Noncommutative Monomial Symmetric Functions

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Classical Quasi-symmetric Noncommutative

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where all $K_{\boldsymbol{\lambda}\boldsymbol{\kappa}} \in \mathbb{N}$	are all K_{IJ} also nonnegative?

- Is there a noncommutative analog of of Cauchy identity and a corresponding scalar product?

$$\sum_{\boldsymbol{\lambda}} m_{\boldsymbol{\lambda}}(x) h_{\boldsymbol{\lambda}}(y) = \sum_{\boldsymbol{\lambda}} s_{\boldsymbol{\lambda}}(x) s_{\boldsymbol{\lambda}}(y) = \sum_{\boldsymbol{\lambda}} z_{\boldsymbol{\lambda}}^{-1} p_{\boldsymbol{\lambda}}(x) p_{\boldsymbol{\lambda}}(y)$$

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Compositions, their Reverses and Conjugates.

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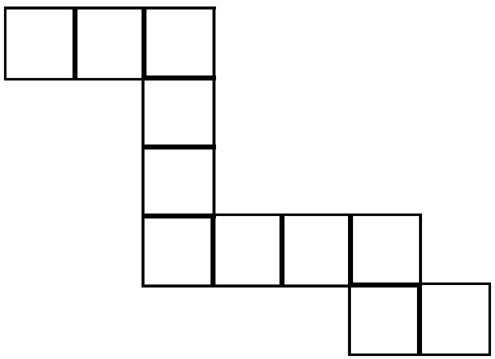
$$I = (3, 1, 1, 4, 2),\ |I| = 11,\ \ell(I) = 5$$

$I =$



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A composition is ordered set of integers: $I = (i_1, \ldots, i_n)$. The sum of all parts is denoted by $|I|$, and the number of parts – by $\ell(I)$. For a composition I define a **reverse** composition $\overline{I} = (i_n, \ldots, i_1)$.

For instance, if $I = (3, 1, 1, 4, 2)$, then $\overline{I} = (2, 4, 1, 1, 3)$.

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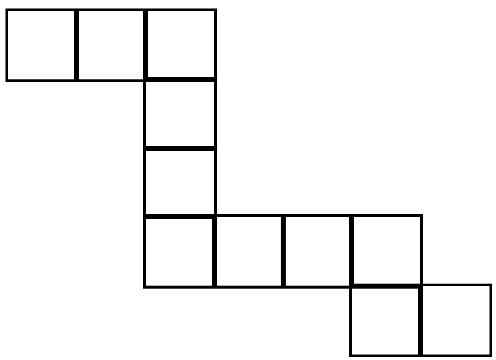
A composition is ordered set of integers: $I = (i_1, \ldots, i_n)$. The sum of all parts is denoted by $|I|$, and the number of parts – by $\ell(I)$. Parts of a **conjugate** composition $\widetilde{I}$ can be read from the diagram of the composition I from left to right and from bottom to top:

$$I = (3, 1, 1, 4, 2) \quad \widetilde{I} = (1, 2, 1, 1, 4, 1, 1)$$

$I =$



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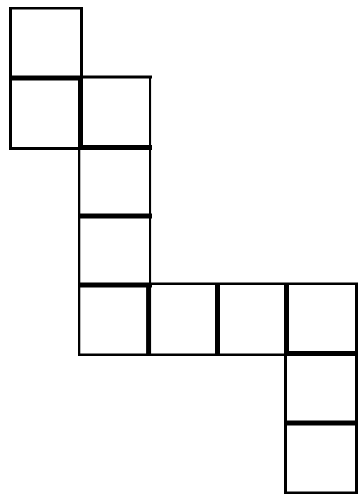

Reverse refinement order.

