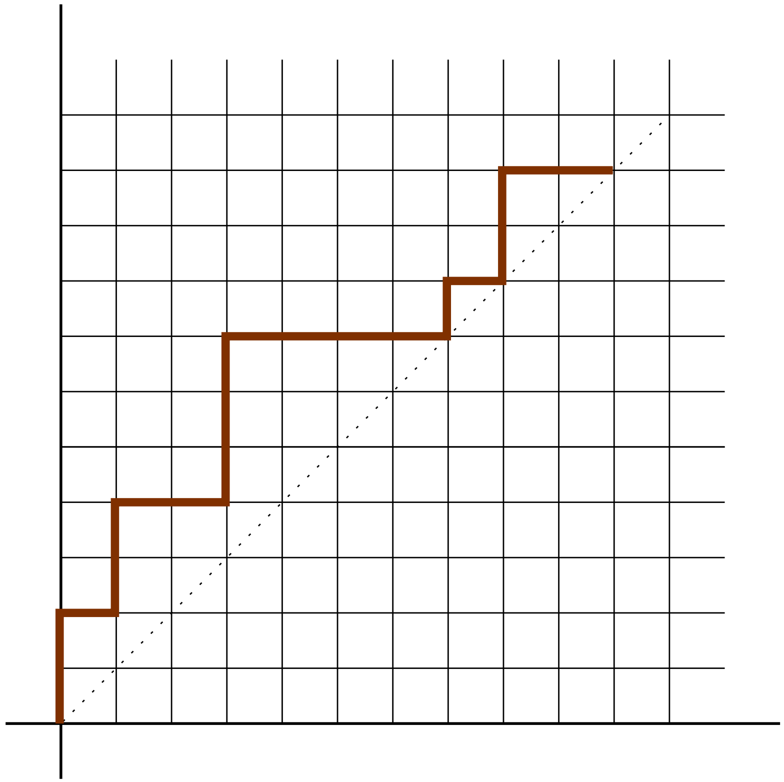
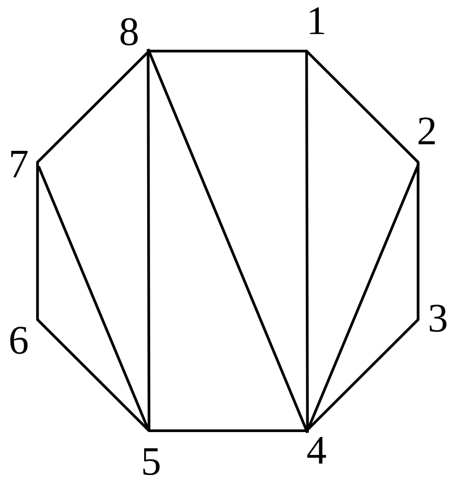
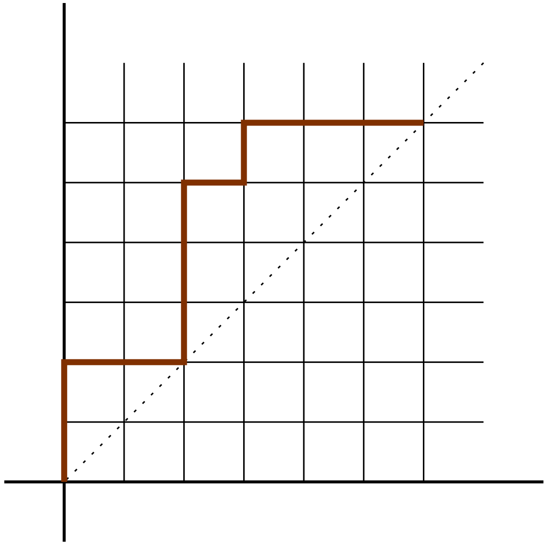


If (m,m) is the final point, we call m the *size* of the path.

Let $\mathcal{D}_m$ be the set of Dyck paths of size m. Then, $|\mathcal{D}_m| = C_m$.

A bijection between triangulations and Dyck paths





For each $j = 3, 4, \ldots, n$:

- draw an N step,
- draw as many E steps as diagonals of the form (i, j) with $i < j$.

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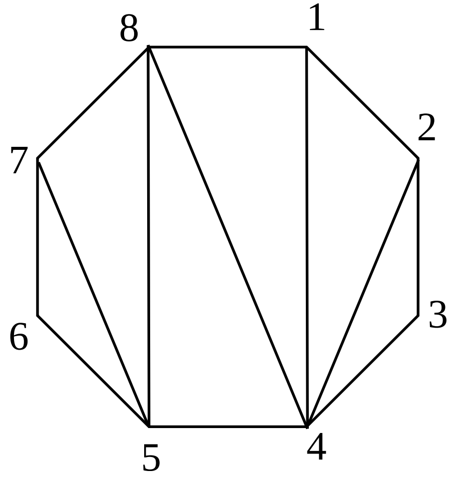
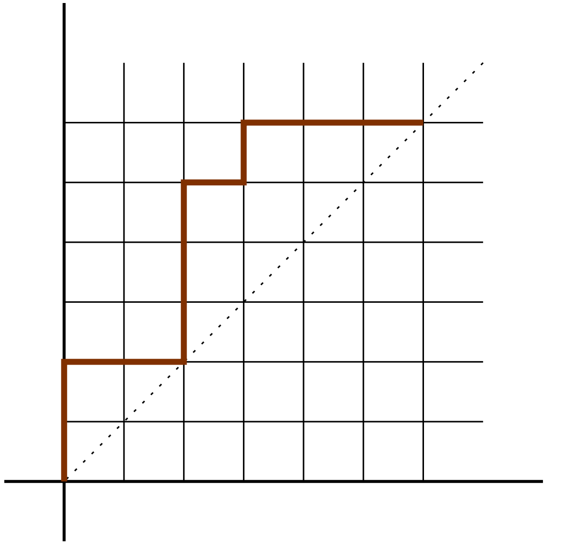


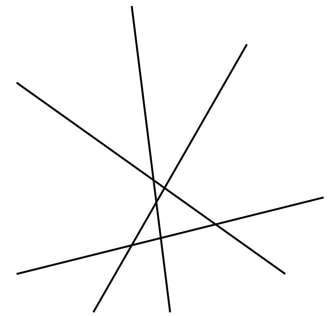


For each $j = 3, 4, \ldots, n$:

- draw an N step,
- draw as many E steps as diagonals of the form (i, j) with $i < j$.

Draw an E step at the end.

Definition. A j-*crossing* is a set of j diagonals where any two of them cross.



Generalized triangulations

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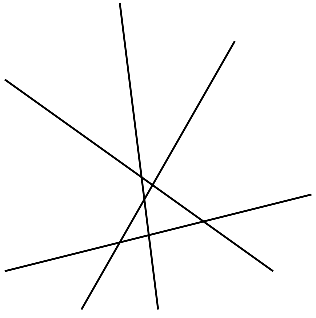

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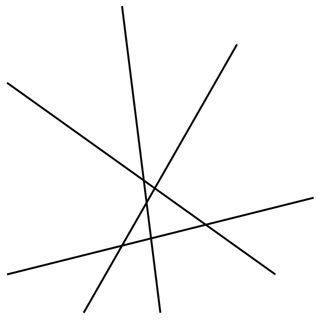
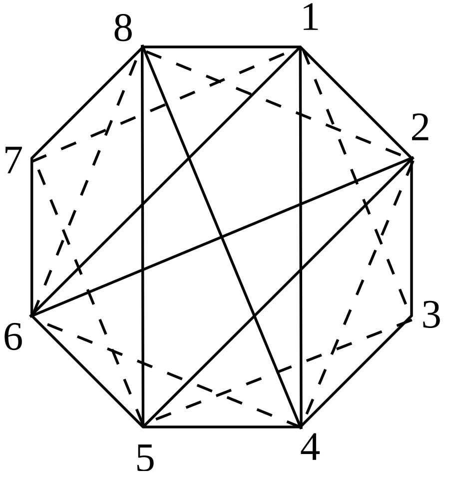


We could have defined a triangulation of $\mathfrak{C}_n$ as a maximal set of diagonals with no 2-crossings.

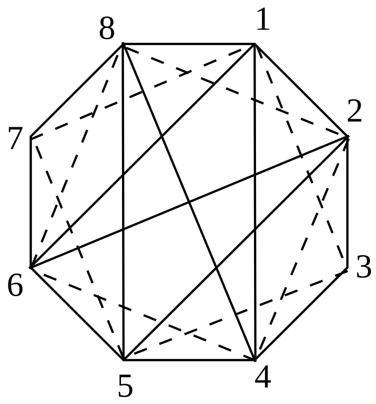
Definition. A k-triangulation of $\mathfrak{C}_n$ is a maximal set of diagonals with no $(k+1)$-crossings.



a 2-triangulation

Facts about generalized triangulations

All the diagonals connecting vertices at distance $\leq k$ belong to every k-triangulation.



