# Non-commutative extensions of classical determinantal identities

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# Overview

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2. Non-commutative determinantal identities
3. Matrix inverse formula
4. MacMahon master theorem
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# Matrix inverse formula

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### Theorem

*For a complex invertible matrix $A = (a_{ij})_{m\times m}$, we have*

$$\left(A^{-1}\right)_{ij} = (-1)^{i+j}\frac{\det A^{ji}}{\det A}.$$

# Matrix inverse formula

We know that

$$(I - A)^{-1} = I + A + A^2 + \dots,$$

so

$$\left((I - A)^{-1}\right)_{ij} = \delta_{ij} + a_{ij} + \sum_k a_{ik} a_{kj} + \dots$$

We can rephrase the matrix inverse formula as follows:

$$\det(I - A) \cdot \left(\delta_{ij} + a_{ij} + \sum_k a_{ik} a_{kj} + \dots\right) = (-1)^{i+j} \det(I - A)^{ji}.$$

# Matrix inverse formula

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Matrix inverse formula says that two power series in $a_{ij}$ are the same, provided that the variables commute.

# MacMahon master theorem

## Theorem (MacMahon 1916)

*Let $A = (a_{ij})_{m\times m}$ be a complex matrix, and let $x_1, \ldots, x_m$ be a set of variables. Denote by $G(\mathbf{r})$ the coefficient of $x_1^{r_1}\cdots x_m^{r_m}$ in*

$$\prod_{i=1}^{m}(a_{i1}x_1 + \ldots + a_{im}x_m)^{r_i}.$$

*Let $t_1, \ldots, t_m$ be another set of variables, and $T = (\delta_{ij}t_i)_{m\times m}$. Then*

$$\sum_{\mathbf{r}\geq\mathbf{0}} G(\mathbf{r})\mathbf{t}^{\mathbf{r}} = \frac{1}{\det(I - TA)}.$$

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The coefficient of $x^2y^0z^2$ in $(y+z)^2(x+z)^0(x+y)^2$ is 1, and the coefficient of $x^2y^3z^1$ in $(y+z)^2(x+z)^3(x+y)^1$ is 3. On the other hand, for

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \qquad T = \begin{pmatrix} t & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & v \end{pmatrix},$$

we have

$$\frac{1}{\det(I - TA)} = \frac{1}{1 - tu - tv - uv - 2tuv} =$$

$$= 1 + \ldots + t^2u^0v^2 + \ldots + 3t^2u^3v^1 + \ldots$$

# MacMahon master theorem

We can take $a_{ij}$ to be variables; each $G(\mathbf{r})$ is then a finite sum of monomials in $a_{ij}$. By taking $t_1 = \ldots = t_m = 1$, MacMahon master theorem gives

$$\sum_{\mathbf{r} \geq \mathbf{0}} G(\mathbf{r}) = \frac{1}{\det(I - A)}.$$

Since $\det(I - A) = 1 - a_{11} - \ldots - a_{mm} + a_{11}a_{22} + \ldots$, the right-hand side is also a power series in $a_{ij}$'s.

# MacMahon master theorem

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# MacMahon master theorem

MacMahon master theorem says that two power series in $a_{ij}$ are the same, provided that the variables commute.

# Sylvester's determinantal identity



## Theorem (Sylvester's identity)

*Let $A = (a_{ij})_{m\times m}$ be a complex matrix; take $n < i, j \le m$ and define*

$$A_0 = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}, a_{i*} = \begin{pmatrix} a_{i1} & \cdots & a_{in} \end{pmatrix}, a_{*j} = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{pmatrix},$$

$$b_{ij} = \det \begin{pmatrix} A_0 & a_{*j} \\ a_{i*} & a_{ij} \end{pmatrix}, \quad B = (b_{ij})_{n+1 \le i,j \le m}$$

*Then*

$$\det A \cdot (\det A_0)^{m-n-1} = \det B.$$

# Sylvester's determinantal identity



If we take $n = 1$ and $m = 3$, the Sylvester's identity says that

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \cdot a_{11} = \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \end{vmatrix}.$$

# Sylvester's determinantal identity

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Sylvester's determinantal identity says that two power series in $a_{ij}$ are the same, provided that the variables commute.

# Non-commutative extensions



- Do these (or similar) identities hold when the variables are not commutative?
- Can we find combinatorial proofs of these identities?
- Can we add parameters and find natural $q$-analogues?

Yes!

(otherwise I would be talking about something else)

# Non-commutative extensions



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- Can we find combinatorial proofs of these identities?
- Can we add parameters and find natural $q$-analogues?

Yes!

(otherwise I would be talking about something else)

# Previous work



- D. Foata, A Noncommutative Version of the Matrix Inversion Formula, *Adv. Math.* 31 (1979), 330–349
- S. Garoufalidis, T. Tq Lê and D. Zeilberger, The Quantum MacMahon Master Theorem, to appear in *Proc. Natl. Acad. of Sci.*
- Yu. I. Manin, Multiparameter quantum deformations of the linear supergroup, *Comm. Math. Phys.* **123** (1989), 163–175

# Non-commutative extensions

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Commutative variables:

$$a_{ik}a_{jl} = a_{jl}a_{ik} \text{ for all } i, j, k, l$$

# Cartier-Foata and right-quantum matrices



Cartier-Foata:

$$\begin{aligned} a_{jl}a_{ik} &= a_{ik}a_{jl} \quad \text{for all } i<j, k<l \\ a_{jl}a_{ik} &= a_{ik}a_{jl} \quad \text{for all } i<j, k>l \\ a_{jk}a_{ik} &= a_{ik}a_{jk} \quad \text{for all } i<j \end{aligned}$$

Right-quantum:

$$\begin{aligned} a_{jk}a_{ik} &= a_{ik}a_{jk} \quad \text{for all } i<j \\ a_{ik}a_{jl} - a_{jk}a_{il} &= a_{jl}a_{ik} - a_{il}a_{jk} \quad \text{for all } i<j, k<l \end{aligned}$$

Cartier-Foata $\Rightarrow$ right-quantum

# $q$-Cartier-Foata and $q$-right-quantum matrices



$q$-Cartier-Foata:

$$\begin{aligned} a_{jl}a_{ik} &= a_{ik}a_{jl} \quad \text{for all } i<j, k<l \\ a_{jl}a_{ik} &= q^2\, a_{ik}a_{jl} \quad \text{for all } i<j, k>l \\ a_{jk}a_{ik} &= q\, a_{ik}a_{jk} \quad \text{for all } i<j \end{aligned}$$

$q$-right-quantum:

$$\begin{aligned} a_{jk}a_{ik} &= q\, a_{ik}a_{jk} \quad \text{for all } i<j \\ a_{ik}a_{jl} - q^{-1}\, a_{jk}a_{il} &= a_{jl}a_{ik} - q\, a_{il}a_{jk} \quad \text{for all } i<j, k<l \end{aligned}$$

$q$-Cartier-Foata $\Rightarrow$ $q$-right-quantum

# **q**-Cartier-Foata and **q**-right-quantum matrices



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**q**-Cartier-Foata:

$$\begin{aligned} a_{jl}a_{ik} &= q_{kl}^{-1}q_{ij}\; a_{ik}a_{jl} \quad \text{for all } \; i<j, k<l \\ a_{jl}a_{ik} &= q_{ij}q_{lk}\; a_{ik}a_{jl} \quad \text{for all } \; i<j, k>l \\ a_{jk}a_{ik} &= q_{ij}\; a_{ik}a_{jk} \quad \text{for all } \; i<j \end{aligned}$$

