Cyclic Sieving Phenomenon for the Generalized Cluster Complexes

Sen-Peng Eu
游森棚

speu@nuk.edu.tw

University of Kaohsiung, Taiwan
高雄大學

• Part of this work is done during my visit at School of Mathematics, University of Minnesota.

• To appear in Advances in Applied Mathematics, jointed with T.Fu.

• This work is mentioned by V. Reiner in his invited talk in 2007 AMS-MAA annual meeting under the title “A new Combinatorics"

Outline of the talk

- Cyclic Sieving Phenomenon
- Cluster complex and Generalized Cluster complex
- The result in type A, idea of proof.
- More Results
- Discussion and Open Problems

Cyclic Sieving Phenomenon

- The notion is by Reiner, Stanton, White (JCTA, 2005)

· X :=a combinatorial structures

· $X(q) \in \mathbb{Z}[q]$, $X(1) = |X|$

· C := a cyclic group acting on X, where $|C| = n$.

- $(X, X(q), C)$ exhibits CSP := for every $c \in C$,

$$[X(q)]_{q=\omega} = |\{x \in X : c(x) = x\}|,$$

where ω is a root of 1, of the same multiplicative order as c.

- Equivalently, write

$$X(q) \equiv a_0 + a_1 q + \dots a_{n-1} q^{n-1} (\bmod q^n - 1),$$

then a_k = orbits whose stablizer order divides k.

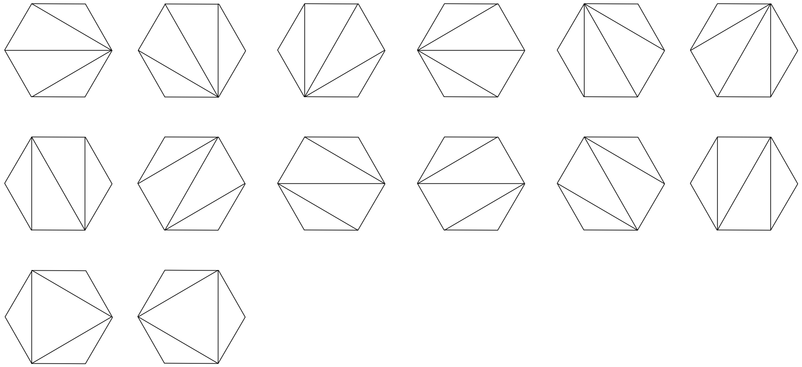
Cyclic Sieving Phenomenon

- For example,

$\cdot\ X :=$ Δ-dissections of a regular hexagon.

$\cdot\ X(q) = \frac{1}{[5]}\begin{bmatrix}8\\4\end{bmatrix} = q^{12}+q^{10}+q^9+2q^8+q^7+2q^6+q^5+2q^4+q^3+q^2+1$

$\cdot\ C := \mathbb{Z}_6$



- Let $c = 3 \in \mathbb{Z}_6$ (turn 180°). Then $\omega = -1$.

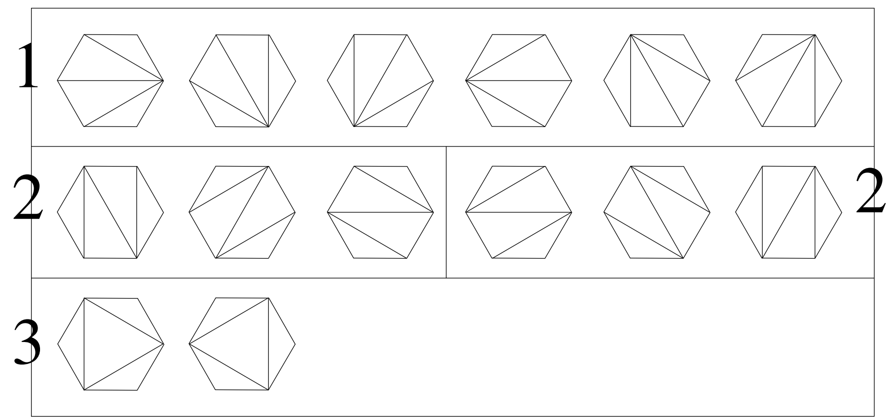
$[X(q)]_{q=-1} = 6 = |\{x \in X : x$ looks the same when turn $180^\circ\}|$.

- There is much information hidden in the generating function.

Cyclic Sieving Phenomenon

- Equivalently,

$$\begin{aligned} X(q) : &= q^{12} + q^{10} + q^{9} + 2q^{8} + q^{7} + 2q^{6} + q^{5} + 2q^{4} + q^{3} + q^{2} + 1 \\ &\equiv 4 + 1q + 3q^{2} + 2q^{3} + 3q^{4} + 1q^{5} \bmod q^{6} - 1 \end{aligned}$$



$4 = \#$ orbits

$1 = \#$ orbits whose stablizer order divides 1

$3 = \#$ orbits whose stablizer order divides 2

$2 = \#$ orbits whose stablizer order divides 3... etc.