Let $I=(i_1,\ldots,i_n), J=(j_1,\ldots,j_k)$, $|J|=|I|$ then I is said to be greater in the **reverse refinement order** (or, simply, **finer**) than J,

$$I \succ J$$

if every part of J can be obtained by summing some consecutive parts of I:

$$J=(i_1+\ldots+i_{p_1},\ldots,i_{p_{s-1}+1}+\ldots+i_{p_s},\ldots,i_{p_{k-1}+1}+\ldots+i_n)$$

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For instance, $(3, 3, 2) = (3, 1 + 2, 2) \prec (3, 1, 2, 2)$,

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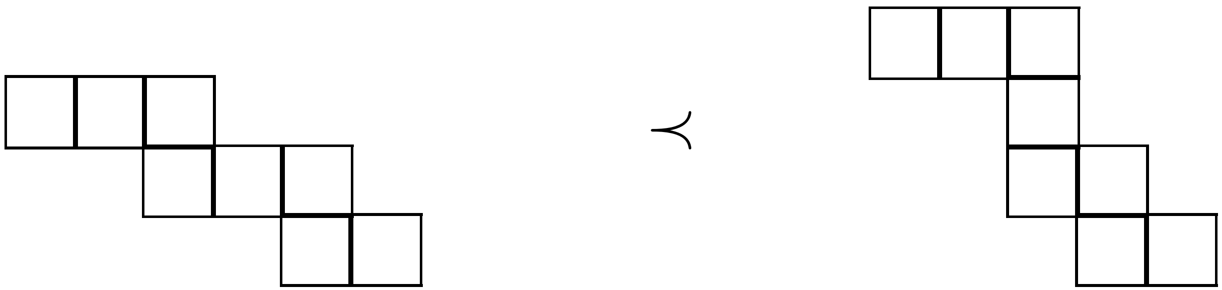

Two Multiplications.

For two compositions $I = (i_1, \ldots, i_{r-1}, i_r)$ and $J = (j_1, j_2, \ldots, j_s)$ one defines two multiplications

$$I \triangleright J = (i_1, \ldots, i_{r-1}, i_r + j_1, j_2, \ldots, j_s),$$

with $\ell(I \triangleright J) = \ell(I) + \ell(J) - 1$

and

$$I \cdot J = (i_1, \ldots, i_r, j_1, \ldots, j_s),$$

with $\ell(I \cdot J) = \ell(I) + \ell(J)$

Quasideterminants.

A quasideterminant (with respect to the bottom left element) of an almost-triangular matrix with free entries a_{ij} and commutative off-diagonal entries b_j is a sum of all weighted paths starting at the bottom row, ending at the first column, taking north $\uparrow$ and east $\leftarrow$ steps and making eastward turns only at the off-diagonal entries.

$$\begin{vmatrix} a_{11} & b_1 & 0 \\ a_{21} & a_{22} & b_2 \\ \boxed{a_{31}} & a_{32} & a_{33} \end{vmatrix}$$

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Noncommutative Symmetric Functions: plan of the review.

Everything is in place to introduce the object of interest : the algebra of noncommutative symmetric functions **NSym**.

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In the original paper, (Gelfand, Krob, Lascoux, Leclerc, Retakh, Thibon (1994)), took noncommutative elementary symmetric functions as generators of **NSym**.

Consider a set of non-commutative power sums (of the first kind) $\Psi^I = \Psi_{i_1} \cdot \ldots \cdot \Psi_{i_n}$ as generators.

As a particular realization, one can consider a (possibly infinite) set of non-commuting variables: $x_1, \ldots, x_n, \ldots$.

Then

$$\Psi_n = \sum_i x_i^n$$

(In particular when all variables are declared to be commutative, $\Psi_n \to p_n$.)

NSym: elementary and homogeneous.