Facts about generalized triangulations

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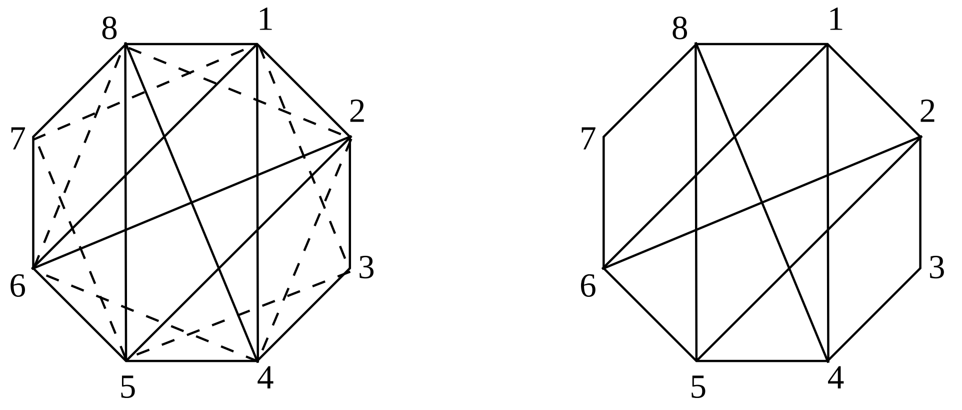

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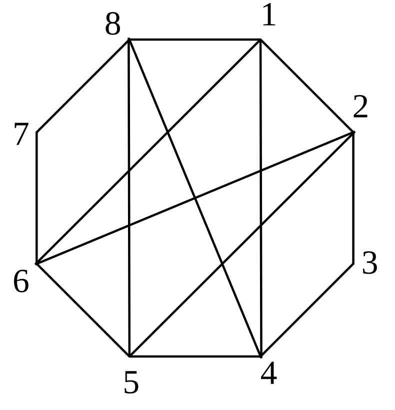

This 2-triangulation has $2(8-2\cdot 2-1) = 6$ diagonals.

Theorem (Nakamigawa, Dress-Koolen-Moulton)**.**

Every k-triangulation of $\mathfrak{C}_n$ has exactly $k(n-2k-1)$ diagonals.

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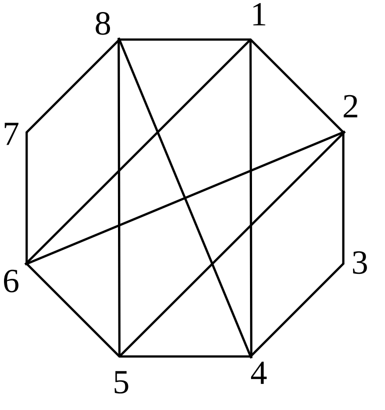

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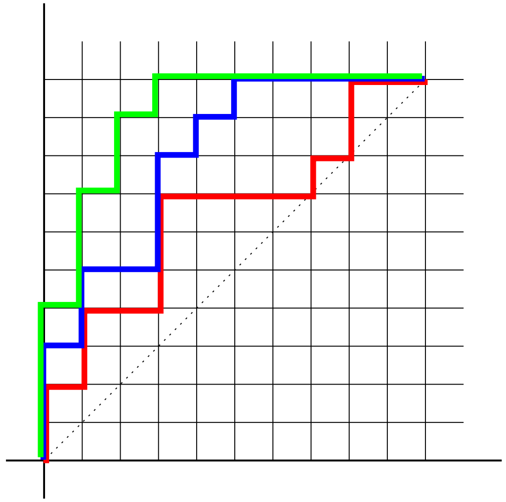
Theorem (Nakamigawa, Dress-Koolen-Moulton)**.**

Every k-triangulation of $\mathfrak{C}_n$ has exactly $k(n-2k-1)$ diagonals.

Theorem (Jonsson)**.** *The number of k-triangulations of $\mathfrak{C}_n$ is*

$$\det(C_{n-i-j})_{i,j=1}^{k} = \begin{vmatrix} C_{n-2} & C_{n-3} & \cdots & C_{n-k} & C_{n-k-1} \\ C_{n-3} & C_{n-4} & \cdots & C_{n-k-1} & C_{n-k-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ C_{n-k-1} & C_{n-k-2} & \cdots & C_{n-2k+1} & C_{n-2k} \end{vmatrix}$$

Theorem (Lindström, Gessel-Viennot)**.** *The number of k-tuples $(P_1, P_2, \ldots, P_k)$ of Dyck paths of size $n - 2k$ such that each P_i never goes below P_{i+1} is given by the same determinant* $\det(C_{n-i-j})_{i,j=1}^{k}$.



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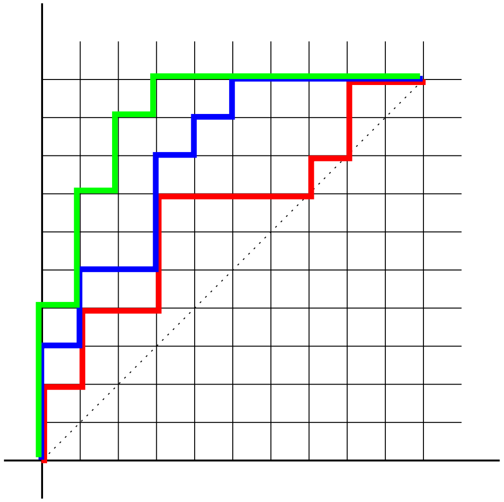

Problem: Find a bijection between k-triangulations and k-tuples of non-crossing Dyck paths.

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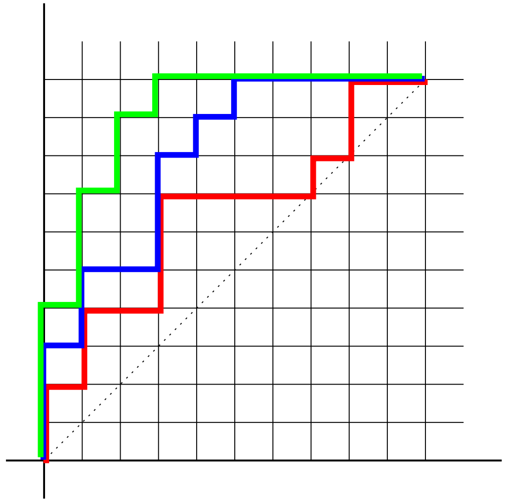
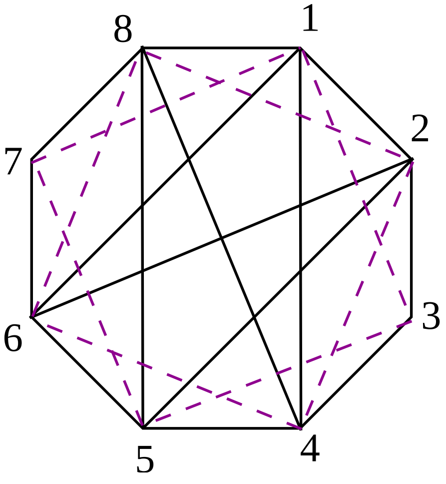
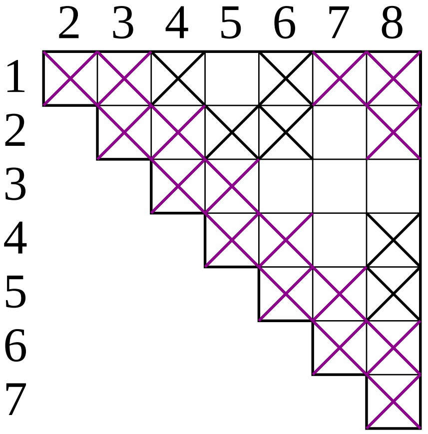


Problem: Find a bijection between k-triangulations and k-tuples of non-crossing Dyck paths.

- $k = 1 \longrightarrow$ known
- $k = 2 \longrightarrow$ we will see it next
- $k \geq 3 \longrightarrow$ open problem

We represent a 2-triangulation as an array:





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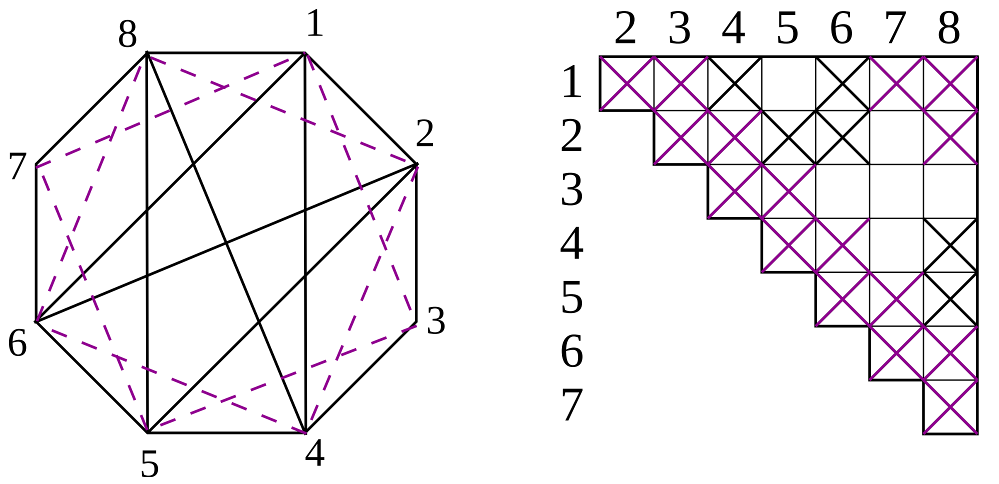

The purple crosses appear in any 2-triangulation, and they can't be part of any 3-crossing.