**q**-right-quantum:

$$\begin{aligned} a_{jk}a_{ik} &= q_{ij}\; a_{ik}a_{jk} \quad \text{for all } \; i<j \\ a_{ik}a_{jl} - q_{ij}^{-1}a_{jk}a_{il} &= q_{kl}q_{ij}^{-1}a_{jl}a_{ik} - q_{kl}a_{il}a_{jk} \; \text{for all } i<j, k<l \end{aligned}$$

**q**-Cartier-Foata $\Rightarrow$ **q**-right-quantum

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# Non-commutative determinant

Given a matrix $A = (a_{ij})_{m\times m}$ with not necessarily commuting entries, we can define its:

- determinant by

$$\det A = \sum_{\sigma\in S_m} (-1)^{\mathrm{inv}(\sigma)} a_{\sigma_1 1}\cdots a_{\sigma_m m}$$

- $q$-determinant by

$$\det_q A = \sum_{\sigma\in S_m} (-q)^{-\operatorname{inv}\sigma} a_{\sigma_1 1}\cdots a_{\sigma_m m}$$

- $\mathbf{q}$-determinant by

$$\det_{\mathbf{q}} A = \sum_{\sigma\in S_m} \left( \prod_{(i,j)\in\mathcal{I}(\sigma)} (-q_{\sigma_j\sigma_i})^{-1} \right) a_{\sigma_1 1}\cdots a_{\sigma_m m}$$

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# Non-commutative determinant

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- $\det(I-A)=\sum_{J\subseteq[m]}(-1)^{|J|}\det A_J$
- $\det_q(I-A)=\sum_{J\subseteq[m]}(-1)^{|J|}\det_q A_J$
- $\det_{\mathbf{q}}(I-A)=\sum_{J\subseteq[m]}(-1)^{|J|}\det_{\mathbf{q}} A_J$

# Matrix inverse formula

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## Theorem

*If $A = (a_{ij})_{m \times m}$ is a Cartier-Foata or right-quantum matrix, we have*

$$\left(\frac{1}{I-A}\right)_{ij} = (-1)^{i+j} \cdot \frac{1}{\det(I-A)} \cdot \det\,(I-A)^{ji}$$

*for all $i, j$.*

# Matrix inverse formula - $q$-cases

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## Theorem

*If $A = (a_{ij})_{m\times m}$ is a $q$-Cartier-Foata or a $q$-right-quantum matrix, we have*

$$\left(\frac{1}{I - A_{[ij]}}\right)_{ij} = (-1)^{i+j}\frac{1}{\det_q(I-A)} \cdot \det_q(I-A)^{ji}$$

*for all $i, j$, where*

$$A_{[ij]} = \begin{pmatrix} q^{-1}a_{11} & \cdots & q^{-1}a_{1j} & a_{1,j+1} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ q^{-1}a_{i-1,1} & \cdots & q^{-1}a_{i-1,j} & a_{i-1,j+1} & \cdots & a_{i-1,m} \\ a_{i1} & \cdots & a_{ij} & qa_{i,j+1} & \cdots & qa_{i,m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mj} & qa_{m,j+1} & \cdots & qa_{mm} \end{pmatrix}.$$

# Matrix inverse formula - **q**-cases

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### Theorem

*If $A = (a_{ij})_{m\times m}$ is a* **q***-Cartier-Foata matrix or a* **q***-right-quantum matrix, we have*

$$\left(\frac{1}{I - A_{[ij]}}\right)_{ij} = (-1)^{i+j}\frac{1}{\det_{\mathbf{q}}(I-A)} \cdot \det_{\mathbf{q}}(I-A)^{ji}$$

*for all $i, j$, where $A_{[ij]}$ is given by a similar formula (involving $a_{ij}$, $q_{ij}$).*

# Language of paths



What is the coefficient of $x_1^{r_1} \cdots x_m^{r_m}$ in

$$(a_{11}x_1 + \ldots + a_{1m}x_m)^{r_1} \cdots (a_{m1}x_1 + \ldots + a_{mm}x_m)^{r_m},$$

where $a_{ij}$ are (not necessarily commuting) variables and $x_i$ commute with $a_{ij}$'s and each other?

# Language of paths



It is the sum of all monomials

$$\underbrace{a_{1*}\cdots a_{1*}}_{r_1}\underbrace{a_{2*}\cdots a_{2*}}_{r_2}\cdots\underbrace{a_{m*}\cdots a_{m*}}_{r_m},$$

so that $*$ represents 1 $r_1$ times, 2 $r_2$ times, etc.

We call such a monomial an *ordered sequence* or *o-sequence* of *type* $(r_1,\ldots,r_m)$.

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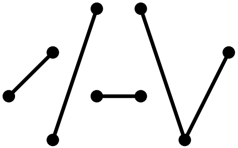
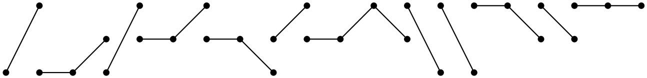
Represent the variable $a_{ij}$ as a step from height $i$ to height $j$, and a monomial $a_{i_1j_1} \cdots a_{i_nj_n}$ as a concatenation of steps.

For example, $a_{23}a_{14}a_{22}a_{41}a_{13}$ becomes

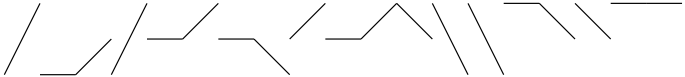


# Language of paths

An o-sequence of type $(r_1, \ldots, r_m)$ is represented by a concatenation of steps so that starting heights are non-decreasing and so that each $i$ appears $r_i$ times as a starting height and $r_i$ times as an ending height.



# A bijection



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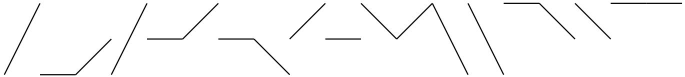

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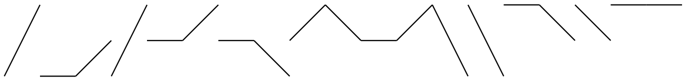

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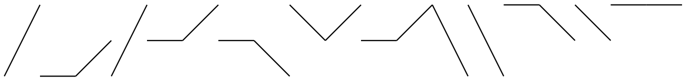

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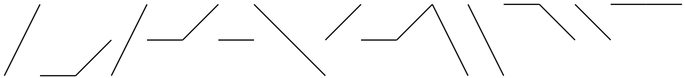

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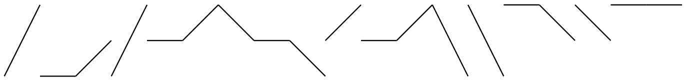

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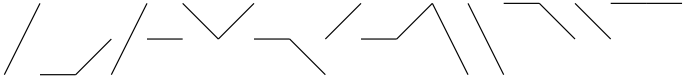

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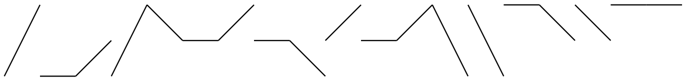

