Clusters, noncrossing partitions and the Coxeter plane

Nathan Reading

NC State University

FPSAC 2007
Nankai University, Tianjin, China

Finite Coxeter groups

Coxeter group W: Generated by a finite set S (with relations).

Motivation

Finite Coxeter groups $\leftrightarrow$ finite groups generated by reflections.
(Also Lie theory, rep. theory, geometric group theory, etc.)

Classical examples

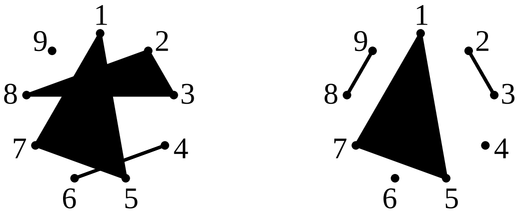
S_n: permutations of $\{1, \ldots, n\}$. $(S = \{(i\ i{+}1)\})$
B_n: "signed" permutations of $\{\pm 1, \ldots, \pm n\}$.
D_n: has a similar description in terms of permutations.

(All) other examples

$I_2(m)$: full (dihedral) symmetry group of regular m-gon.
H_3: full symmetry group of icosahedron/dodecahedron.
F_4, H_4: symmetry groups of 4-dimensional regular polytopes.
E_6, E_7, E_8.

Noncrossing partitions (Kreweras, 1972 & many others 1996-2002)

Write $1, \ldots, n$ cyclically. Set partitions are crossing or noncrossing.



This is the S_n case of a general algebraic construction.
(Set partitions = equivalence relations $\leftrightarrow$ sets of transpositions.)

The general definition is algebraic, not via planar diagrams.
Analog of set partitions: certain collections of reflections.
Algebraic criterion $\rightarrow$ certain partitions are "noncrossing."

A pivotal role is played by a (the) Coxeter element $c = \prod S$.

Noncrossing partitions (continued)

Planar diagrams for noncrossing partitions for B_n and D_n:

B_n:
Write $1, \ldots, n, (-1), \ldots, (-n)$ cyclically. The "type B noncrossing partitions" are those classical noncrossing partitions which have central symmetry.

D_n:
A similar, slightly more complicated picture:
Place ± 1 at the origin, write $2, \ldots, n, (-2), \ldots, (-n)$ cyclically.
Criterion for noncrossing is essentially "blocks don't cross."

Clusters (Fomin and Zelevinsky, 2003)

Clusters: max'l sets of "pairwise compatible almost positive roots."
Almost positive roots: (more or less) correspond to reflections.
Def. of compatibility: "altered" Coxeter element plays a key role.
Generalized associahedron: polytope with vertices $\leftrightarrow$ clusters.

S_n:

Almost positive roots for $S_n \leftrightarrow$ diagonals of an $(n+2)$-gon.
Compatible $\leftrightarrow$ diagonals don't cross.
Clusters are triangulations of the $(n+2)$-gon.

B_n:

Clusters are centrally symmetric triangulations of a $(2n+2)$-gon.

D_n:

Clusters are not quite as easily described (a slightly more complicated model on a $2n$-gon).

Central questions

Why are models available for S_n, B_n, D_n only?

Why are the models **planar**?

Can we find (planar) models in other cases?

Central questions

Why are models available for S_n, B_n, D_n only?

Small orbits. ($\sim$ the Coxeter number $h =$ the order of c.)

Why are the models **planar**?

Can we find (planar) models in other cases?

Central questions

Why are models available for S_n, B_n, D_n only?

Small orbits. ($\sim$ the Coxeter number $h =$ the order of c.)

Why are the models **planar**?

They are based on projections of small orbits to the Coxeter plane. (A certain plane fixed, as a set, by c.)

Can we find (planar) models in other cases?

Central questions

Why are models available for S_n, B_n, D_n only?

Small orbits. ($\sim$ the Coxeter number $h =$ the order of c.)

Why are the models **planar**?

They are based on projections of small orbits to the Coxeter plane. (A certain plane fixed, as a set, by c.)

Can we find (planar) models in other cases?

Yes for compatibility, sometimes for noncrossing partitions.

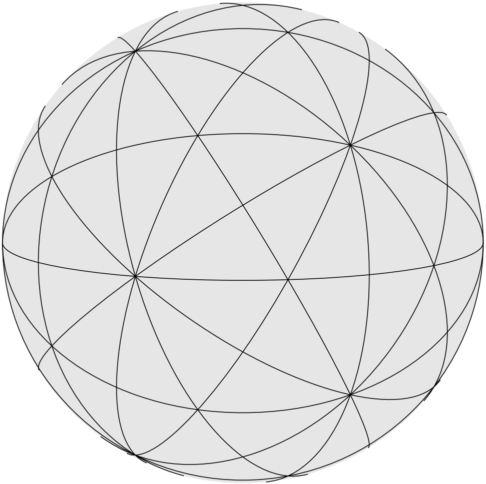
Motivation for planar models

1. Realize noncrossing partitions as combinatorial objects s.t. the algebraic symmetry acts as a natural combinatorial symmetry.
2. Realize clusters (and generalized clusters) as combinatorial objects with the defining symmetry acting as some natural combinatorial symmetry. (Cf. Eu's talk.)
3. Generalize the combinatorics occurring in diagrams for clusters $\rightarrow$ new combinatorial models for cluster algebras of infinite type. (Cf. Fomin's talk.)
4. Generalize the beautiful fiber-polytope constructions for S_n- and B_n-associahedra.

The Coxeter plane

A certain plane P fixed, as a set, by the Coxeter element c.
The action of c on P is by h-fold rotation.