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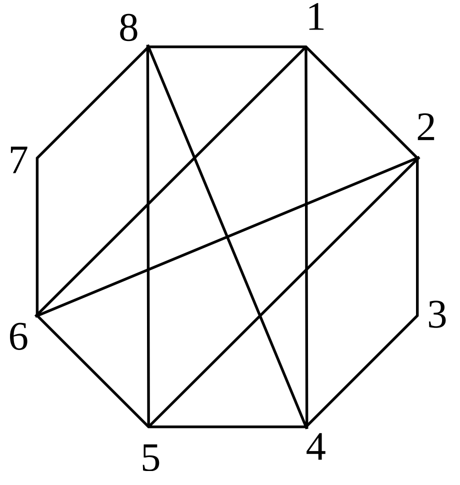

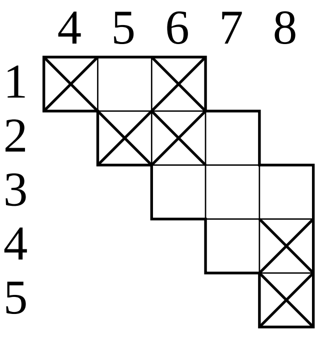

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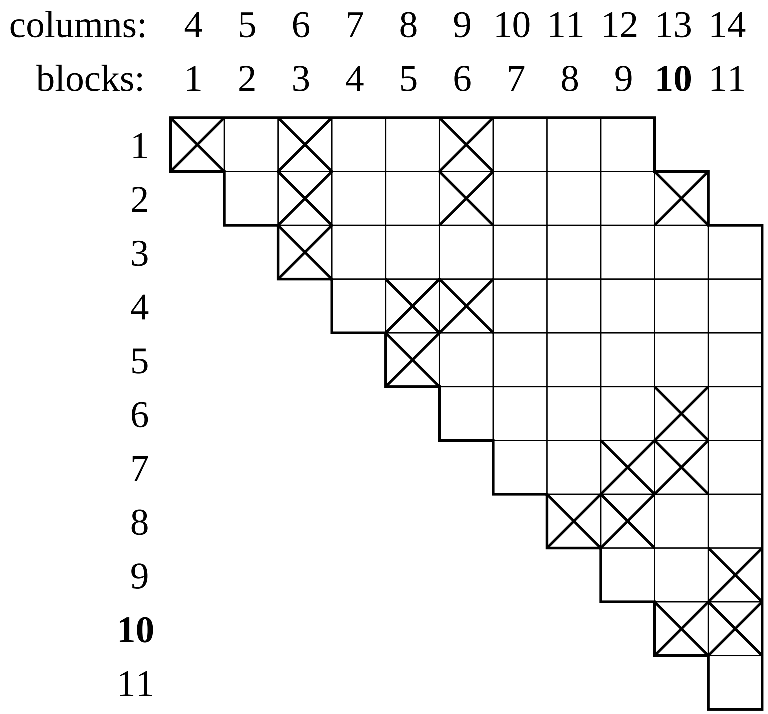
We omit them for simplicity.

Now there are exactly $2n - 10$ diagonals.

The bijection: introduction

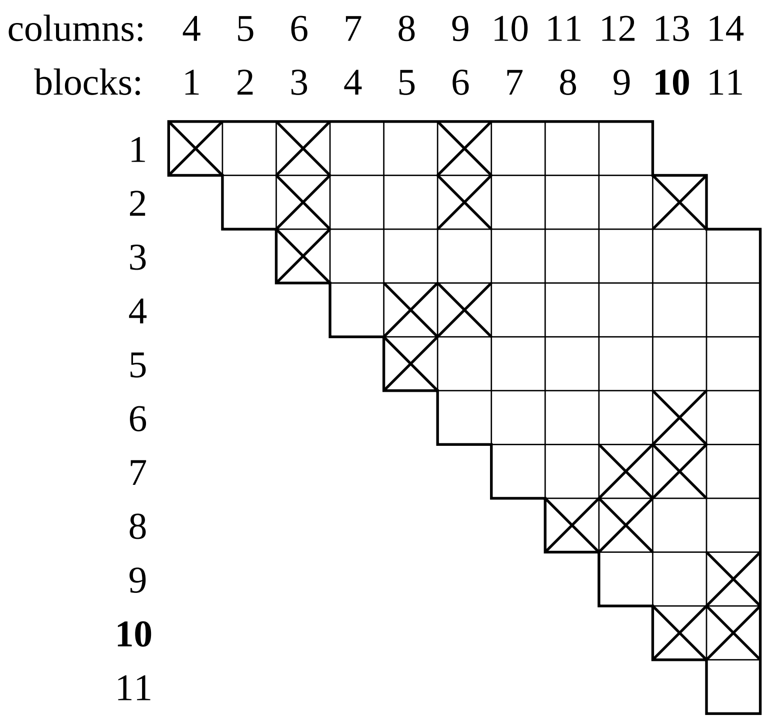
We will construct a bijection between 2-triangulations and pairs of non-crossing Dyck paths.

Given a 2-triangulation, first we give an algorithm to color half of the crosses blue and the other half red.



At each iteration, one cross will be colored red and another blue, and two blocks will be merged.

The bijection (part I): coloring stage



$r = 10$

Repeat until all crosses have been colored:

- Let r be the largest index so that row r has a cross in block r.
- Color blue the leftmost uncolored cross in block r.
- Merge blocks $r - 2$ and $r - 1$.
  (If $r = 2$, we consider that block 1 disappears when it is merged with “block 0".)
- Color red the rightmost uncolored cross in the merged block.

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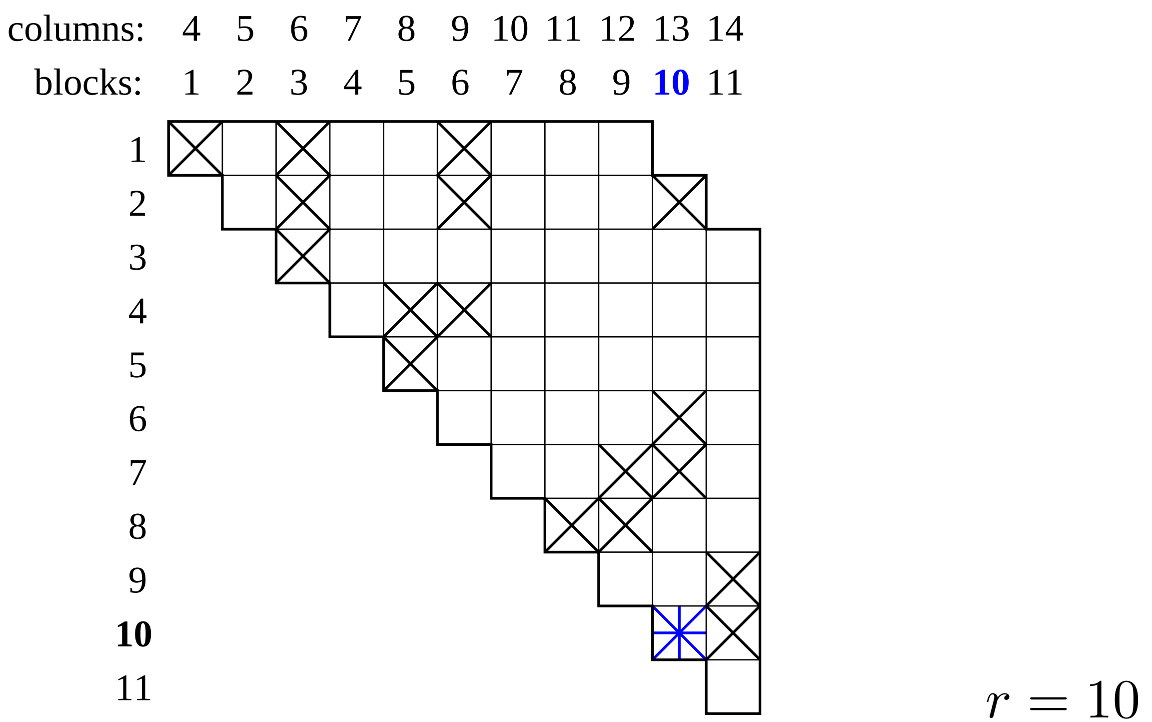

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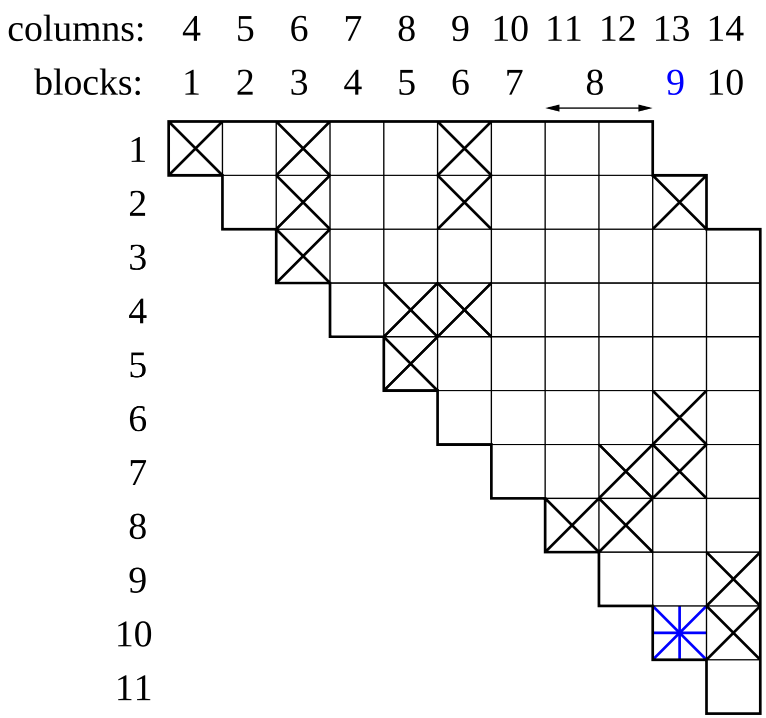