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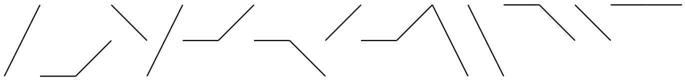

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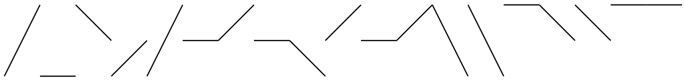

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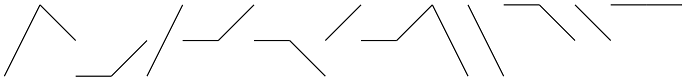

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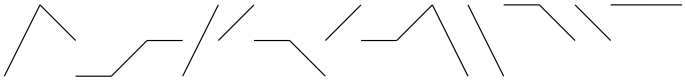

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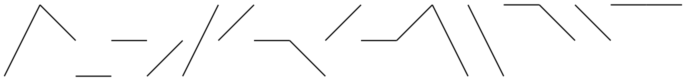

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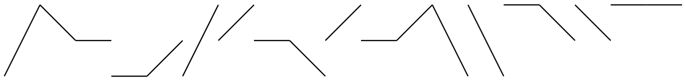

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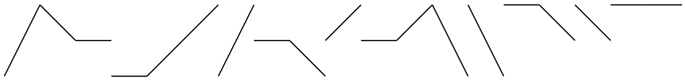

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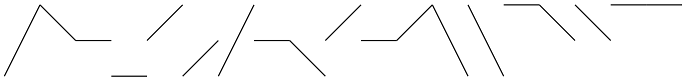

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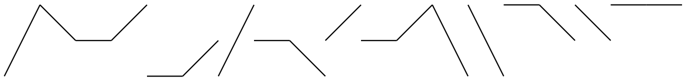

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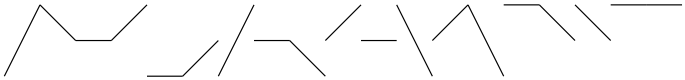

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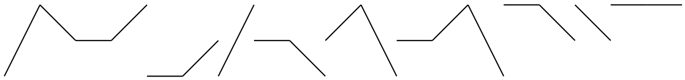

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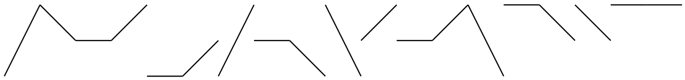

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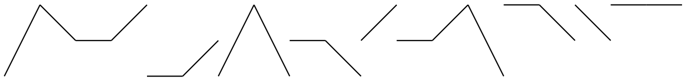

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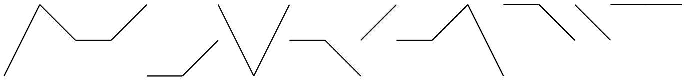

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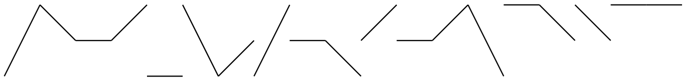
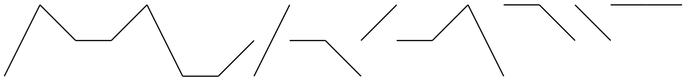


# A bijection





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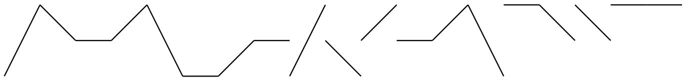

# A bijection



# A bijection



# A bijection

# A bijection

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# A bijection



# Path sequences



A *path sequence* or *p-sequence* is a concatenation of a lattice path from $(0, 1)$ to $(x_1, 1)$ that never goes below $y = 1$ or above $y = m$, a lattice path from $(x_1, 2)$ to $(x_2, 2)$ that never goes below $y = 2$ or above $y = m$, a lattice path from $(x_2, 3)$ to $(x_3, 3)$ that never goes below $y = 3$ or above $y = m$, etc.

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We have established a bijection $\varphi$ from the set of o-sequences to the set of p-sequences so that $\varphi(\alpha)$ is a rearrangement of $\alpha$.

# Classical MacMahon master theorem



The sum of all paths from 1 to 1 is given by

$$(I + A + A^2 + \dots)_{11} = \left(\frac{1}{I - A}\right)_{11},$$

the sum of all paths from 2 to 2 that avoid 1 is given by

$$\left(\frac{1}{I - A^{11}}\right)_{22},$$

etc.

# Classical MacMahon master theorem

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etc.

# Classical MacMahon master theorem

Therefore the sum of all p-sequences is given by

$$\left(\frac{1}{I-A}\right)_{11}\left(\frac{1}{I-A^{11}}\right)_{22}\left(\frac{1}{I-A^{12,12}}\right)_{33}\cdots\frac{1}{1-a_{mm}}$$

$$=\frac{\det(I-A)^{11}}{\det(I-A)}\cdot\frac{\det(I-A)^{12,12}}{\det(I-A)^{11}}\cdots\frac{1}{1-a_{mm}}$$

$$=\frac{1}{\det(I-A)}$$

This finishes the proof of MacMahon master theorem.

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# Classical MacMahon master theorem

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# Cartier-Foata master theorem

Since:

- the bijection $\varphi$ never switches steps that begin at the same height, and
- the matrix inverse formula holds for Cartier-Foata matrices,

the same proof gives the following theorem.

# Cartier-Foata master theorem



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- the bijection $\varphi$ never switches steps that begin at the same height, and
- the matrix inverse formula holds for Cartier-Foata matrices,

the same proof gives the following theorem.

# Cartier-Foata master theorem



## Theorem (Cartier-Foata master theorem)

*Let $A = (a_{ij})_{m\times m}$ be a Cartier-Foata matrix. Denote by $G(\mathbf{r})$ the coefficient of $x_1^{r_1}\cdots x_m^{r_m}$ in*

$$\prod_{i=1}^{m}(a_{i1}x_1 + \ldots + a_{im}x_m)^{r_i}.$$

*Then*

$$\sum_{\mathbf{r}\geq\mathbf{0}} G(\mathbf{r}) = \frac{1}{\det(I-A)}.$$

# Right-quantum master theorem



Can we extend the theorem to the case when $A$ is right-quantum?

# Right-quantum master theorem



Yes, but we need something extra for the proof.