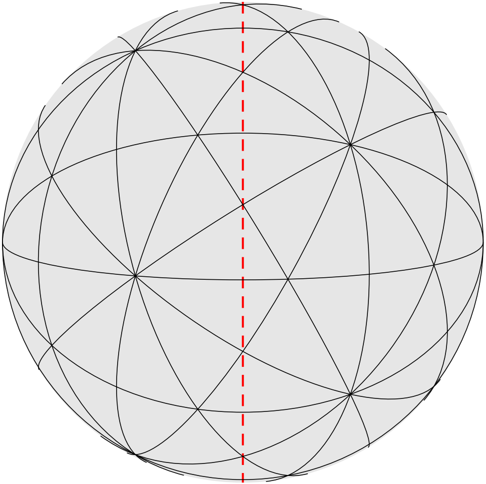
$W = H_3$ (the full symmetry group of the icosahedron), $h = 10$



The Coxeter plane

A certain plane P fixed, as a set, by the Coxeter element c.
The action of c on P is by h-fold rotation.

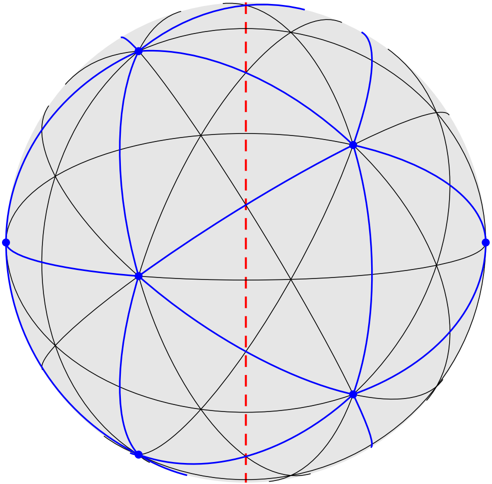
$W = H_3$ (the full symmetry group of the icosahedron), $h = 10$



The Coxeter plane

A certain plane P fixed, as a set, by the Coxeter element c.
The action of c on P is by h-fold rotation.

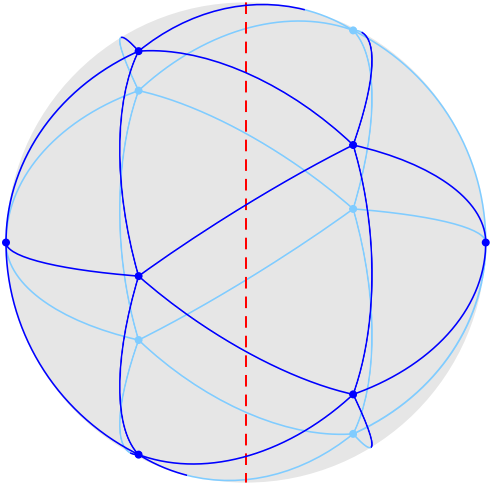
$W = H_3$ (the full symmetry group of the icosahedron), $h = 10$



The Coxeter plane

A certain plane P fixed, as a set, by the Coxeter element c.
The action of c on P is by h-fold rotation.

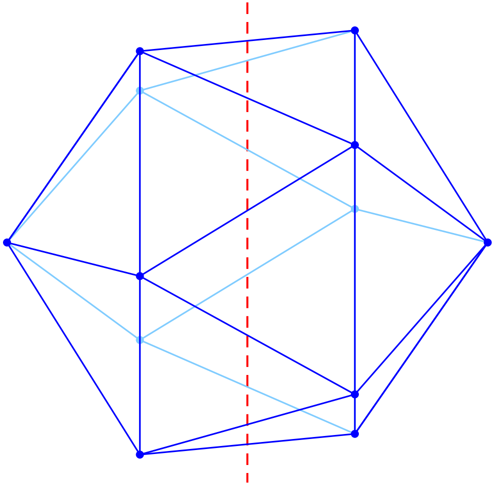
$W = H_3$ (the full symmetry group of the icosahedron), $h = 10$



The Coxeter plane

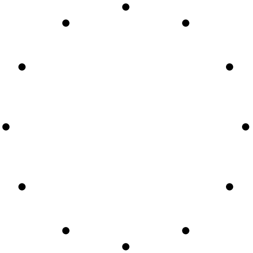
A certain plane P fixed, as a set, by the Coxeter element c.
The action of c on P is by h-fold rotation.

$W = H_3$ (the full symmetry group of the icosahedron), $h = 10$



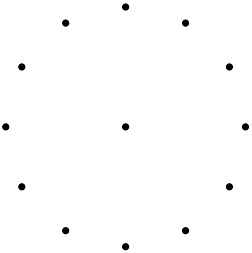
Projecting an orbit to the Coxeter plane

Take a smallest nontrivial orbit o of W. Project orthogonally to P.



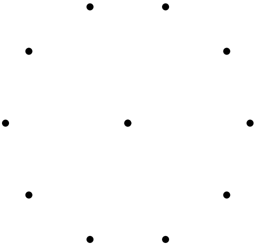
S_{12}

$|o| = 12, \ h = 12$



D_7

$|o| = 14, \ h = 12$



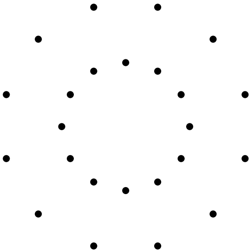
H_3

$|o| = 12, \ h = 10$

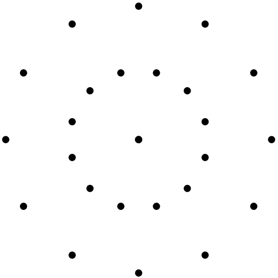
Projections are simple because $|o| \approx h$.

Projecting an orbit to the Coxeter plane (continued)

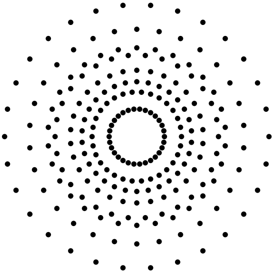
When $|o| \gg h$, the projections are necessarily more complicated.