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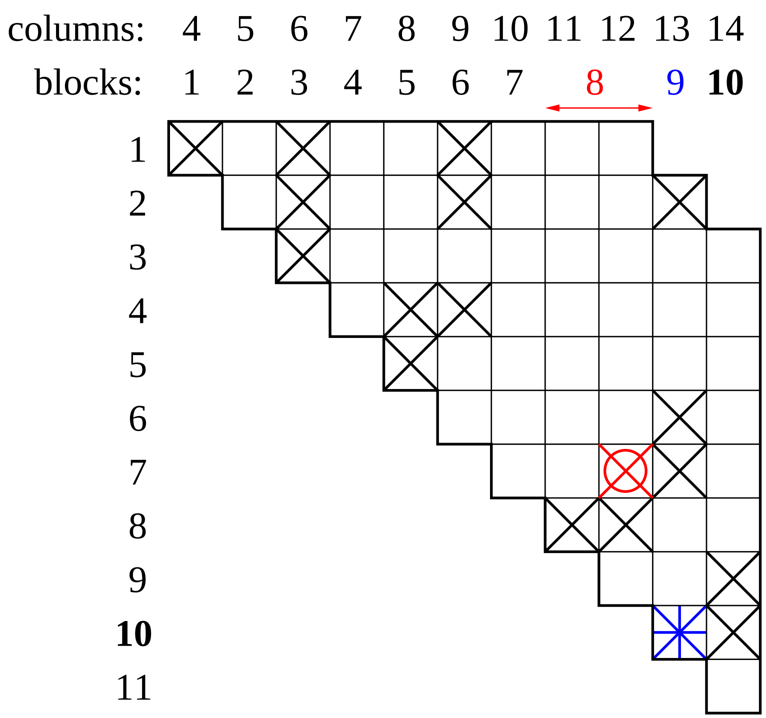

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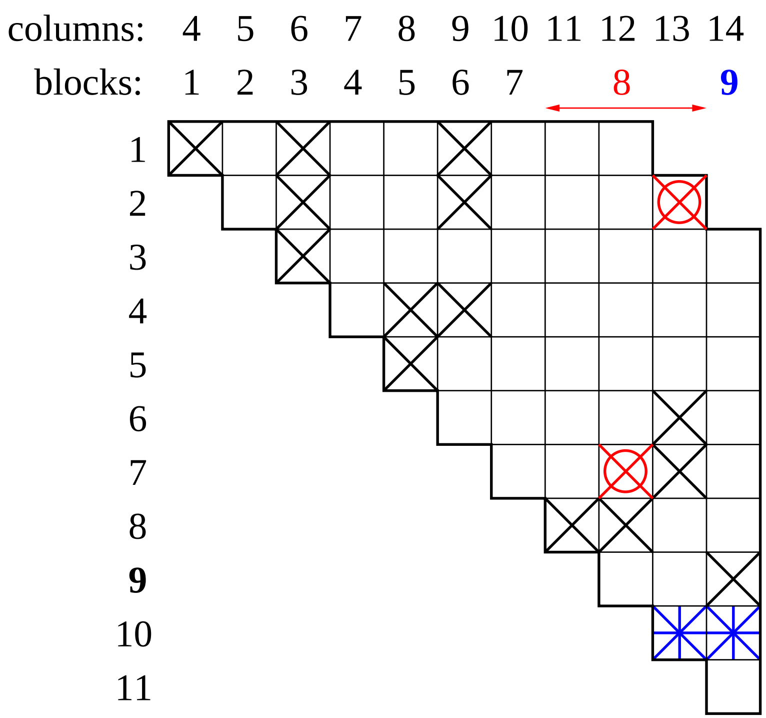

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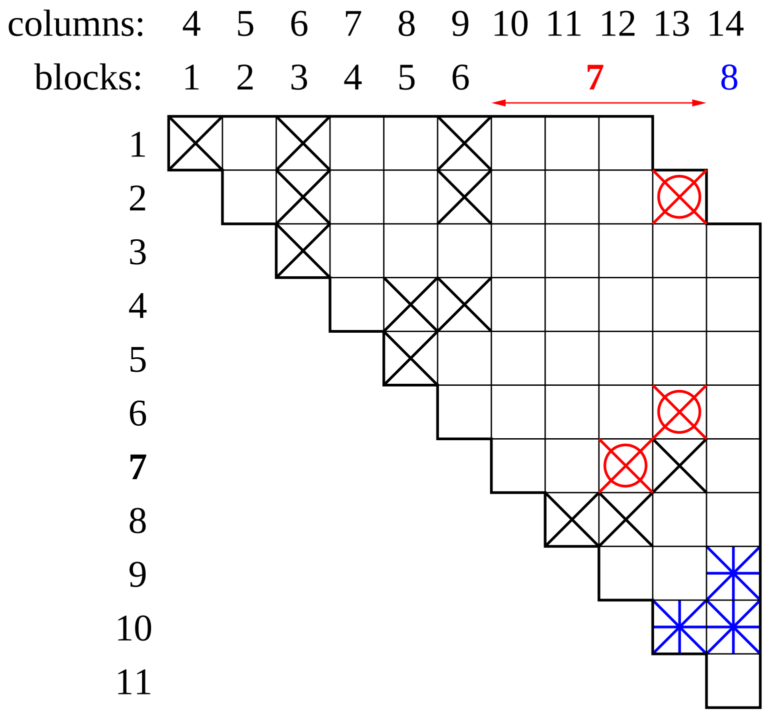

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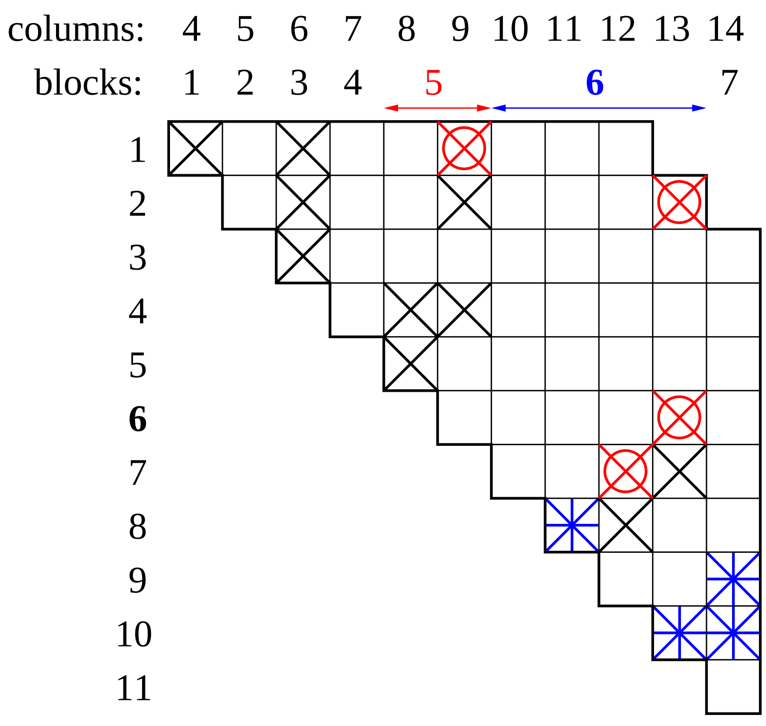

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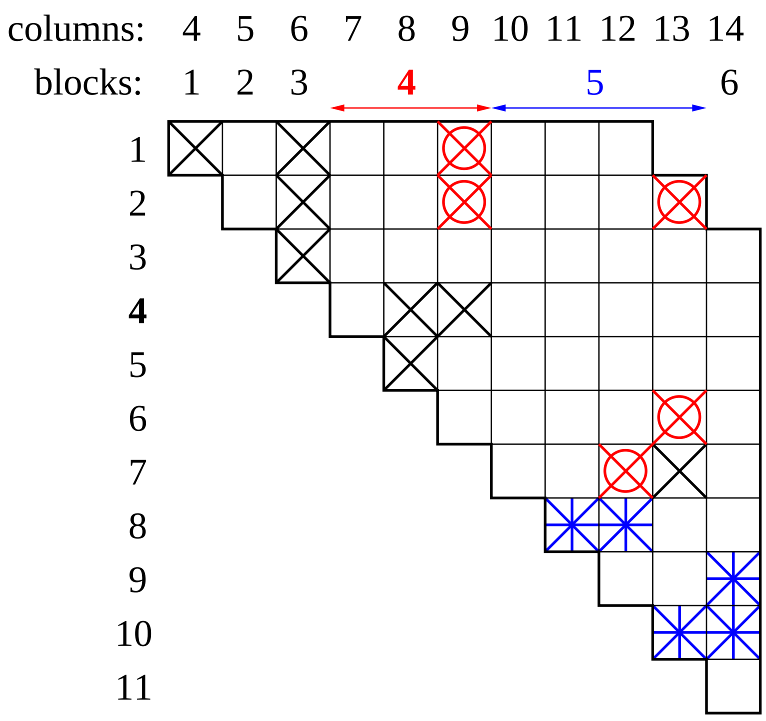