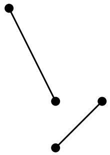
# Right-quantum master theorem



Since

$$a_{jk}a_{ik} = a_{ik}a_{jk},$$

we can switch steps that end on the same height:



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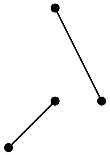

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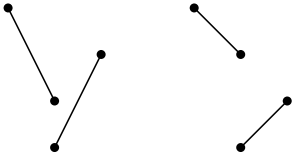
But since

$$a_{ik}a_{jl} + a_{il}a_{jk} = a_{jl}a_{ik} + a_{jk}a_{il},$$

we have to make other switches *simultaneously*, in pairs:



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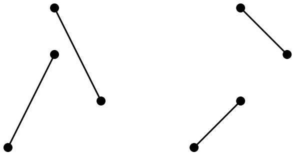

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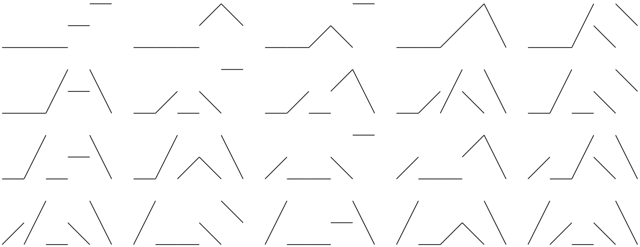

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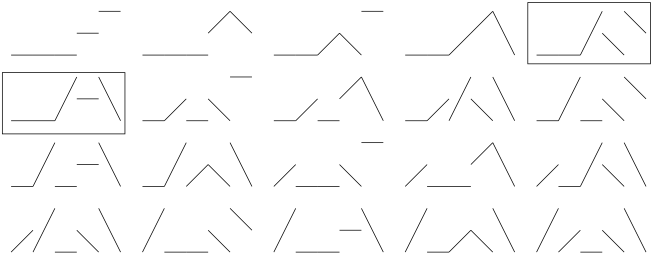
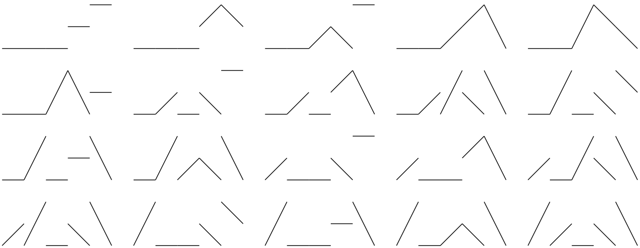


# Right-quantum master theorem





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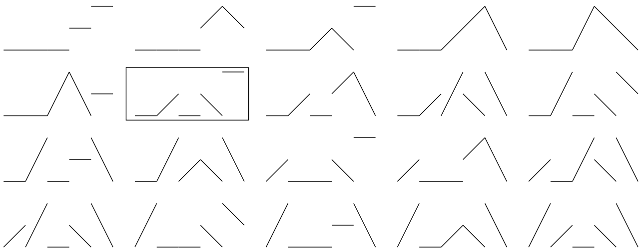

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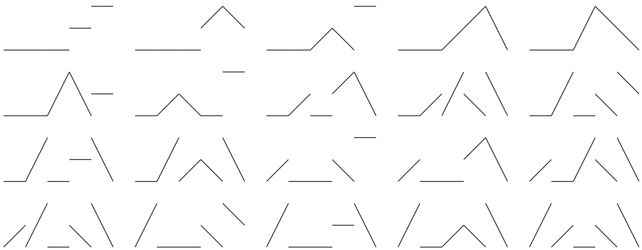

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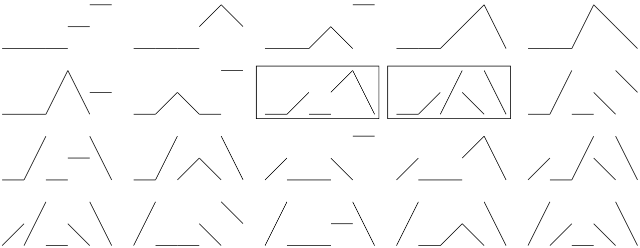

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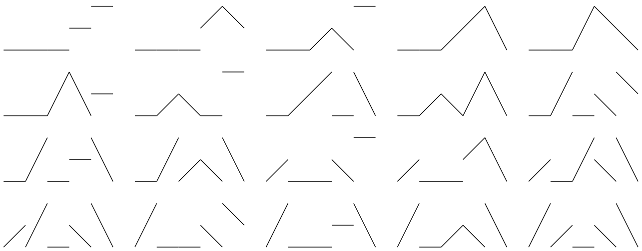
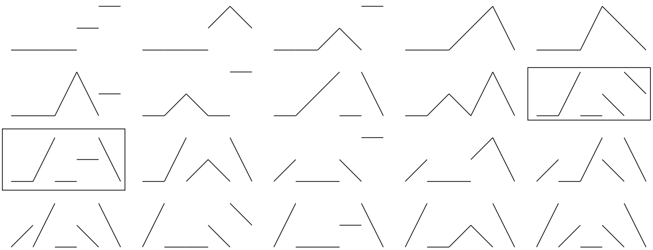


# Right-quantum master theorem





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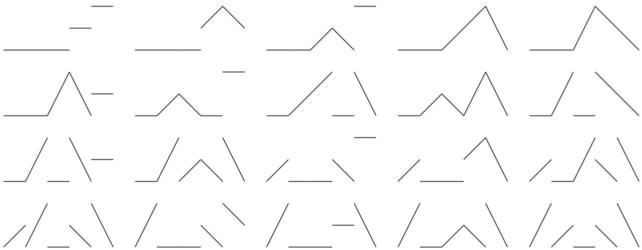

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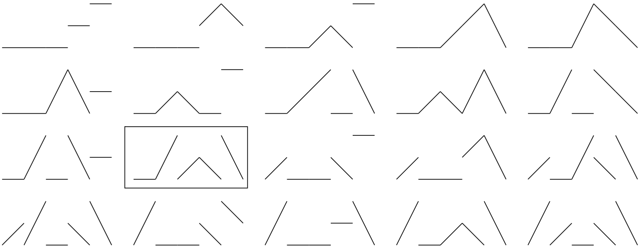

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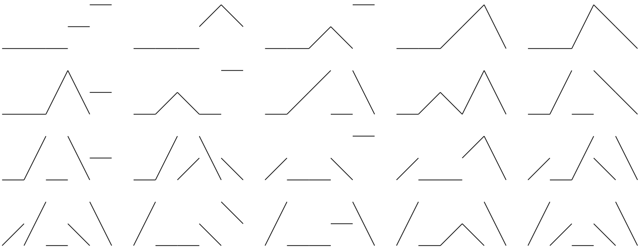

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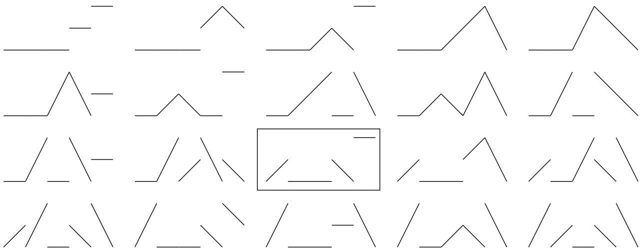

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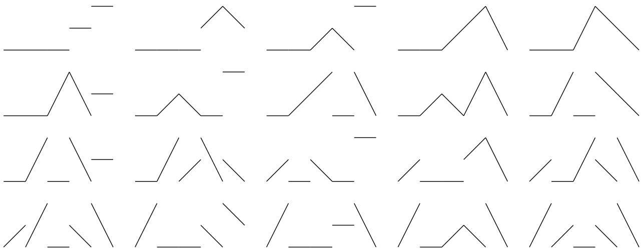

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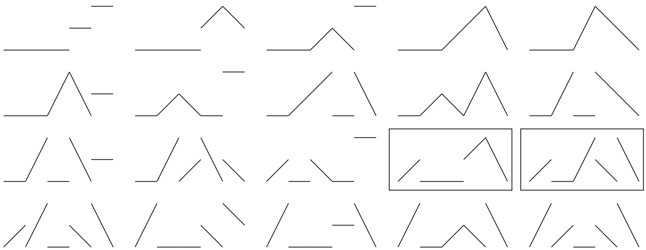