F_4
$|o| = 24, \ h = 12$



E_6
$|o| = 27, \ h = 12$



E_8
$|o| = 240, \ h = 30$

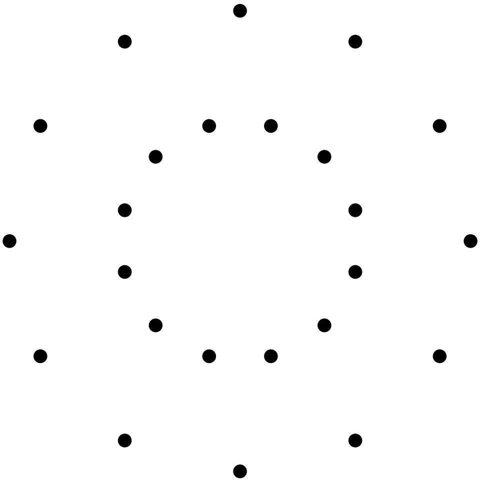
Projecting partitions to the Coxeter plane

Reflection $t \longrightarrow$ matching on $o \longrightarrow$ matching on projection of o.
Diagram of t: straight-line drawing of this matching in P.

Projecting partitions to the Coxeter plane

Reflection $t \longrightarrow$ matching on $o \longrightarrow$ matching on projection of o.
Diagram of t: straight-line drawing of this matching in P.

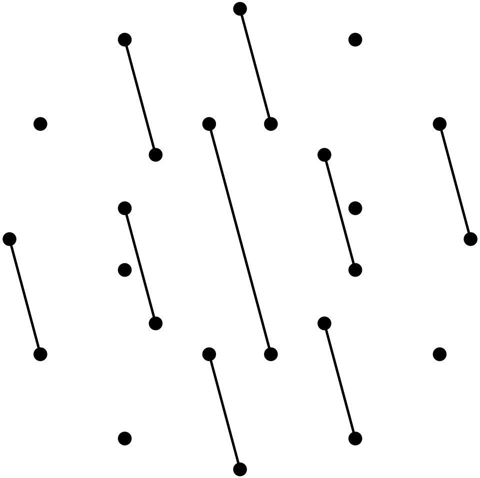
Example: a reflection in F_4



Projecting partitions to the Coxeter plane

Reflection $t \longrightarrow$ matching on $o \longrightarrow$ matching on projection of o.
Diagram of t: straight-line drawing of this matching in P.

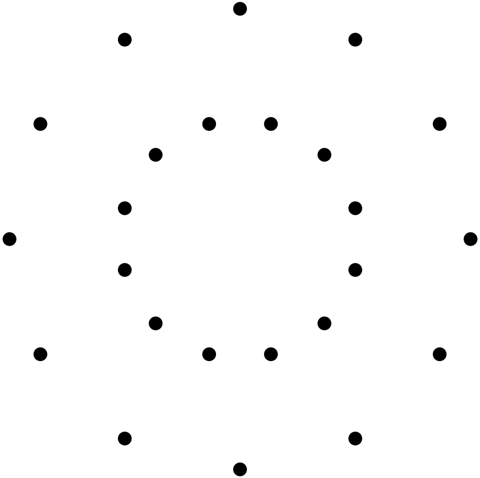
Example: a reflection in F_4



Projecting partitions to the Coxeter plane

Reflection $t \longrightarrow$ matching on $o \longrightarrow$ matching on projection of o.
Diagram of t: straight-line drawing of this matching in P.
Diagram for a partition: union of the diagrams of the reflections.
(This defines a set partition of the projected orbit.)

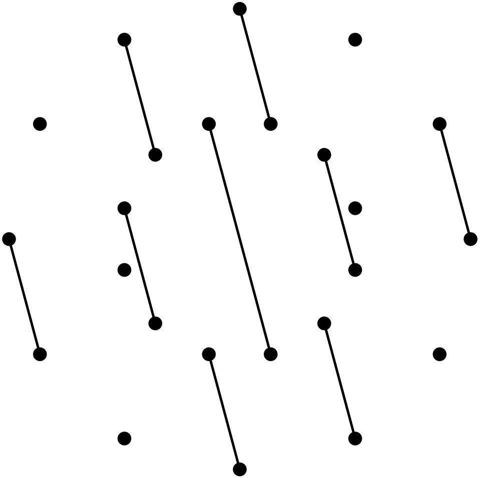
Example: a partition in F_4



Projecting partitions to the Coxeter plane

Reflection $t \longrightarrow$ matching on $o \longrightarrow$ matching on projection of o.
Diagram of t: straight-line drawing of this matching in P.
Diagram for a partition: union of the diagrams of the reflections.
(This defines a set partition of the projected orbit.)

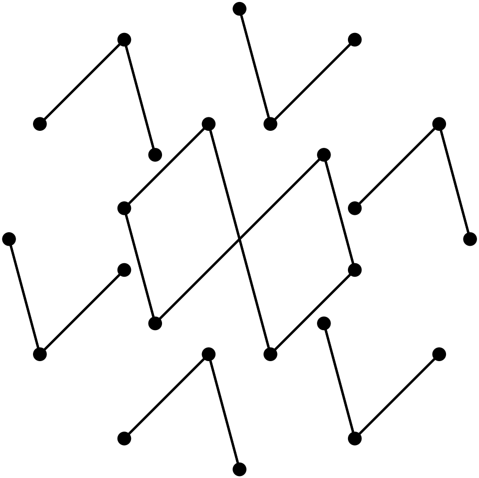
Example: a partition in F_4



Projecting partitions to the Coxeter plane

Reflection $t \rightarrow$ matching on $o \rightarrow$ matching on projection of o.
Diagram of t: straight-line drawing of this matching in P.
Diagram for a partition: union of the diagrams of the reflections.
(This defines a set partition of the projected orbit.)