Define **elementary symmetric** functions Λ_n:

$$\Lambda_n = \frac{(-1)^{n-1}}{n} \begin{vmatrix} \Psi_1 & 1 & 0 & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \Psi_{n-1} & \Psi_{n-2} & \dots & \dots & n-1 \\ \boxed{\Psi_n} & \Psi_{n-1} & \dots & \dots & \Psi_1 \end{vmatrix}$$

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and **complete symmetric** functions S_n:

$$S_n = \frac{1}{n} \begin{vmatrix} \Psi_1 & -(n-1) & 0 & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \Psi_{n-1} & \Psi_{n-2} & \dots & \dots & -1 \\ \boxed{\Psi_n} & \Psi_{n-1} & \dots & \dots & \Psi_1 \end{vmatrix}$$

NSym: ribbon Schur functions.

For every composition $I = (i_1, \ldots, i_n)$ ribbon Schur functions are defined as

$$R_I = (-1)^{\ell(I)-1} \begin{vmatrix} S_{i_n} & 1 & 0 & \ldots & \ldots \\ S_{i_n+i_{n-1}} & S_{i_{n-1}} & 1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ S_{i_n+\ldots+i_2} & S_{i_{n-1}+\ldots+i_2} & \ldots & S_{i_2} & 1 \\ \boxed{S_{i_n+\ldots+i_1}} & S_{i_{n-1}+\ldots+i_1} & \ldots & \ldots & S_{i_1} \end{vmatrix}$$

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Remarkably, the multiplication of ribbon Schur is very simple:

$$R_I \cdot R_J = R_{I.J} + R_{I \triangleright J}$$

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In the noncommutative setting can also introduce a map ω such that:

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Under this map,

$$\omega\left(\Lambda_n\right) = S_n$$

and

$$\omega\left(R_I\right) = R_{I^\sim}$$

Noncommutative Monomial and Fundamental Functions.

At this point I would like introduce new personae in **NSym**:

- Noncommutative monomial (and forgotten) symmetric functions.

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Noncommutative Monomial Symmetric Functions.

Define **noncommutative monomial symmetric function** corresponding to a composition $I = (i_1, \ldots, i_n)$ as a quasideterminant of an n by n matrix:

$$M^I = \frac{(-1)^{n-1}}{n} \begin{vmatrix} \Psi_{i_n} & 1 & 0 & \ldots & 0 & 0 \\ \Psi_{i_{n-1}+i_n} & \Psi_{i_{n-1}} & 2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Psi_{i_2+\ldots+i_n} & \ldots & \ldots & \ldots & \Psi_{i_2} & n-1 \\ \boxed{\Psi_{i_1+\ldots+i_n}} & \ldots & \ldots & \ldots & \Psi_{i_1+i_2} & \Psi_{i_1} \end{vmatrix}$$

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where n is the length of I. In particular

$$M^{1^n} = \Lambda_n$$

where Λ_n is an elementary symmetric function.

Noncommutative Forgotten Symmetric Functions.

Also define **noncommutative forgotten symmetric function** corresponding to a composition $I = (i_1, \ldots, i_n)$ as an n by n quasideterminant:

$$F^I = \frac{1}{n} \begin{vmatrix} \Psi_{i_n} & -(n-1) & 0 & \ldots & 0 & 0 \\ \Psi_{i_{n-1}+i_n} & \Psi_{i_{n-1}} & -(n-2) & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Psi_{i_2+\ldots+i_n} & \ldots & \ldots & \ldots & \Psi_{i_2} & -1 \\ \boxed{\Psi_{i_1+\ldots+i_n}} & \ldots & \ldots & \ldots & \Psi_{i_1+i_2} & \Psi_{i_1} \end{vmatrix}$$

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In particular

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where S_n a homogeneous symmetric function.

Properties of Monomial Symmetric Functions.

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Commutative limit of M^I, i.e. $\Psi_n \to p_n$:

$$m_{\boldsymbol{\lambda}} = \sum_{I=\sigma(\boldsymbol{\lambda})} M^I,$$

where the sum is over all distinct permutations of parts of $\boldsymbol{\lambda}$.