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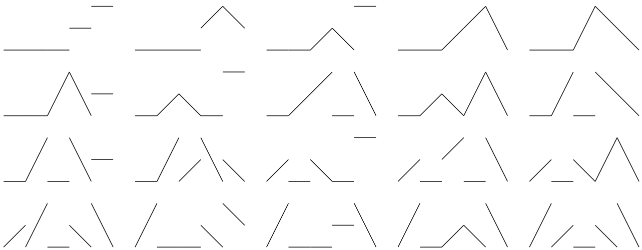

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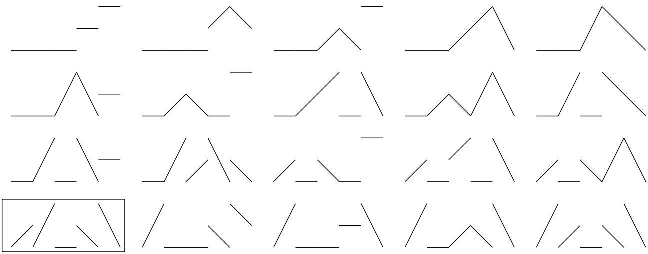

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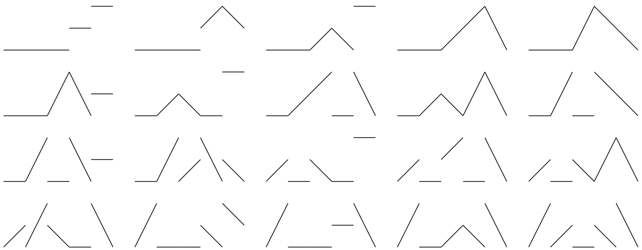

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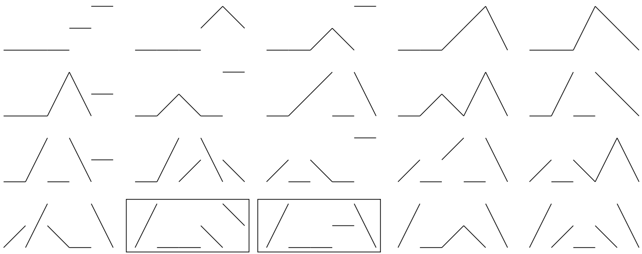

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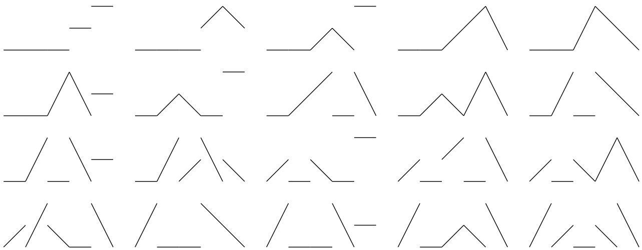


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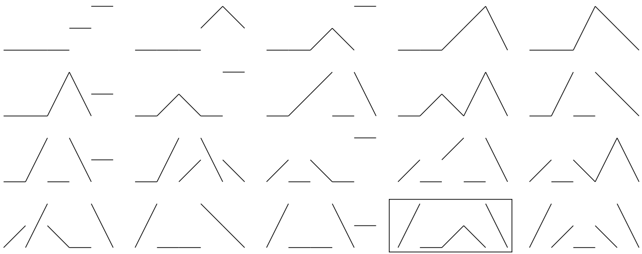

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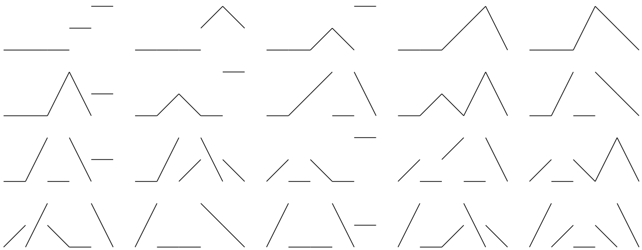

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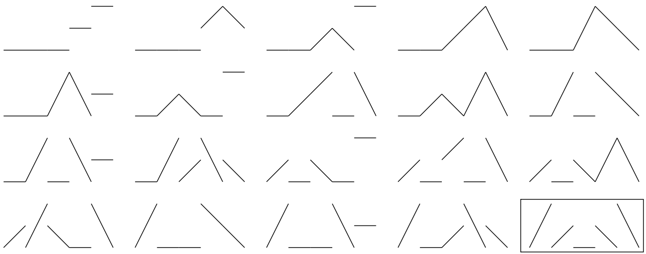

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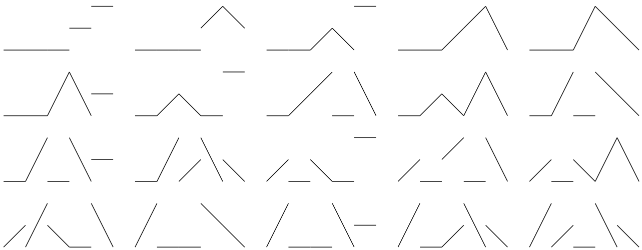

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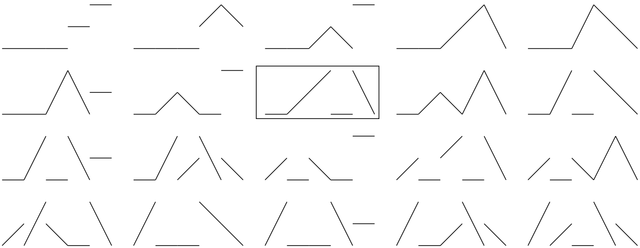

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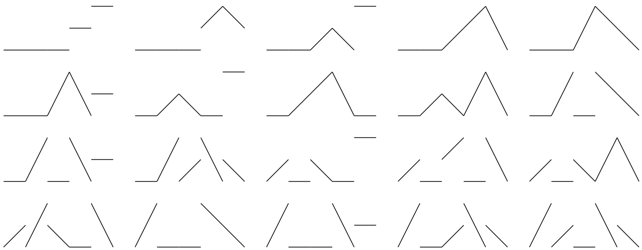

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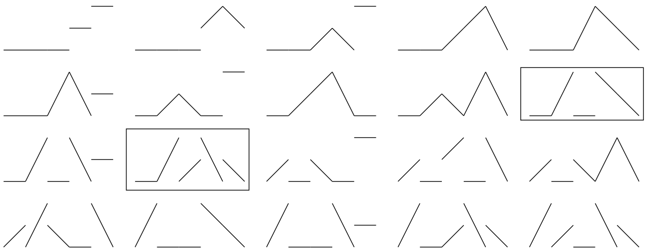

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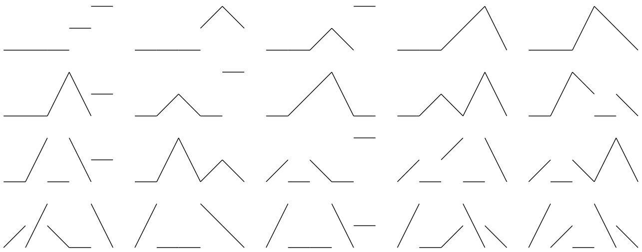