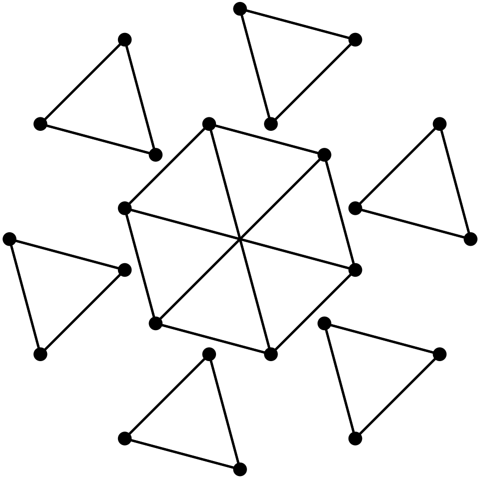
Example: a partition in F_4



Projecting partitions to the Coxeter plane

Reflection $t \longrightarrow$ matching on $o \longrightarrow$ matching on projection of o.
Diagram of t: straight-line drawing of this matching in P.
Diagram for a partition: union of the diagrams of the reflections.
(This defines a set partition of the projected orbit.)

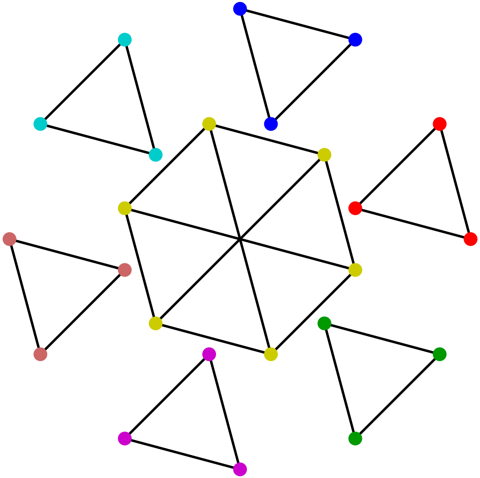
Example: a partition in F_4



Projecting partitions to the Coxeter plane

Reflection $t \rightarrow$ matching on $o \rightarrow$ matching on projection of o.
Diagram of t: straight-line drawing of this matching in P.
Diagram for a partition: union of the diagrams of the reflections.
(This defines a set partition of the projected orbit.)

Example: a partition in F_4



Noncrossing partitions in S_n, B_n, D_n

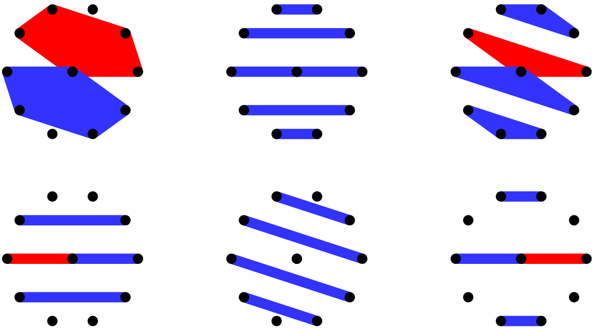
Applying this construction to S_n, B_n and D_n, we get (the usual) nice planar diagrams.

Noncrossing partitions in S_n, B_n, D_n

Applying this construction to S_n, B_n and D_n, we get (the usual) nice planar diagrams. We expect nice things to happen for H_3, too, because $h = 10$ and H_3 has an orbit with 12 elements.

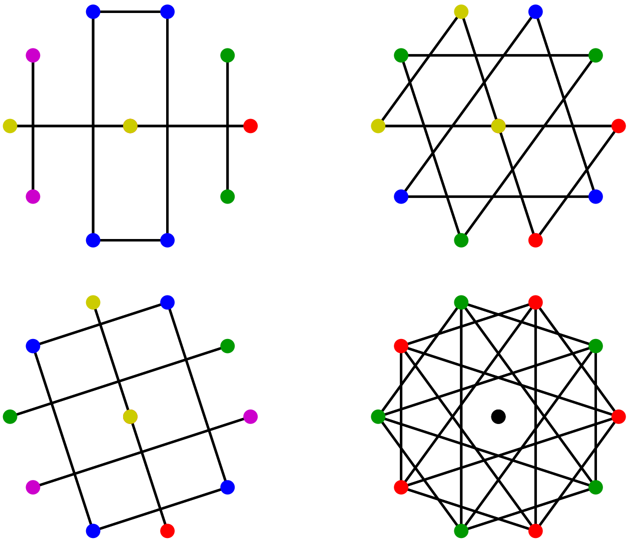
Noncrossing partitions in S_n, B_n, D_n and H_3!

Applying this construction to S_n, B_n and D_n, we get (the usual) nice planar diagrams. We expect nice things to happen for H_3, too, because $h = 10$ and H_3 has an orbit with 12 elements. Indeed, in H_3 we can determine noncrossing partitions by a simple criterion: Relative interiors of blocks may not intersect!



c-Orbit representatives of noncrossing partitions in H_3.

Crossing partitions in H_3

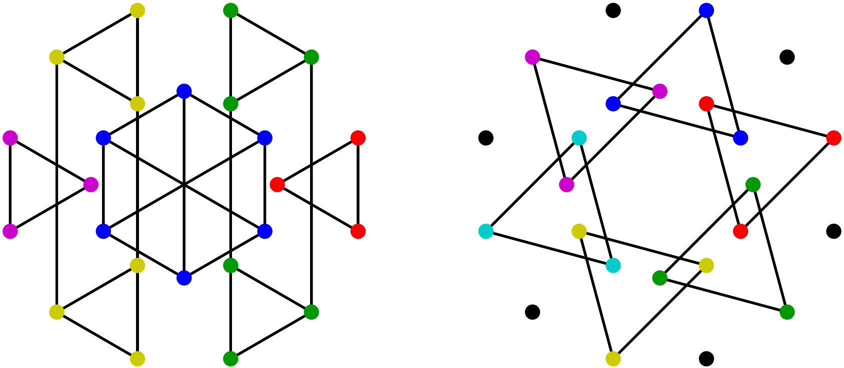


c-Orbit representatives of crossing partitions in H_3.