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For every composition I one can define, by analogy with Gessel's fundamental quasi-symmetric functions, **fundamental noncommutative** symmetric function

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Multiplication of fundamental symmetric functions when $I = (n)$ (or dually $J = 1^n$):

$$L^n \cdot L^J = \sum_{M \succeq J} \binom{n + \ell(J) - 1}{\ell(M)} L^{n \cdot M} + \binom{n + \ell(J) - 1}{\ell(M) - 1} L^{n \triangleright M}$$

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Noncommutative Kostka numbers are nonnegative integers.

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Example.

$$R_{k,1^r} = \binom{k+r-1}{r} \sum_{|I|=k} M^{I\cdot 1^r}$$

Sketch of calculation: Consider the expansion in M^I of $S_n \Lambda_r$ and the fact that $S_n \Lambda_r = R_{k,1^r} + R_{k+1,1^{r-1}}$.

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Conjecture 2
Kostka-Gessel numbers are nonnegative integers.

Note 1: Only nonnegativity in the above conjectures requires proof. The fact that these numbers are integers follows from rules of multiplication of M^Is and L^Is respectively.
Note 2: Nonnegativity of Kostka-Gessel numbers implies that of noncommutative Kostka numbers.

Examples of expansions of R_I in L^J.

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Sketch of the calculation:

$$R_{nm} = S_n S_m - S_{n+m} = L^n \cdot L^m - L^{n \triangleright m}$$

Noncommutative Cauchy identity and pairing.

Proposition.
Given two noncommutative alphabets X and Y, the following identity is true:

$$\sum_I M^I(X)S^I(Y) = \sum_I L^I(X)R_I(Y)$$

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Then define a noncommutative pairing in **NSym** by requiring that M^I and S^I are dual to each other.

$$\langle M^I | S^J \rangle = \delta_{IJ},$$

it follows that

$$\langle L^I | R_J \rangle = \delta_{IJ}$$

Some properties of the pairing.

Furthermore, ω is an isometry, i.e. for any two functions $H, G \in$ **NSym**

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where p_k are such that for each M

$$M = (i_1 + \ldots + i_{p_1}, \ldots, i_{p_{k-1}+1} + \ldots + i_{p_k}, \ldots, i_{p_s} + \ldots + i_n)$$

In particular

$$\langle \Psi^I | \Psi^I \rangle = \left(\prod_{k=1}^{\ell(I)} i_k \right) \ell(I)!$$

Kostka and Kostka-Gessel numbers and pairing.

With help of the pairing, both of conjectures about nonnegativity of noncommutative Kostka numbers and Kostka-Gessel numbers can be restated as follows:

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Conjecture 2.

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A noncommutative identity.

In the Exercise 10, Ch. I, §5 of Macdonald, it is shown that

$$\sum_{|\boldsymbol{\lambda}|=n} X^{\ell(\boldsymbol{\lambda})-1} m_{\boldsymbol{\lambda}} = \sum_{k=0}^{n-1} s_{n-k,1^k} \left(X-1\right)^k$$

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$$\Psi_n = \sum_{k=0}^{n-1} (-1)^k R_{1^k,n-k} \text{ and } \sum_{|I|=n} L^I = \sum_{k=0}^{n-1} R_{1^k n-k}$$

at $X=0$ (GKLLRT, 1994); at $X=2$ (B.-C.-V. Ung, 1998)

	Classical	Quasi-symmetric	Noncommutative
monomial	$m_{\boldsymbol{\lambda}}$	M_I	M^I
power sums	p_n and $p_{\boldsymbol{\lambda}}$		Ψ_n and Ψ^I
elementary	e_n and $e_{\boldsymbol{\lambda}}$		Λ_n and Λ^I
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- Combinatorial and representation-theoretic interpretation of noncommutative Kostka and Kostka-Gessel numbers.
- Recall that in the classical theory, $s_{\boldsymbol{\lambda}} = \sum_{\boldsymbol{\kappa} \leq \boldsymbol{\lambda}} m_{\boldsymbol{\kappa}}$, where $\geq$ is the dominance order.