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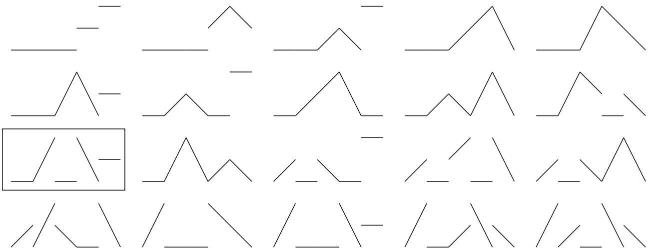

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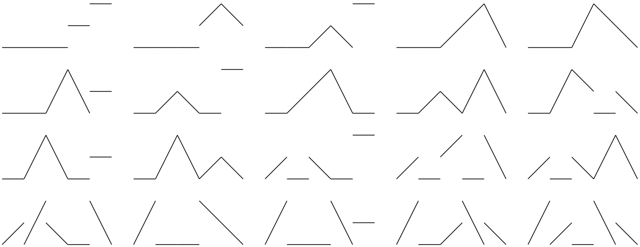

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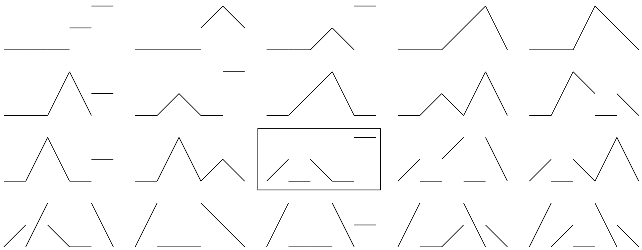

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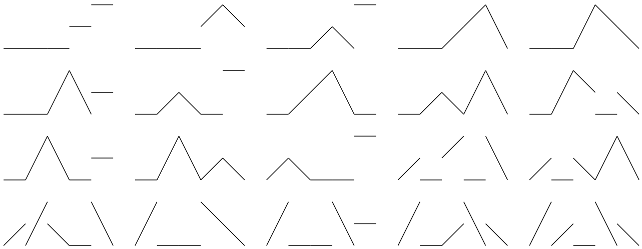

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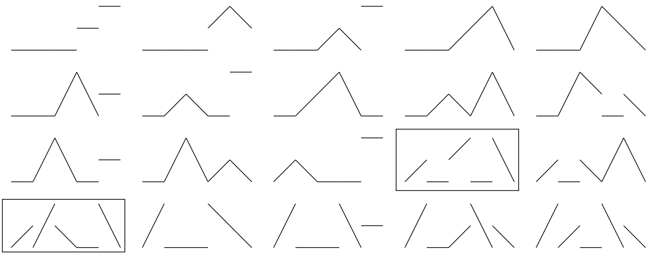

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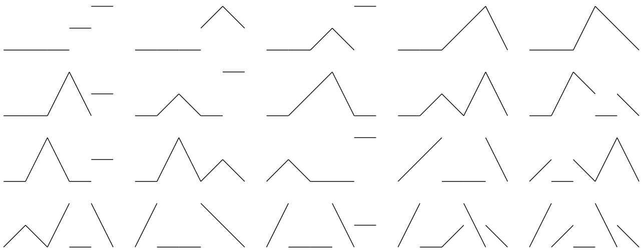

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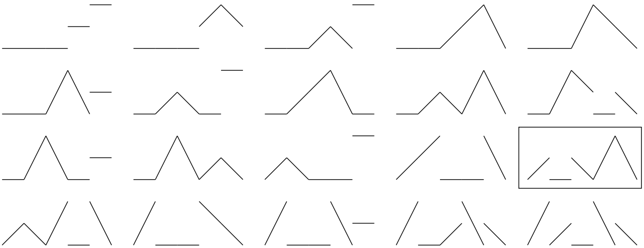

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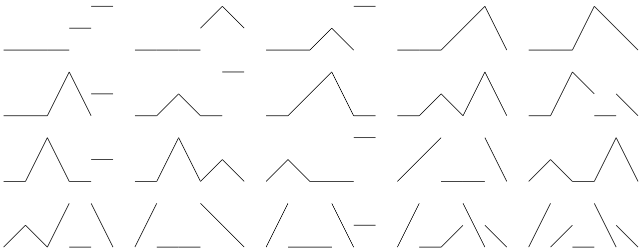

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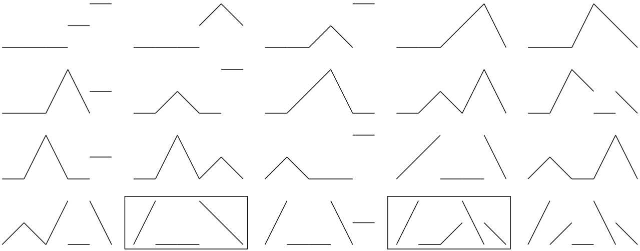

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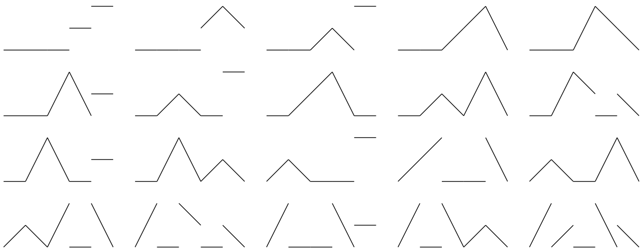

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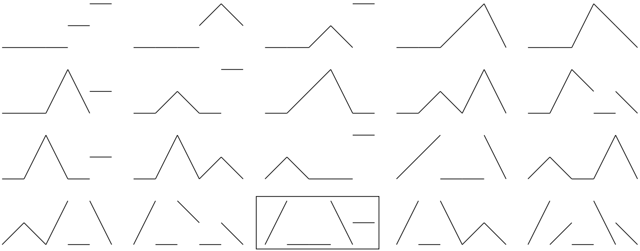

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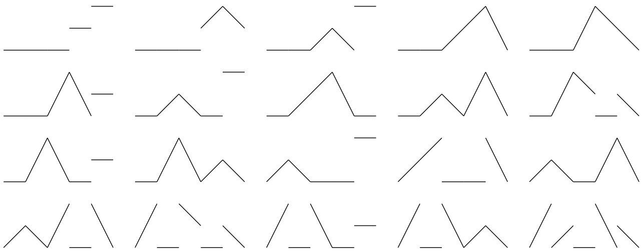

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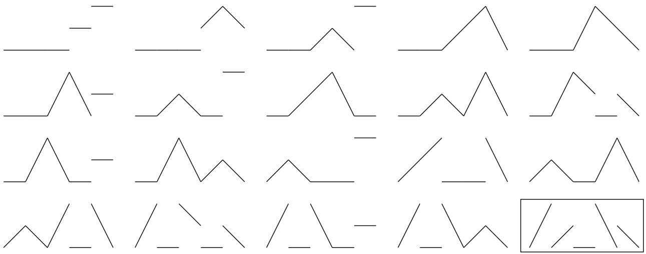

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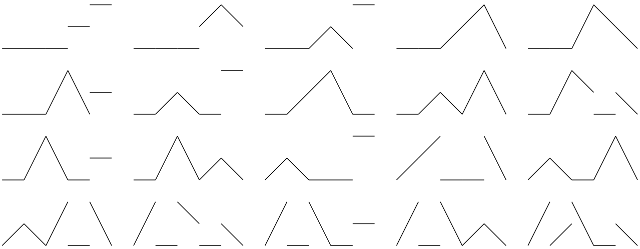