Crossing and noncrossing partitions in F_4

In the Coxeter groups whose smallest orbit o has $|o| \gg h$,
a general criterion for crossing/noncrossing partitions is lacking.

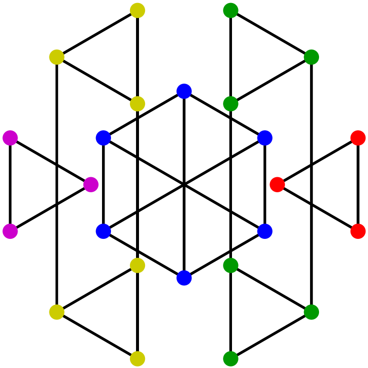
Two partitions in F_4 ($|o| = 24$, $h = 12$)



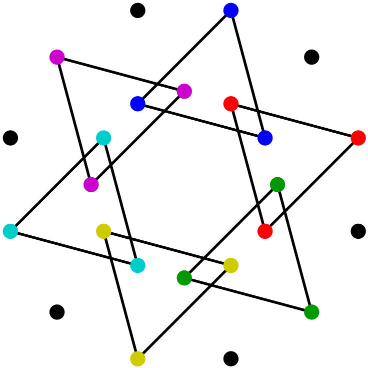
Crossing and noncrossing partitions in F_4

In the Coxeter groups whose smallest orbit o has $|o| \gg h$,
a general criterion for crossing/noncrossing partitions is lacking.

Two partitions in F_4 ($|o| = 24$, $h = 12$)



Crossing



Noncrossing

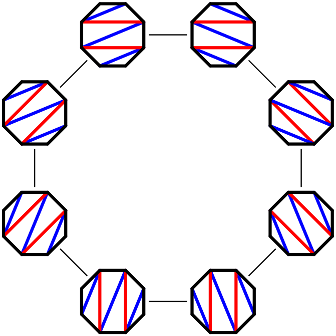
Diagrams for compatibility

There is a uniform way to alter the projected orbit (h-gons become $(h+2)$-gons) and define diagrams for "almost positive roots."
When $|o| \approx h$, things work nicely.

Diagrams for compatibility

There is a uniform way to alter the projected orbit (h-gons become $(h+2)$-gons) and define diagrams for “almost positive roots.” When $|o| \approx h$, things work nicely.

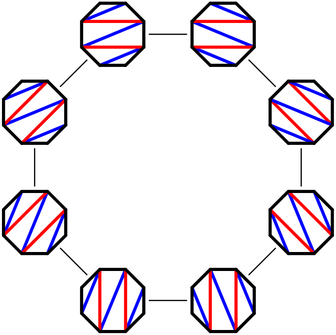
Example: The G_2-associahedron



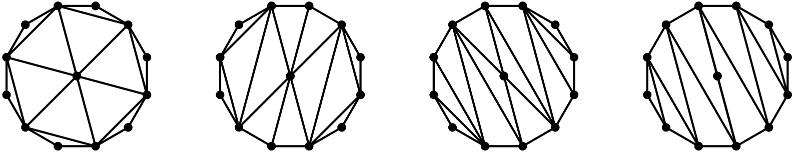
Diagrams for compatibility

There is a uniform way to alter the projected orbit (h-gons become $(h+2)$-gons) and define diagrams for "almost positive roots."
When $|o| \approx h$, things work nicely.

Example: The G_2-associahedron



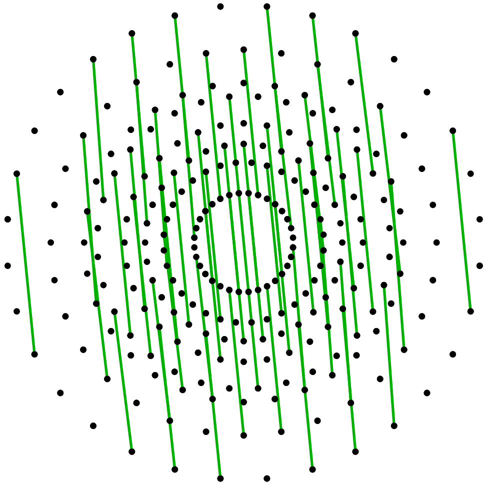
Example: Clusters for H_3



Diagrams for compatibility (continued)

When $|o| \gg h$, things really **want** to work nicely...

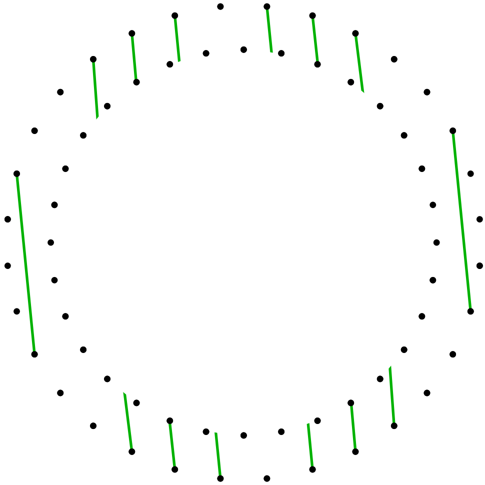
A root in E_8



Diagrams for compatibility (continued)

When $|o| \gg h$, things really **want** to work nicely...