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- Is there a notion of order that stipulates which M^I (or/and L^I) occur in the ribbon Schur expansion?
- What are proper generalizations of the noncommutative theory to the q and q,t settings?

Reverse refinement order.

Let $I = (i_1, \ldots, i_n), J = (j_1, \ldots, j_k)$, $|J| = |I|$ then I is said to be no less than J in the **reverse refinement order**,

$$I \succeq J$$

if every part of J can be obtained by summing some consecutive parts of I:

$$J = (i_1 + \ldots + i_{p_1}, \ldots, i_{p_{s-1}+1} + \ldots + i_{p_s}, \ldots, i_{p_{k-1}+1} + \ldots + i_n)$$

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For instance, $(3, 3, 2) \preceq (3, 1, 2, 2)$

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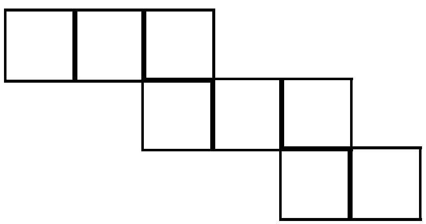
$(3,3,2)\preceq(3,1,2,2)$, $lp((3122),(332))=i_1\cdot i_3\cdot i_4=3\cdot 2\cdot 2$.

$p_1=1$
$p_2=3$
$p_3=4$



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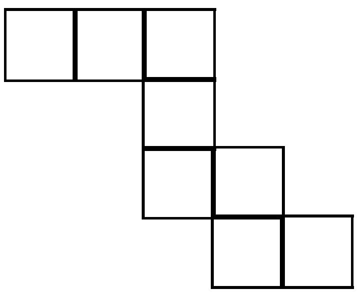

Quasideterminants.

The **quasideterminant** of an $n \times n$ almost triangular matrix with free entries a_{ij} and integers b_i on the off-diagonal is polynomial in its entries and can be written as:

$$Q_n = \begin{vmatrix} a_{11} & b_1 & 0 & \dots & \dots & \dots & \dots \\ a_{21} & a_{22} & b_2 & 0 & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{j1} & a_{j2} & \dots & a_{jj} & b_j & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n-1\ 1} & a_{n-1\ 2} & \dots & \dots & \dots & \dots & b_{n-1} \\ \boxed{a_{n1}} & a_{n2} & \dots & a_{nj} & \dots & \dots & a_{nn} \end{vmatrix} =$$

$$= \sum_{n \geq j_1 > \ldots > j_k > 1} (-1)^{k+1} a_{nj_1} b^{-1}_{j_1 - 1} a_{j_1 - 1\ j_2} b^{-1}_{j_2 - 1} a_{j_2 - 1\ j_3} \ldots b^{-1}_{j_k - 1} a_{j_k - 1\ 1}$$

An identity between quasideterminants.

There is an identity between a quasideterminant of a matrix with off-diagonal elements $-b_{n-1}, \ldots, -b_1$ and a sum of quasideterminants of the same matrix with off-diagonal elements $b_1, \ldots, b_{n-1}$.

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$T_J = \prod_{s=1}^{k} T_{j_s}$.
Then the following identity is true

$$\frac{1}{n} Q_n(-(n-1), \dots, -1) = \sum_J \frac{(-1)^{n-\ell(J)-1}}{n-\ell(J)} T_J Q_n(1, \dots, n-1),$$

where the sum is over all subsets $J \subseteq [1, 2, \dots, n-1]$.

Example of the kaleidoscopic identity.