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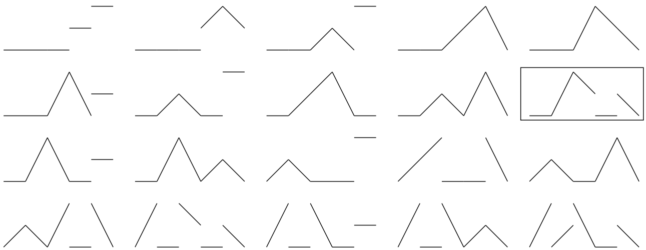

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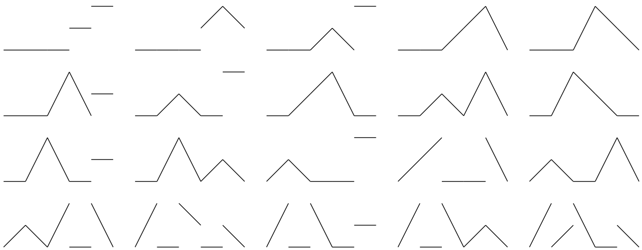

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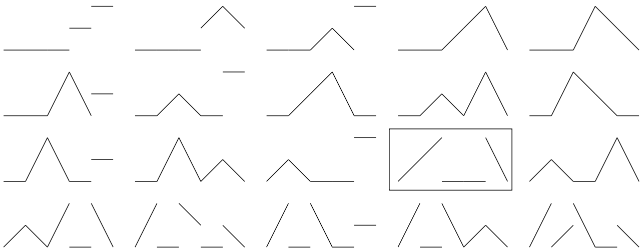

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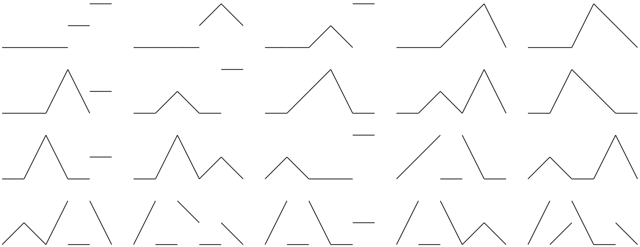
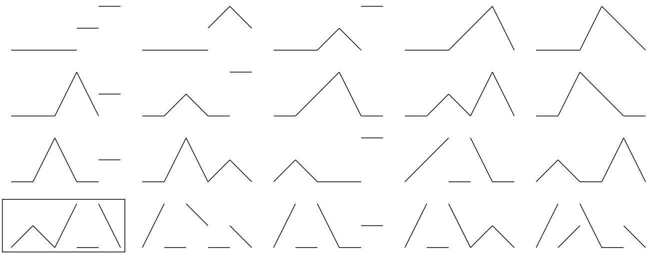


# Right-quantum master theorem





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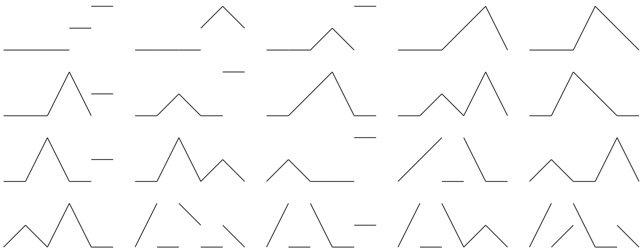

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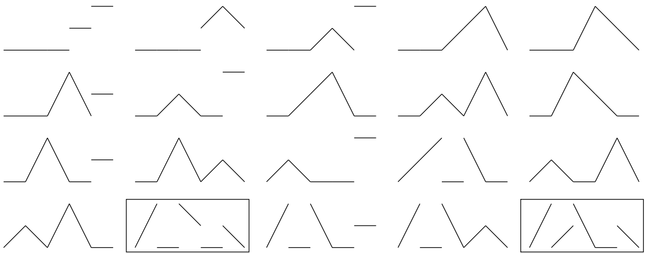
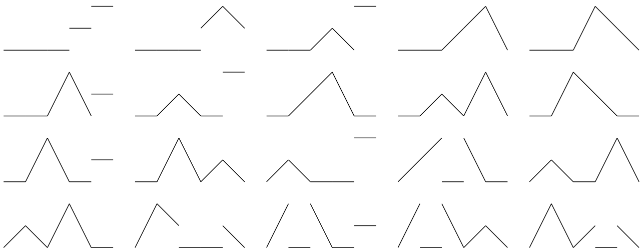


# Right-quantum master theorem





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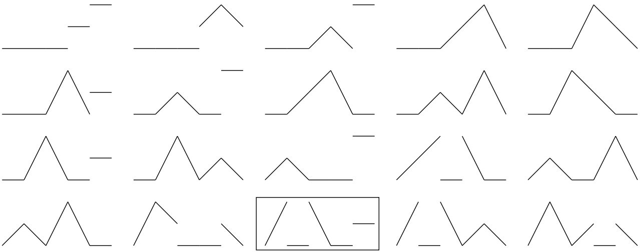

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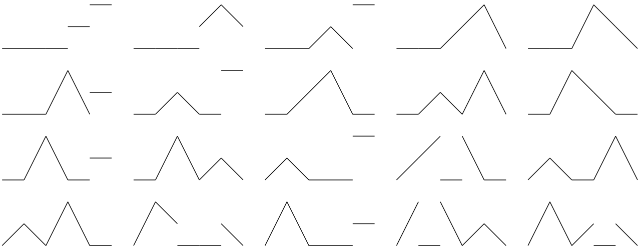

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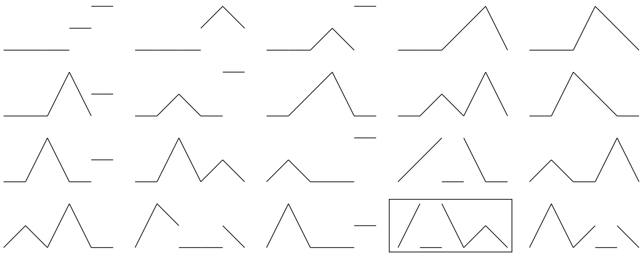

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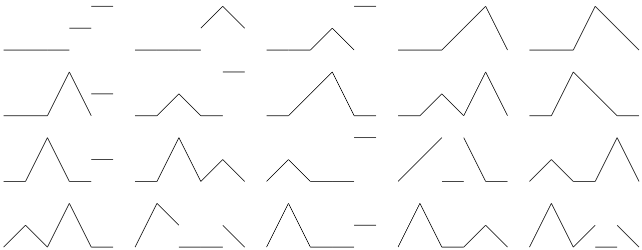

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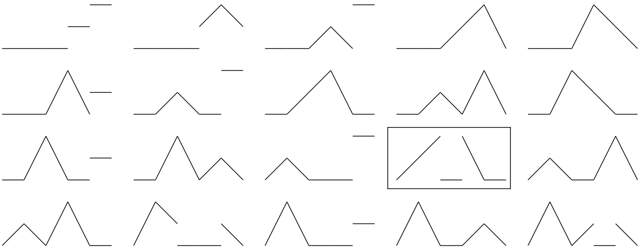