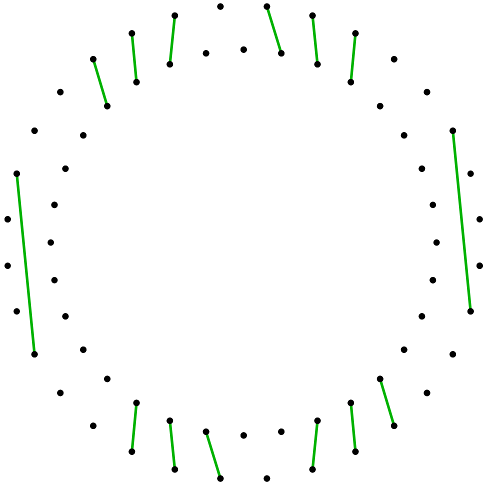
A root in E_8



Diagrams for compatibility (continued)

When $|o| \gg h$, things really **want** to work nicely...

A root in E_8 (altered)

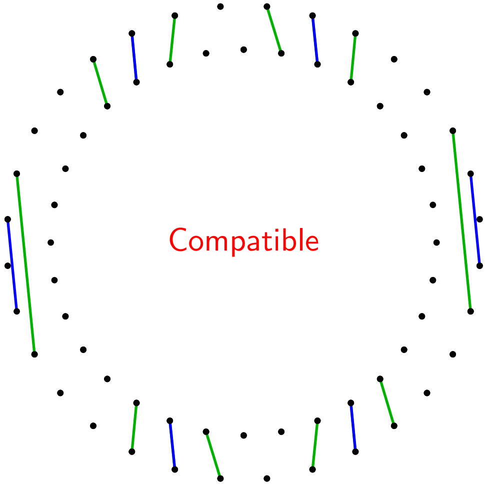


Diagrams for compatibility (continued)

When $|o| \gg h$, things really **want** to work nicely...

A root in E_8 (altered)

And another altered root

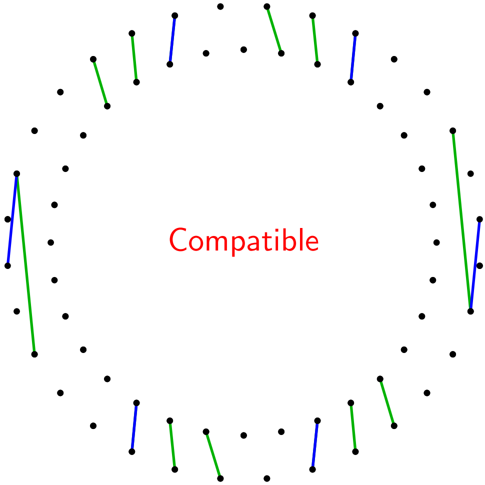


Diagrams for compatibility (continued)

When $|o| \gg h$, things really **want** to work nicely...

A root in E_8 (altered)

And another altered root

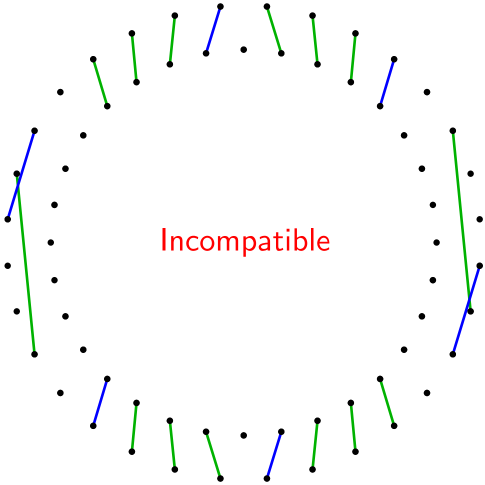


Diagrams for compatibility (continued)

When $|o| \gg h$, things really **want** to work nicely...

A root in E_8 (altered)

And another altered root

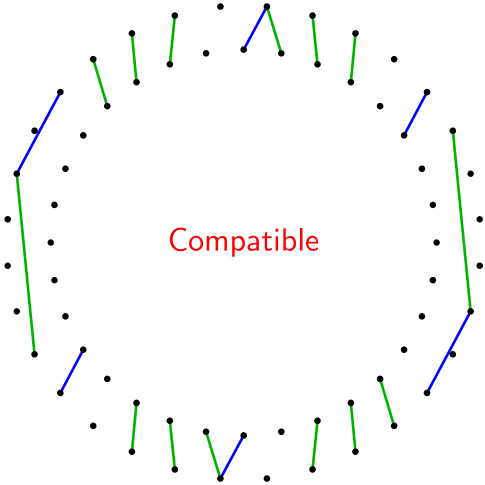


Diagrams for compatibility (continued)

When $|o| \gg h$, things really **want** to work nicely...

A root in E_8 (altered)

And another altered root

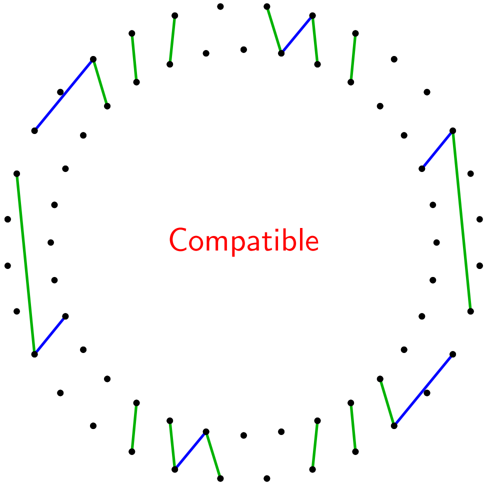


Diagrams for compatibility (continued)

When $|o| \gg h$, things really **want** to work nicely...

A root in E_8 (altered)

And another altered root

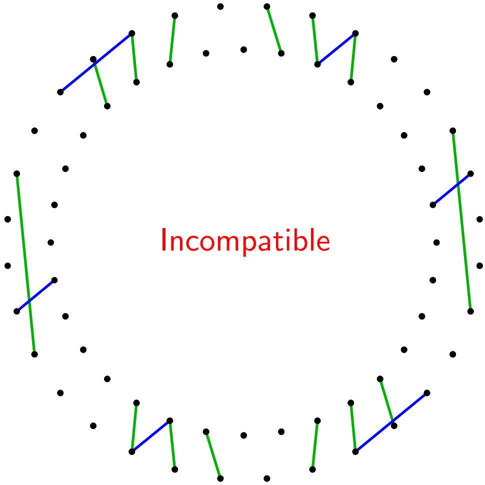


Diagrams for compatibility (continued)

When $|o| \gg h$, things really **want** to work nicely...