$r = 4$

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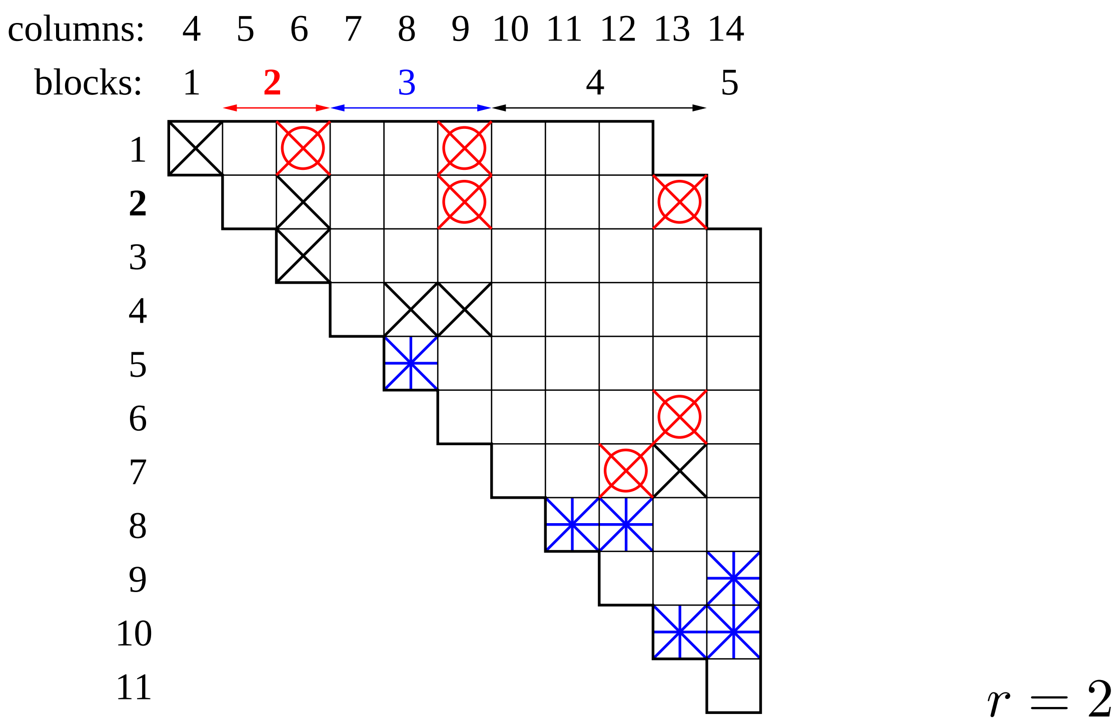

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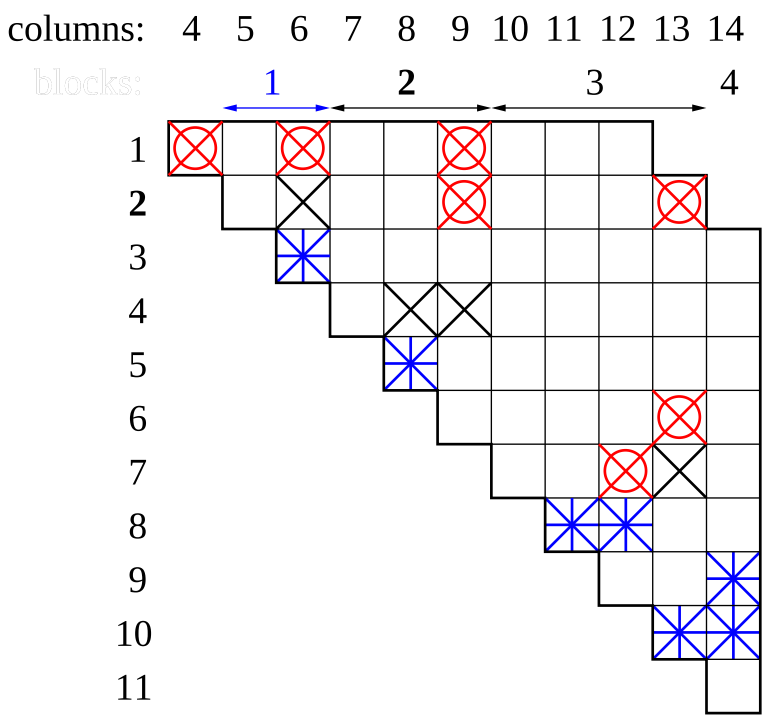

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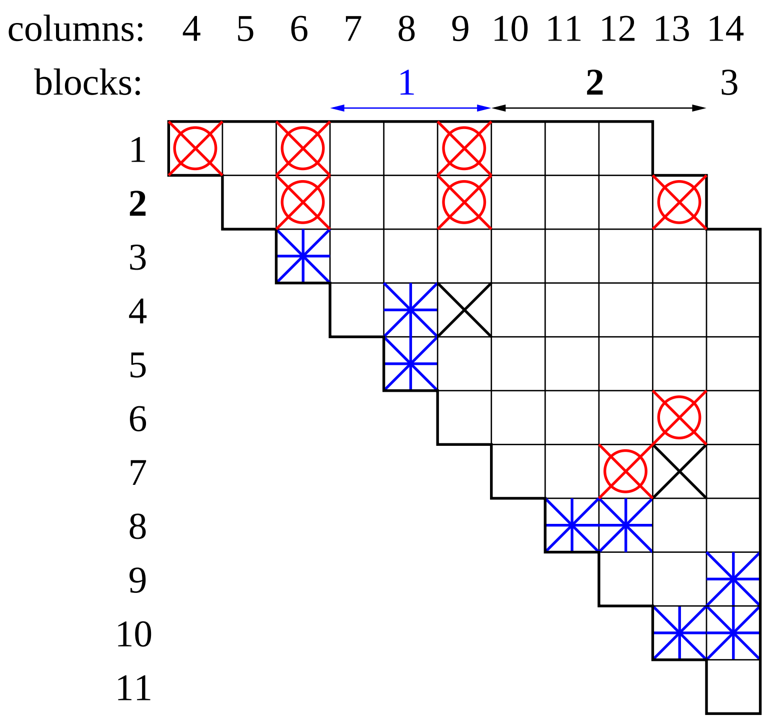

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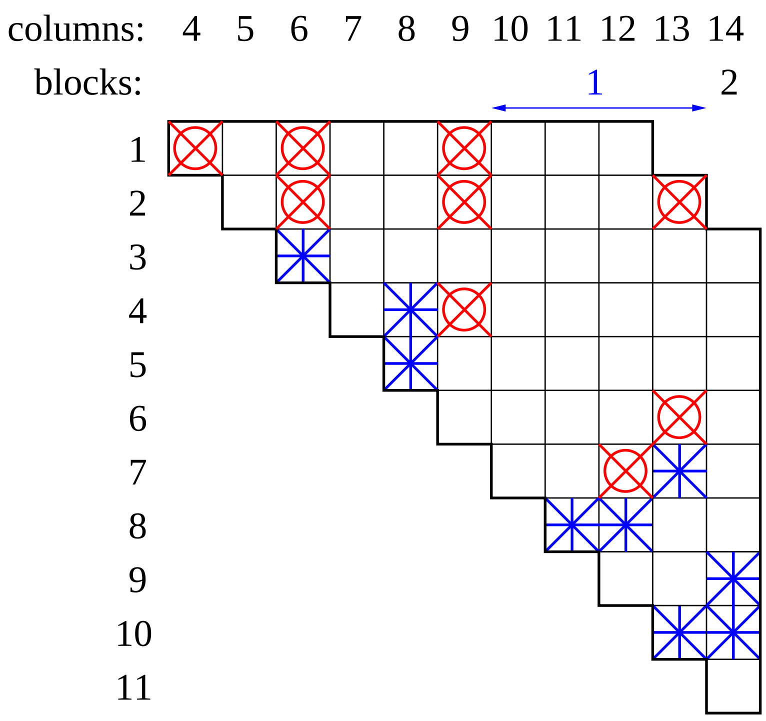