CSP on dissections

Theorem. (Reiner, Stanton, White, 2005)

$\cdot$ X := triangulation of $(n+2)$-gon.
$\cdot$ $X(q) := \dfrac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q$.
$\cdot$ C := cyclic group of order $n+2$, by rotation.

Then $(X, X(q), C)$ exhibits CSP.

- It still works when using fewer diagonals....

CSP on dissections

Theorem. (Reiner, Stanton, White, 2005)

$\cdot\ X :=$ dissections of $(n+2)$-gon using k diagonals.

$$\cdot\ X(q) := \frac{1}{[k+1]_q} \begin{bmatrix} n+k+1 \\ k \end{bmatrix}_q \begin{bmatrix} n-1 \\ k \end{bmatrix}_q .$$

$\cdot\ C :=$ cyclic group of order $n+2$, by rotation.

Then $(X, X(q), C)$ exhibits CSP.

- We will see, this is the type A_{n-1} , $s = 1$ case of our results.
- In our language,

  k-faces of the cluster complex $\Delta^1(A_{n-1})$ exhibits CSP.
- What is the cluster complex $\Delta^1(\Phi)$?
- What is the generalized cluster complex $\Delta^s(\Phi)$?

Cluster Complex $\Delta(\Phi)$

- Developed by Fomin and Zelevinsky(2002, Ann. Math.).

  From a Root system $\Phi \rightsquigarrow$ construct a cluster complex $\Delta(\Phi)$

Step 1: Take a root system Φ, consider the **ground set** $\Phi_{\geq -1}$.

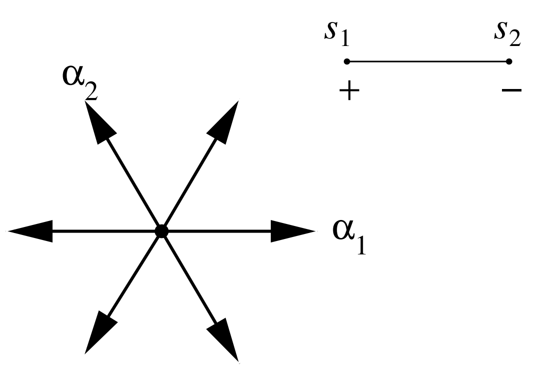
Step 2: Define two **involutions** $\tau_{\pm}$ on $\Phi_{\geq -1}$.

Step 3: Define a **cyclic group** $\Gamma := \langle \tau_- \tau_+ \rangle$ acting on $\Phi_{\geq -1}$.

Step 4: Define **compatibility** of roots under the action of Γ

Step 5: Define the **Cluster complex** $\Delta(\Phi)$ by compatibility.

- Take $\Phi = A_2$ as an example.



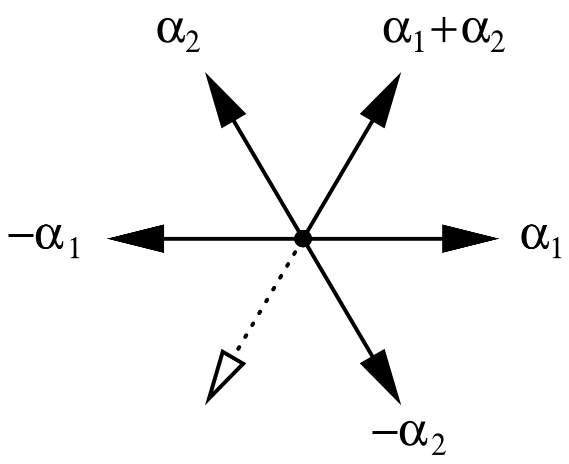
Step 1: The ground set

Step 1: Ground set:

$$\Phi_{\geq -1} := \Phi_{>0} \cup \Phi_{=-1},$$

$\Phi_{>0}$:= positive roots, $\Phi_{=-1}$:=negative simple roots

A_2:



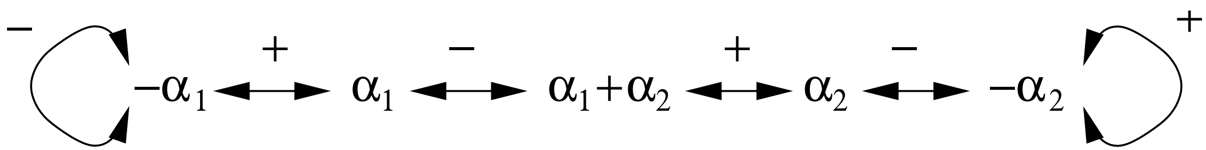
Step 2: Two involutions $\tau_\pm$ on $\Phi_{\geq -1}$

Step 2: Define the involutions $\tau_\pm : \Phi_{\geq -1} \to \Phi_{\geq -1}$ by

$$\tau_\epsilon(\alpha) = \begin{cases} \alpha & \text{if } \alpha = -\alpha_i, \text{ for } i \in I_{-\epsilon}, \\ \left(\prod_{i \in I_\epsilon} s_i\right)(\alpha) & \text{otherwise,} \end{cases}$$

for $\epsilon \in \{+, -\}$.