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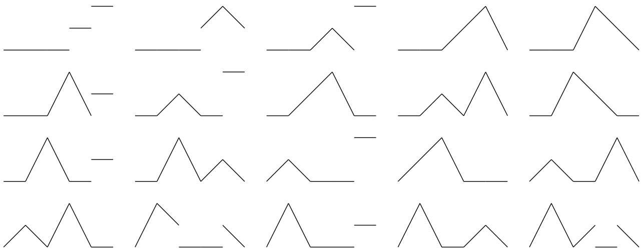

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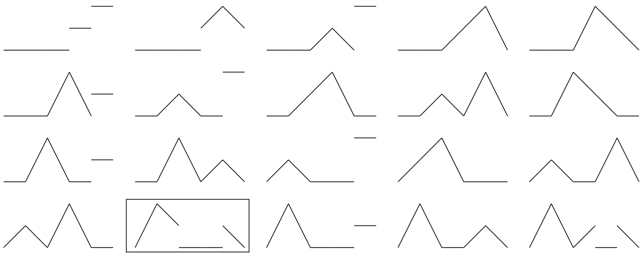
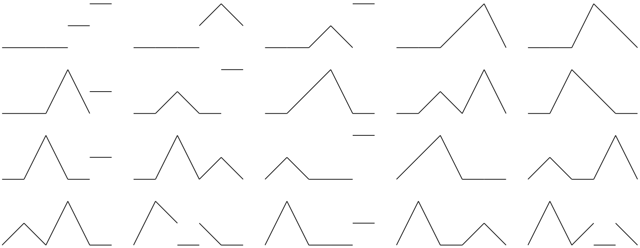


# Right-quantum master theorem





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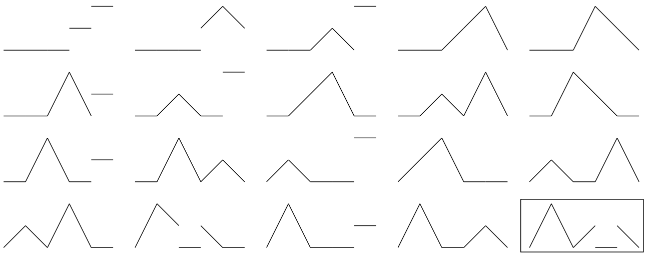

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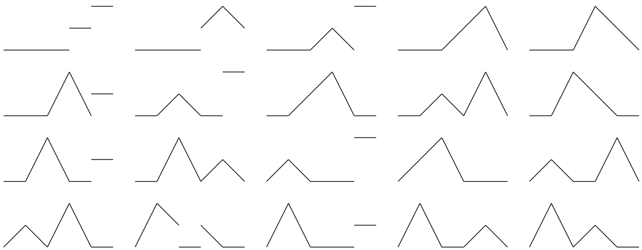
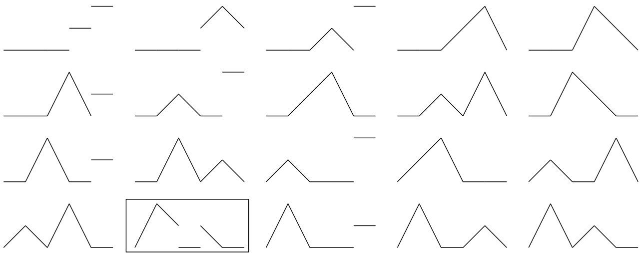


# Right-quantum master theorem





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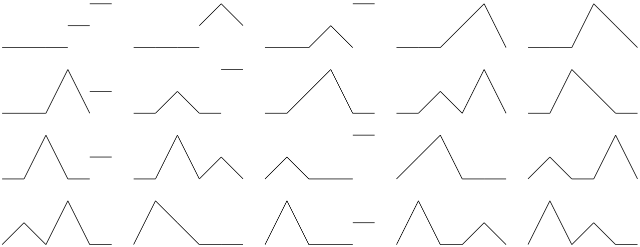

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**Theorem (right-quantum master theorem)**

*Let $A = (a_{ij})_{m\times m}$ be a right-quantum matrix. Denote by $G(\mathbf{r})$ the coefficient of $x_1^{r_1}\cdots x_m^{r_m}$ in*

$$\prod_{i=1}^{m}(a_{i1}x_1 + \ldots + a_{im}x_m)^{r_i}.$$

*Then*

$$\sum_{\mathbf{r}\geq\mathbf{0}} G(\mathbf{r}) = \frac{1}{\det(I-A)}.$$

# Weighted analogue



If we assume that

$$x_j x_i = q x_i x_j \text{ for all } i < j,$$

that $A$ is $q$-right-quantum and that $x_i$'s commute with $a_{ij}$'s, then careful bookkeeping of the weights shows the following.

# Weighted analogue

## Theorem ($q$-right-quantum master theorem)

*Denote the coefficient of* $x_1^{r_1} \cdots x_m^{r_m}$ *in*

$$\prod_{i=1}^{m} (a_{i1}x_1 + \ldots + a_{im}x_m)^{r_i}$$

*by* $G(\mathbf{r})$. *Then*

$$\sum_{\mathbf{r} \geq \mathbf{0}} G(\mathbf{r}) = \frac{1}{\det_q(I - A)}.$$

# Multiparameter analogue



If we assume that

$$x_j x_i = q_{ij} x_i x_j \text{ for all } i < j,$$

that $A$ is **q**-right-quantum and that $x_i$'s commute with $a_{ij}$'s, then we have the following.

# Multiparameter analogue

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## Theorem ($\mathbf{q}$-right-quantum master theorem)

*Denote the coefficient of* $x_1^{r_1} \cdots x_m^{r_m}$ *in*

$$\prod_{i=1}^{m} (a_{i1}x_1 + \ldots + a_{im}x_m)^{r_i}$$

*by* $G(\mathbf{r})$. *Then*

$$\sum_{\mathbf{r} \geq \mathbf{0}} G(\mathbf{r}) = \frac{1}{\det_{\mathbf{q}}(I - A)}.$$

# Non-commutative Sylvester's identity



Similar techniques prove the following theorem.

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### Theorem ($\mathbf{q}$-right-quantum Sylvester's theorem)

*Let $A = (a_{ij})_{m \times m}$ be a $\mathbf{q}$-right-quantum matrix, and choose $n < m$. Let $A_0, a_{i*}, a_{*j}$ be defined as above, and let*

$$c_{ij}^{\mathbf{q}} = -\det_{\mathbf{q}}^{-1}(I - A_0) \cdot \det_{\mathbf{q}} \begin{pmatrix} I - A_0 & -a_{*j} \\ -a_{i*} & -a_{ij} \end{pmatrix},$$

$$C^{\mathbf{q}} = (c_{ij}^{\mathbf{q}})_{n+1 \le i,j \le m}.$$

*Suppose $q_{ij} = q_{i'j'}$ for all $i, i' \le n$ and $j, j' > n$. Then*

$$\det_{\mathbf{q}}^{-1}(I - A_0) \cdot \det_{\mathbf{q}}(I - A) = \det_{\mathbf{q}}(I - C^{\mathbf{q}}).$$

# References



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- M. Konvalinka, Non-commutative Sylvester's determinantal identity, *Electron. J. Combin.*, vol. 14 (2007), Article 42
- M. Konvalinka, A generalization of Foata's fundamental transformation and its applications to the right-quantum algebra, preprint (2007)