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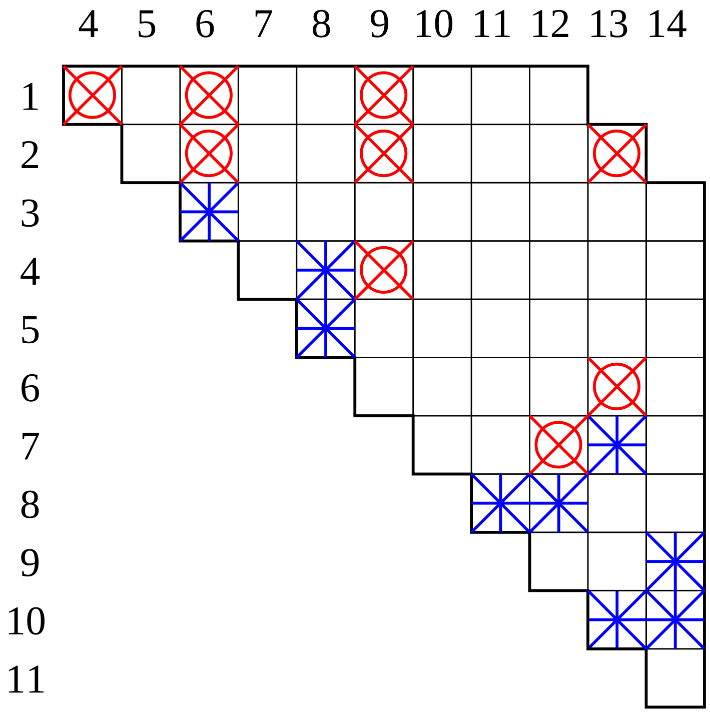
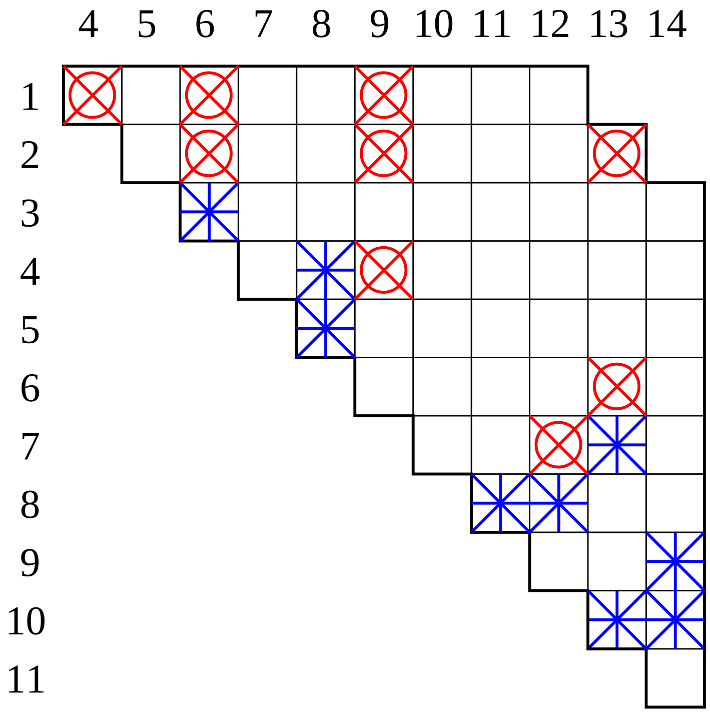
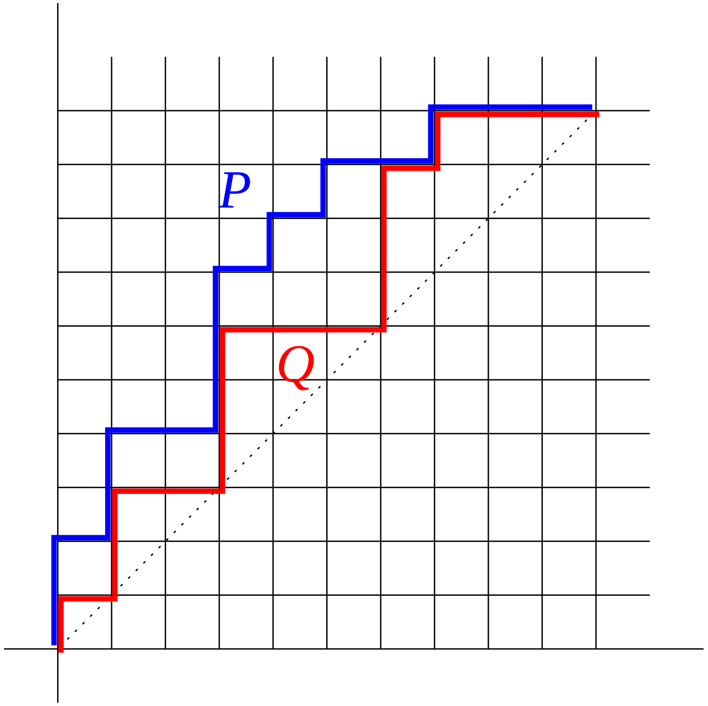
The bijection (part II): from colored crosses to paths



α_i := $\#$ blue crosses in column i
β_i := $\#$ red crosses in column i

The bijection (part II): from colored crosses to paths





$\alpha_i :=$ # blue crosses in column i

$\beta_i :=$ # red crosses in column i

Define

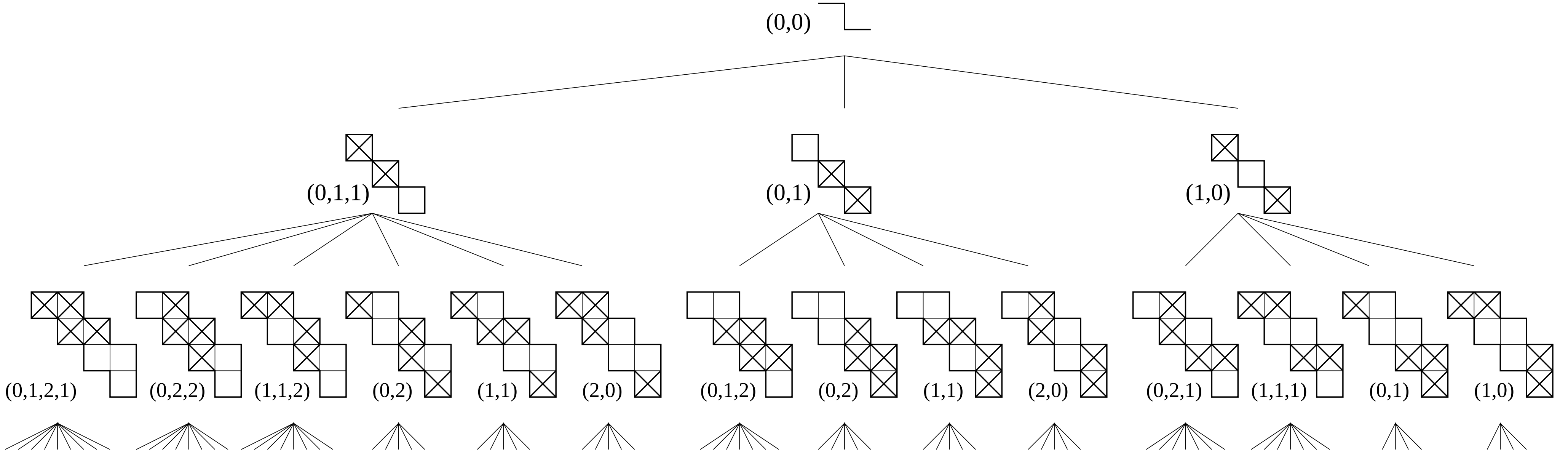
$$P = NE^{\alpha_5}NE^{\alpha_6}\cdots NE^{\alpha_{n-1}}NE^{\alpha_n}E$$

$$Q = NE^{\beta_4}NE^{\beta_5}\cdots NE^{\beta_{n-2}}NE^{\beta_{n-1}}E$$

A generating tree for 2-triangulations

How the bijection is obtained:

- Construct a generating tree for 2-triangulations.
- Construct a generating tree for pairs of non-crossing Dyck paths.
- Give an isomorphism between the generating trees.



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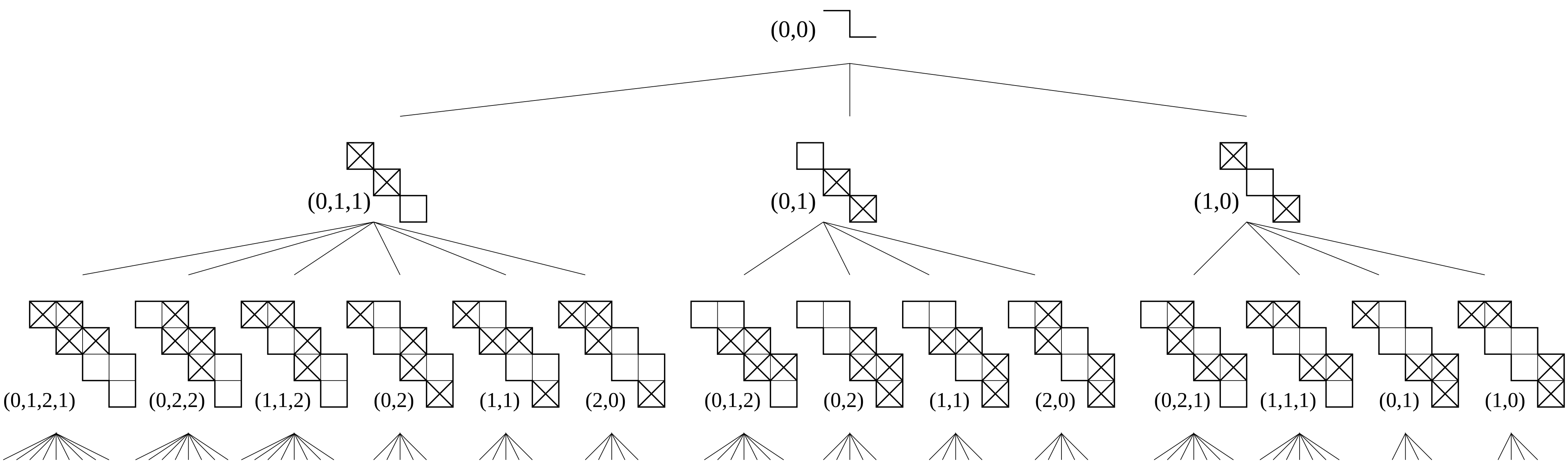
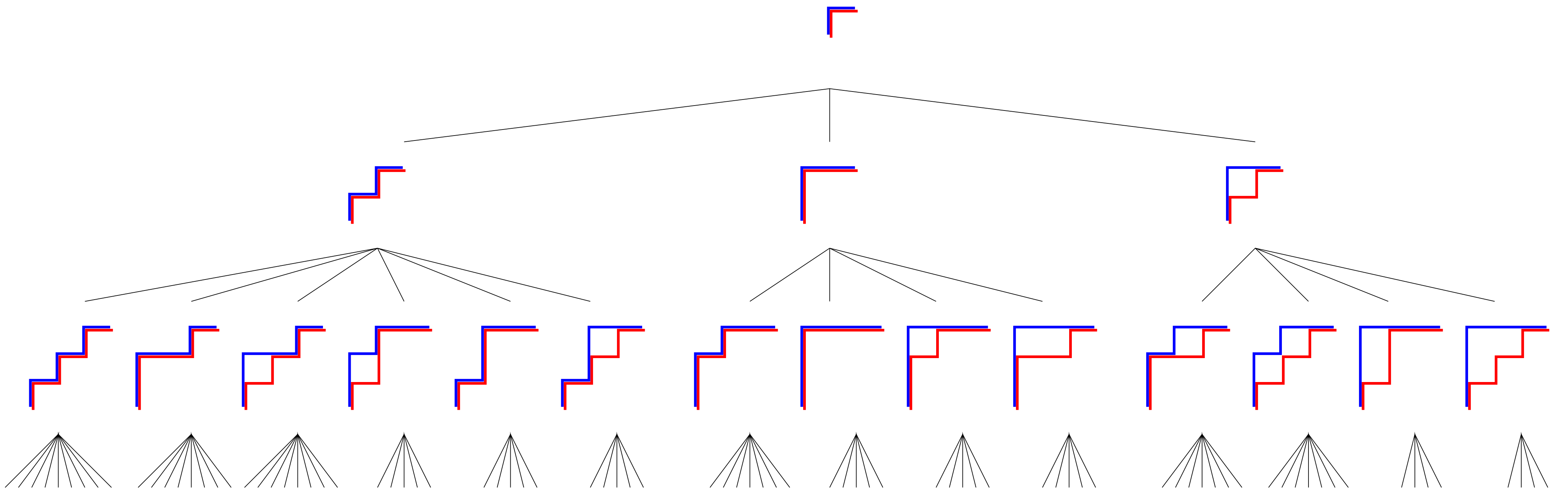