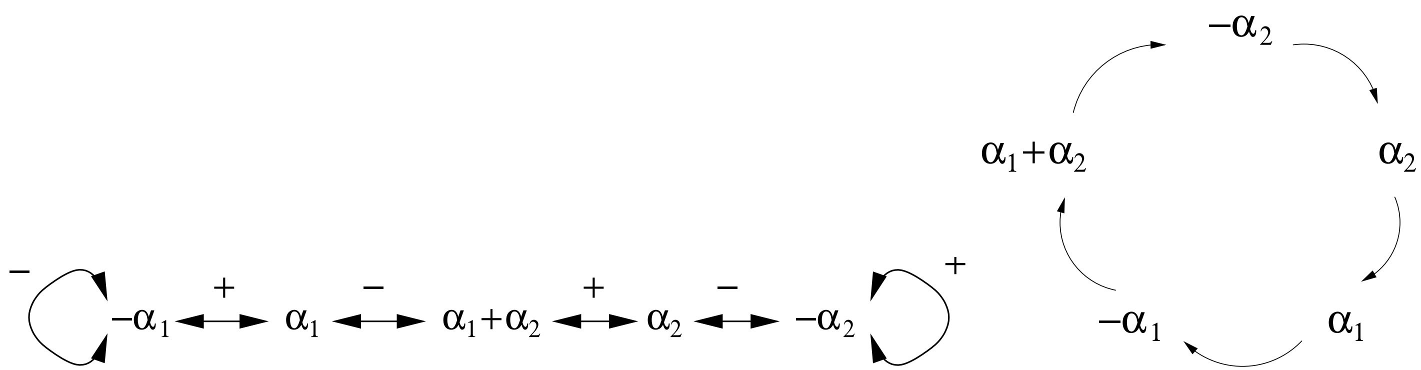
A_2:



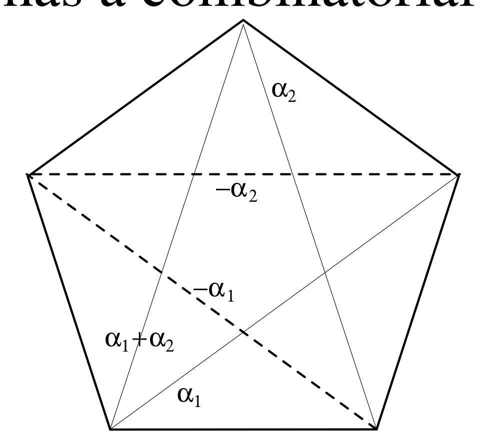
Step 3: cyclic group Γ acting on $\Phi_{\geq -1}$

Step 3: Define **cyclic group** $\Gamma := \langle \tau_- \tau_+ \rangle$, acting on $\Phi_{\geq -1}$

A_2:



Which has a combinatorial model:



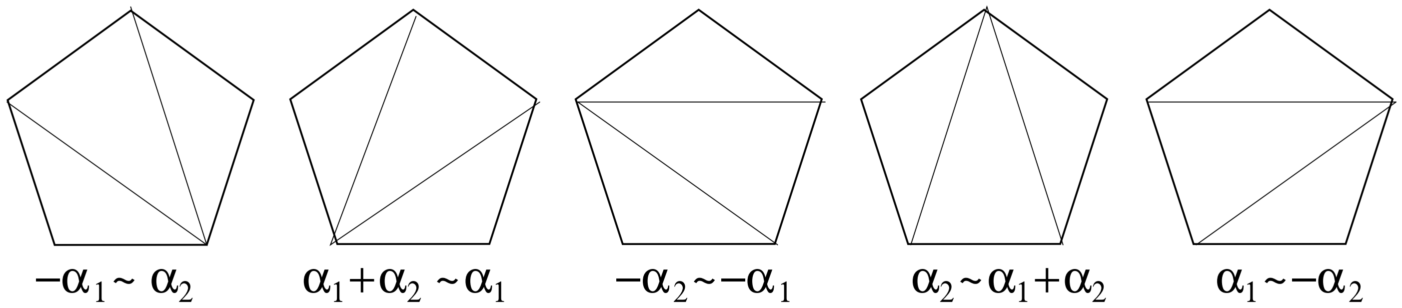
Step 4: Define compatibility

Step 4: Define **compatibility**

(i) $-\alpha_i \sim \beta \iff$ expansion of β does not involve α_i.