Consider a four by four quasideterminant Q_4 and its kaleidoscopic expansion:

$$\frac{1}{4}\begin{vmatrix} a_{11} & -3 & 0 & 0 \\ a_{21} & a_{22} & -2 & 0 \\ a_{31} & a_{32} & a_{33} & -1 \\ \boxed{a_{41}} & a_{42} & a_{43} & a_{44} \end{vmatrix} = \left(-\frac{1}{4}T_\emptyset + \frac{1}{3}(T_1+T_2+T_3) - \frac{1}{2}(T_1T_2+T_1T_3+T_2T_3) + \right.$$

$$\left. + T_1T_2T_3T_4\right) Q_4 = -\frac{1}{4}\begin{vmatrix} a_{11} & 1 & 0 & 0 \\ a_{21} & a_{22} & 2 & 0 \\ a_{31} & a_{32} & a_{33} & 3 \\ \boxed{a_{41}} & a_{42} & a_{43} & a_{44} \end{vmatrix} + \frac{1}{3}\begin{vmatrix} a_{21} & 1 & 0 \\ a_{31} & a_{33} & 2 \\ \boxed{a_{41}} & a_{43} & a_{44} \end{vmatrix} + \frac{1}{3}\begin{vmatrix} a_{11} & 1 & 0 \\ a_{31} & a_{32} & 2 \\ \boxed{a_{41}} & a_{42} & a_{44} \end{vmatrix} +$$

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Quasideterminants in the commutative limit:

If one sets all a_{ij} to be commutative, then the quasideterminant becomes a ratio of the determinant of the same matrix to the minor obtained by crossing out the first column and the last row:

$$Q_n = \begin{vmatrix} a_{11} & b_1 & 0 & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{j1} & a_{j2} & \dots & b_j & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \boxed{a_{n1}} & \dots & \dots & \dots & a_{nn} \end{vmatrix} = \frac{\begin{vmatrix} a_{11} & b_1 & 0 & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{j1} & a_{j2} & \dots & b_j & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \dots & \dots & \dots & a_{nn} \end{vmatrix}}{\prod_{i=1}^{n-1} b_i}$$

Commutative limit of M^I.

Recall that there is an imbedding of $\mathbf{Sym} \hookrightarrow \mathbf{QSym}$, in particular

$$m_{\boldsymbol{\lambda}} = \sum_{I \sim \boldsymbol{\lambda}} M_I,$$

where $I \sim \boldsymbol{\lambda}$ means all compositions I that can be obtained by permuting parts of a partition $\boldsymbol{\lambda}$.
In the commutative limit, the sum of M^I with $I \sim \boldsymbol{\lambda}$ goes over to the **augmented monomial** symmetric function:

$$u_{\boldsymbol{\lambda}} m_{\boldsymbol{\lambda}} = \sum_{I=\sigma(\boldsymbol{\lambda})} M^I,$$

where $u_{\boldsymbol{\lambda}} = \prod_{i \geq 1} m_i(\boldsymbol{\lambda})!$ with $m_i(\boldsymbol{\lambda})$ being the number of parts of $\boldsymbol{\lambda}$ equal to i and the sum is over **all** permutations of parts of $\boldsymbol{\lambda}$.

The third part of Cauchy identity.

$$M^I = \sum_{J \preceq I} \frac{(-1)^{\ell(I)-\ell(J)}}{\prod_{k=0}^{s-1}(\ell(I) - p_k)} \Psi^J, \text{ where } s = \ell(J)$$

And

$$S^I = \sum_{K \succeq I} \frac{1}{\pi_u(K, I)} \Psi^K$$

Therefore

$$\sum_I M^I(X) S^I(Y) = \sum_{I,\ K \succeq I \succeq J} \frac{(-1)^{\ell(I)-\ell(J)}}{\prod_{k=0}^{\ell(J)-1}(\ell(I) - p_k)\pi_u(K, I)} \Psi^J(X) \Psi^K(Y)$$

From noncommutative pairing to the Hall scalar product.

$$\sum_I M^I S^I \to \sum_I M^I h^I = \sum_\lambda \left(\sum_{I=\sigma(\boldsymbol{\lambda})} M^I \right) h_\lambda = \sum_{\boldsymbol{\lambda}} m_{\boldsymbol{\lambda}} h_{\boldsymbol{\lambda}}$$