Generating rule:

$$(d_1, d_2, \ldots, d_s) \longrightarrow \begin{array}{l} \{(i, d_j - i + 1, d_{j+1} + 1, d_{j+2}, \ldots, d_s) \;:\; 1 \leq j \leq s-1,\ 0 \leq i \leq d_j\} \\ \cup\, \{(i, d_s - i + 1) \;:\; 0 \leq i \leq d_s + 1\}. \end{array}$$

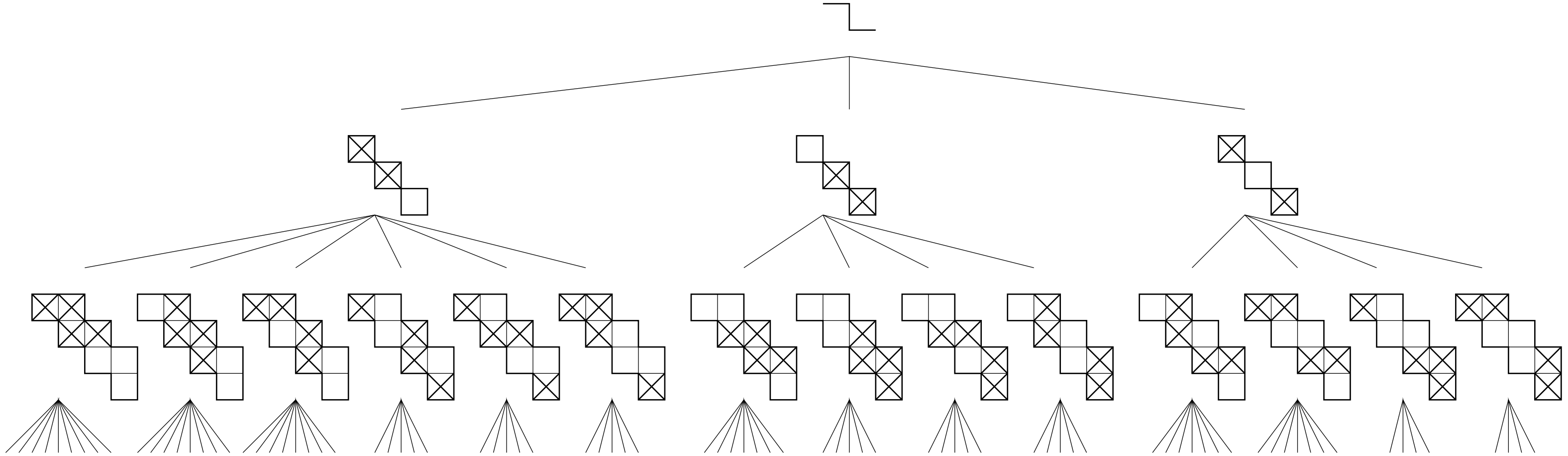
A generating tree for pairs of non-crossing Dyck paths

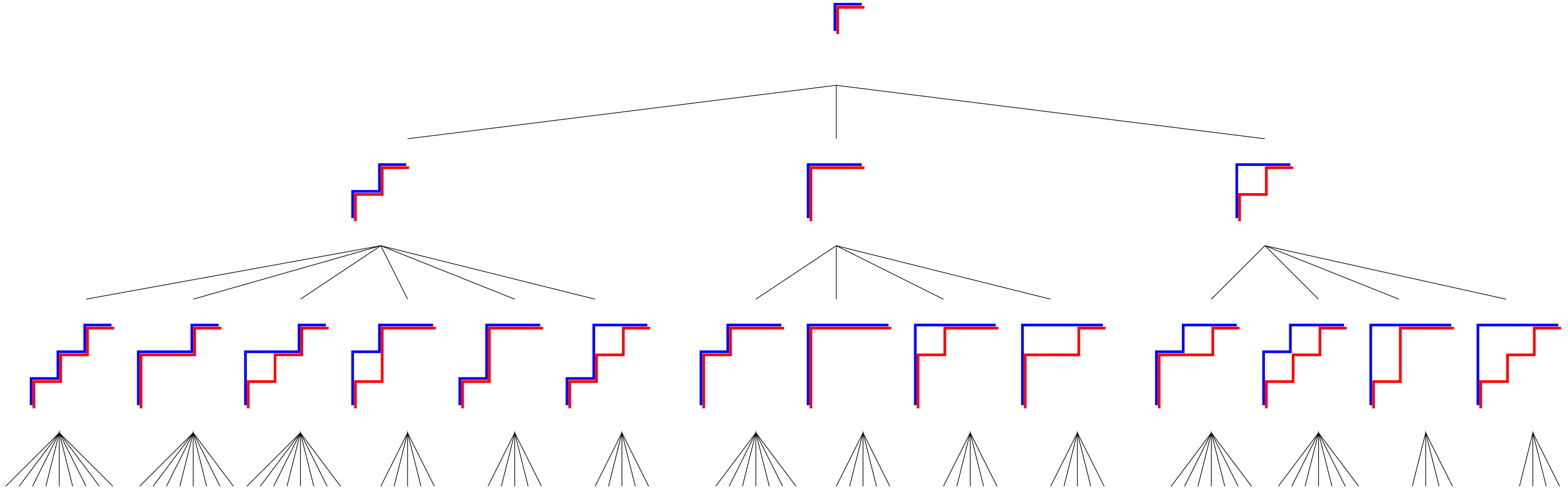


The nodes at level ℓ represent pairs of paths of size $\ell + 1$.

The two generating trees are isomorphic

We can give generating rules for 2-triangulations and for pairs of Dyck paths that yield isomorphic generating trees.





The two generating trees are isomorphic

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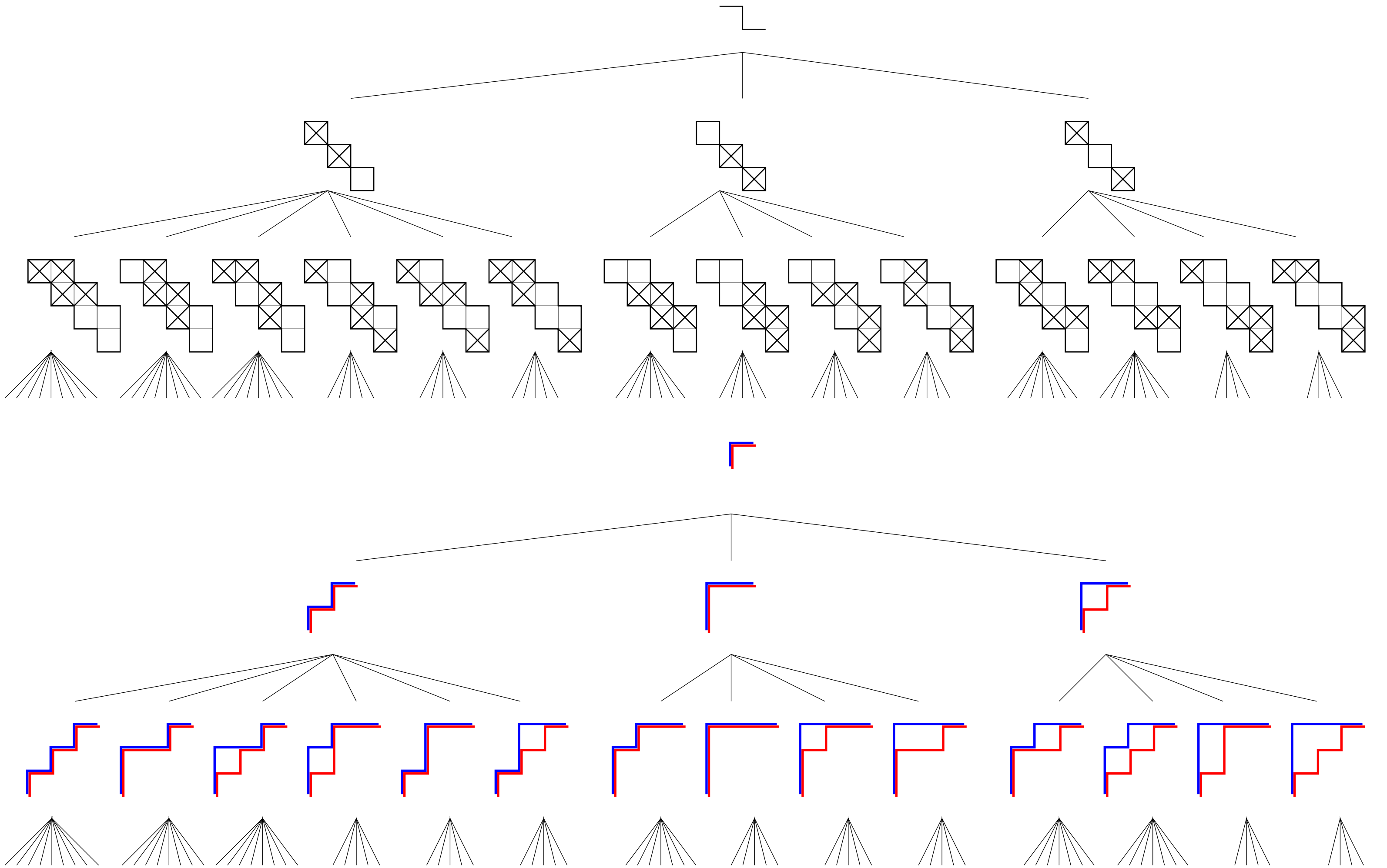
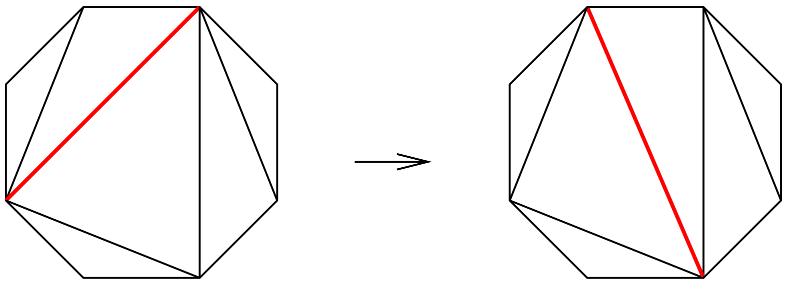