A root in E_8 (altered)

And another altered root

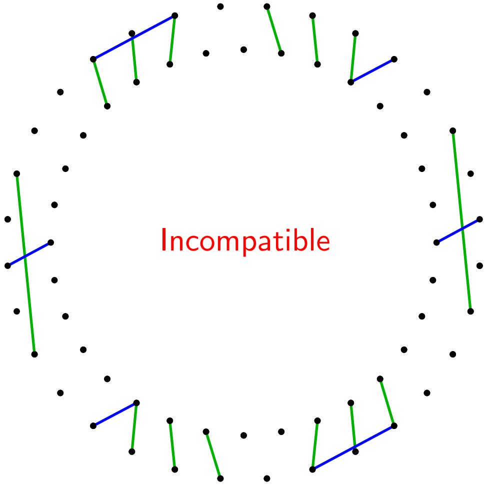


Diagrams for compatibility (continued)

When $|o| \gg h$, things really **want** to work nicely...

A root in E_8 (altered)

And another altered root

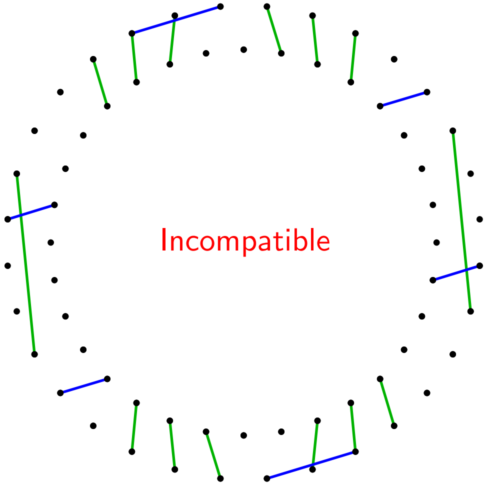


Diagrams for compatibility (continued)

When $|o| \gg h$, things really **want** to work nicely...

A root in E_8 (altered)

And another altered root

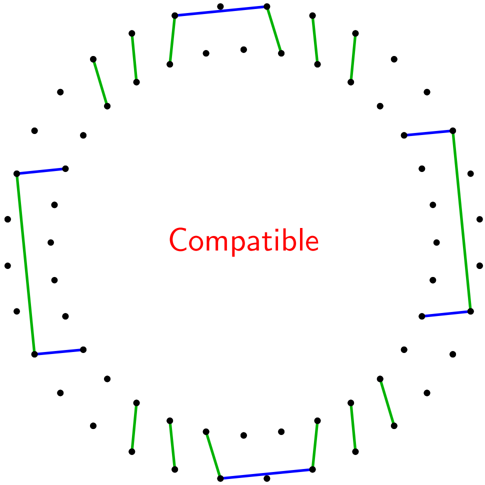


Diagrams for compatibility (continued)

When $|o| \gg h$, things really **want** to work nicely...

A root in E_8 (altered)

And another altered root

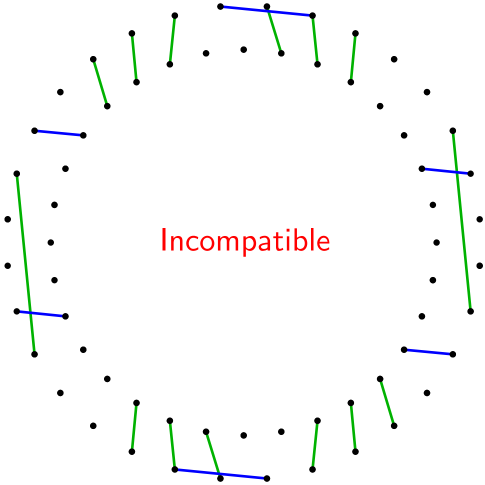


Diagrams for compatibility (continued)

When $|o| \gg h$, things really **want** to work nicely...

A root in E_8 (altered)

And another altered root

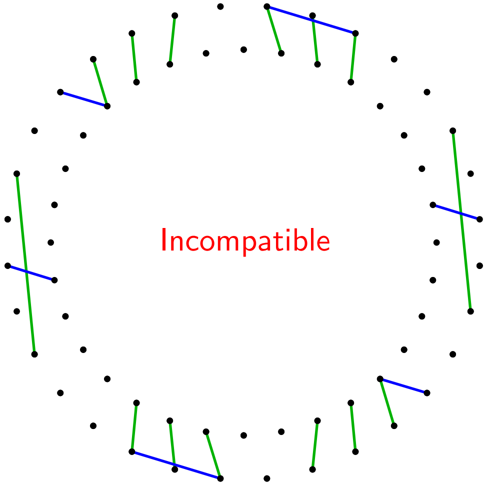


Diagrams for compatibility (continued)

When $|o| \gg h$, things really **want** to work nicely...

A root in E_8 (altered)

And another altered root

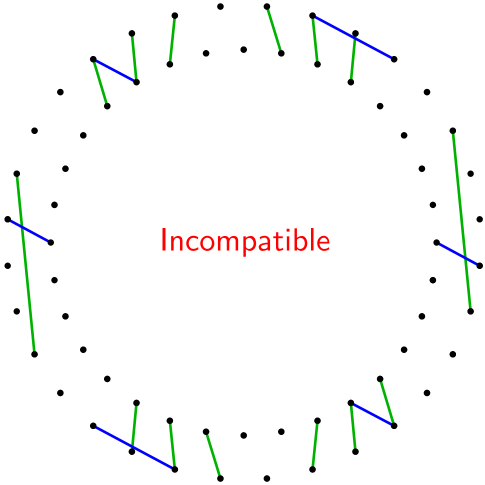


Diagrams for compatibility (continued)

When $|o| \gg h$, things really **want** to work nicely...

A root in E_8 (altered)

And another altered root

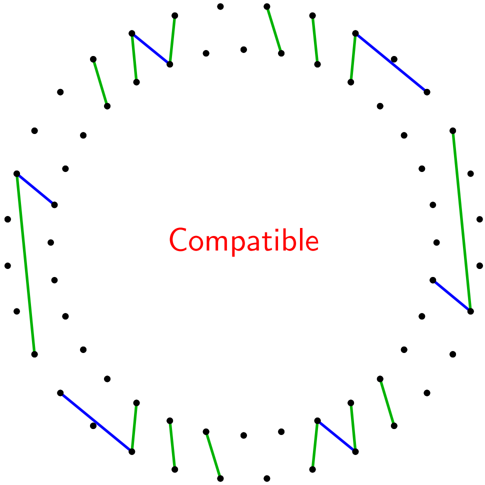


Diagrams for compatibility (continued)

When $|o| \gg h$, things really **want** to work nicely...

A root in E_8 (altered)

And another altered root

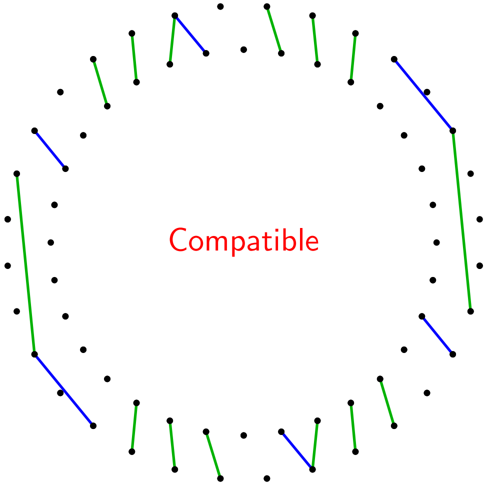


Diagrams for compatibility (continued)

When $|o| \gg h$, things really **want** to work nicely...

A root in E_8 (altered)

And another altered root

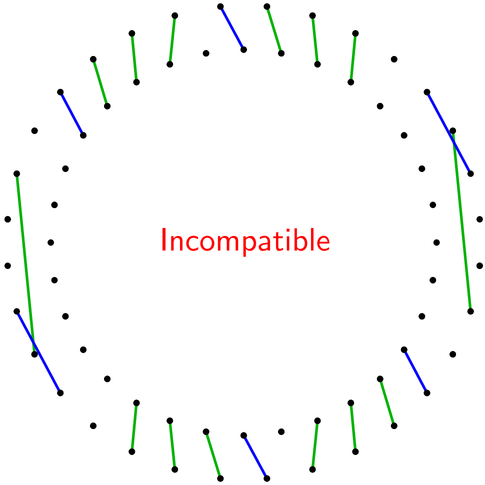


Diagrams for compatibility (continued)

When $|o| \gg h$, things really **want** to work nicely...

A root in E_8 (altered)

And another altered root

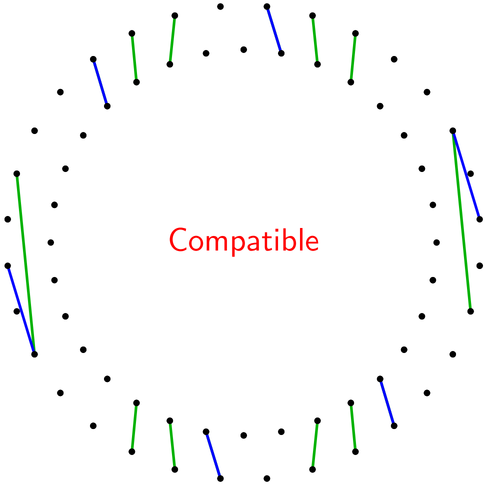


Diagrams for compatibility (continued)

When $|o| \gg h$, things really **want** to work nicely...

A root in E_8 (altered)

And another altered root

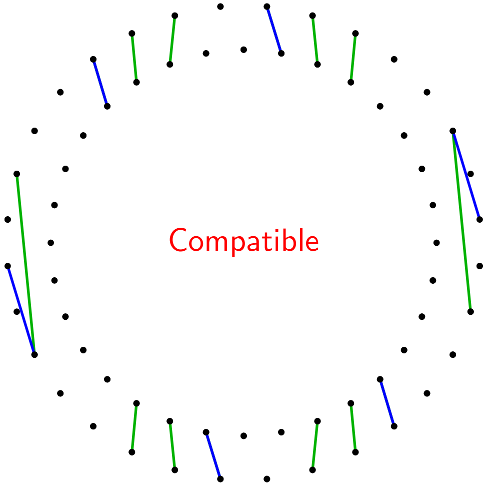


Diagrams for compatibility (continued)

When $|o| \gg h$, things really **want** to work nicely...

A root in E_8 (altered)

And another altered root



With very few additional (ad hoc) alterations (in E_6, E_7, E_8, F_4), we obtain compatibility diagrams for all finite Coxeter groups.

Closing thoughts

The ideal:

Ideally, we want a completely uniform construction and a completely uniform criterion in both settings.

What we have:

What we have is a completely uniform construction in both settings, and so far no uniform criterion in either setting.

In the compatibility setting, we also have a non-uniform alteration of the construction which leads to a very nice criterion.

Heuristically:

Because we start with a construction that reproduces the classical combinatorial models, this work suggests that combinatorial models for crossing/noncrossing or compatibility for exceptional Coxeter groups cannot be much simpler than what is described here.