The bijection we described is the one induced by the isomorphism of generating trees.

Open problem 1: Polytope of k-triangulations

For 1-triangulations, we have *diagonal flips*:



There is a polytope, the **associahedron**, whose vertices correspond to 1-triangulations and whose edges correspond to diagonal flips.

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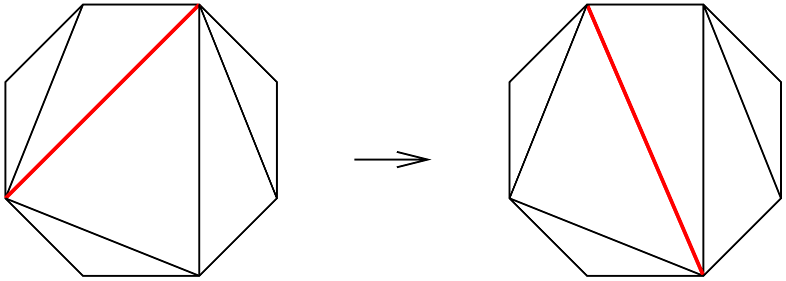

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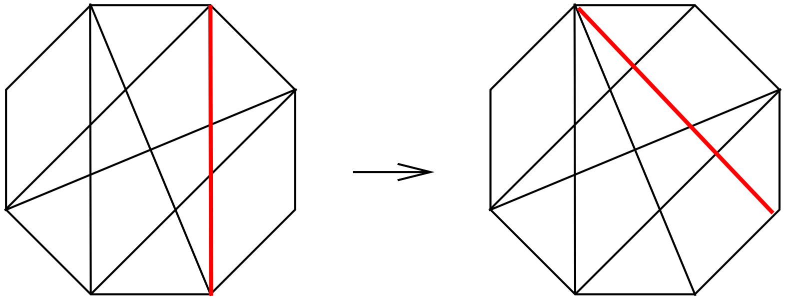
For k-triangulations, we also have *diagonal flips*:



Fact: If we remove any diagonal, there a unique way to put it back to get another k-triangulation.

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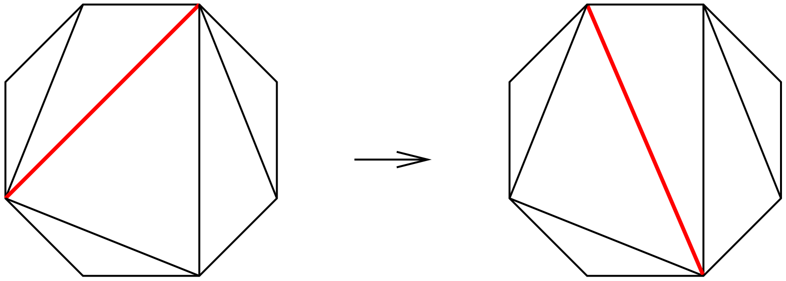

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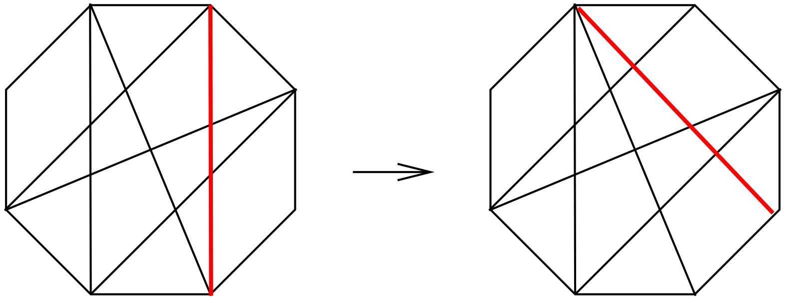


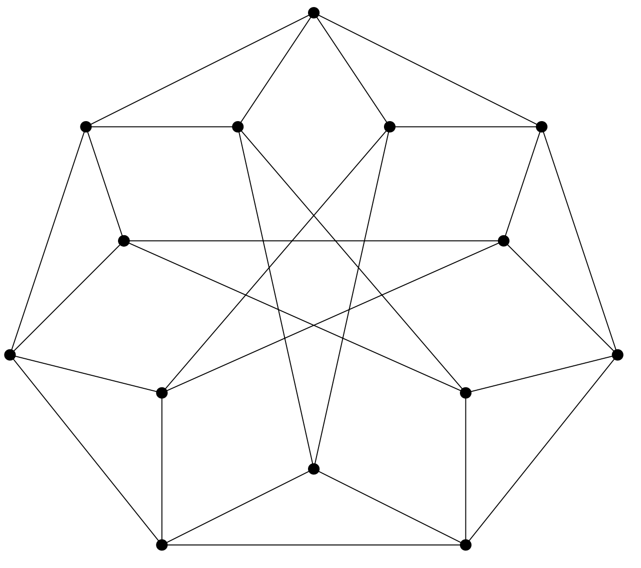
Fact: If we remove any diagonal, there a unique way to put it back to get another k-triangulation.

Open problem: Is there a polytope whose vertices correspond to k-triangulations and whose edges correspond to diagonal flips?

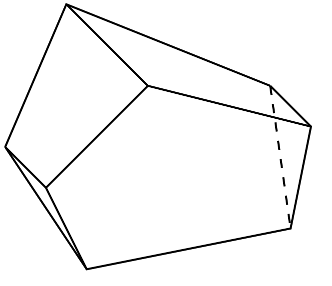
Open problem 1: Polytope of k-triangulations

It should be a simple polytope of dimension $k(n-2k-1)$.

For example, for $k=2$ and $n=7$, the graph of diagonal flips is



It can be realized as a a cyclic polytope in dimension 4, whose 3-dimensional facets are



Open problem 2: Bijection for arbitrary k

Is there an analogous bijection between k-triangulations and k-tuples of non-crossing Dyck paths, for $k \geq 3$?

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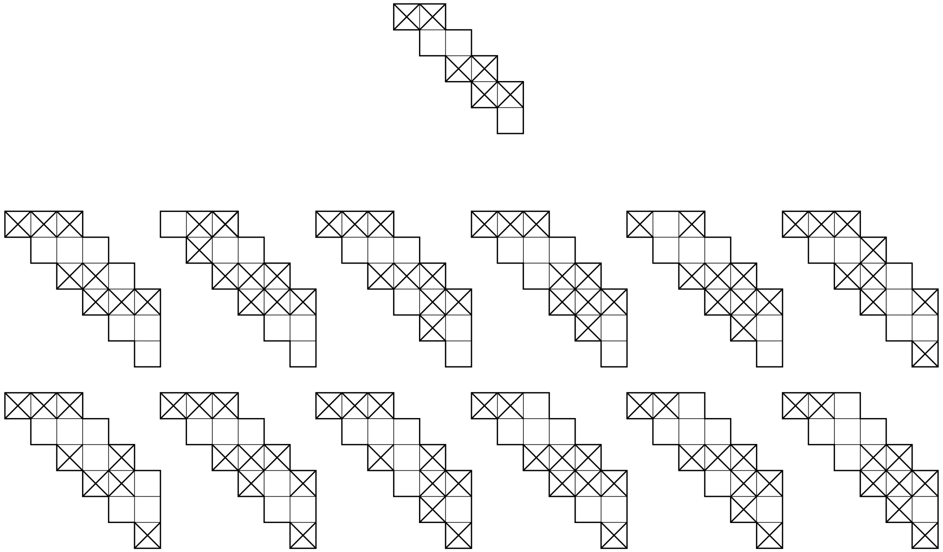
Partial progress:



- The same idea of splitting columns can be used to construct a generating tree for k-triangulations.
- However, it is not clear what is the corresponding operation to generate children of a k-tuple of Dyck paths that would give an isomorphic generating tree.