(ii) $\alpha \sim \beta \iff \Gamma(\alpha) \sim \Gamma(\beta)$;

A_2:

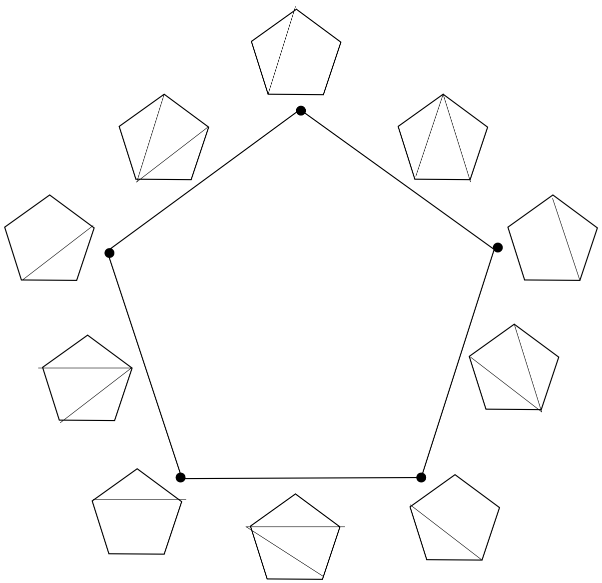


Exactly the 'noncrossing diagonals'!

Step 5: Cluster complex $\Delta(\Phi)$

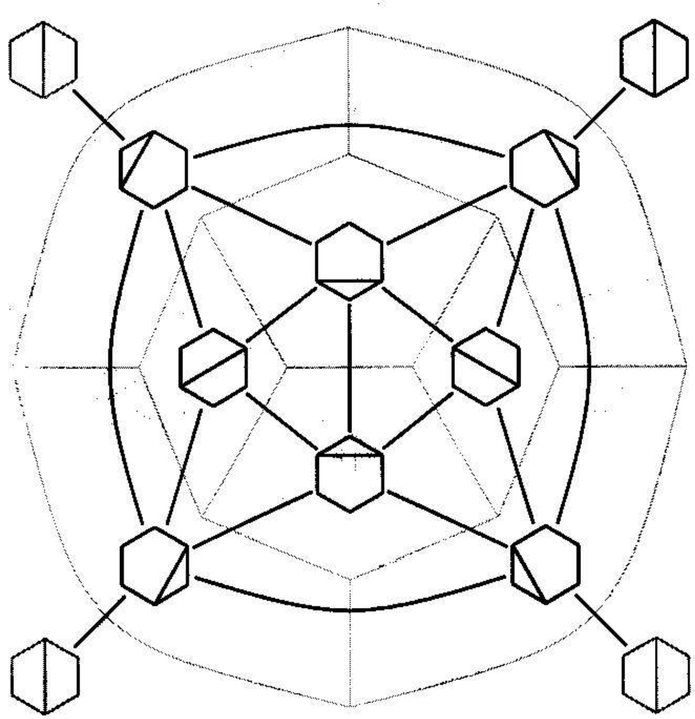
Step 5: Define the **Cluster complex** $\Delta(\Phi)$ by compatibility.

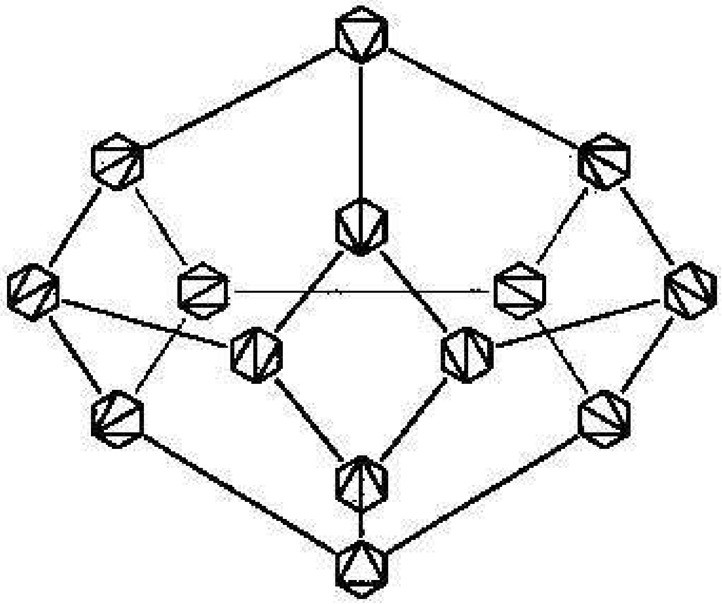
A_2: $\Delta(A_2)$



Step 5: Cluster complex $\Delta(\Phi)$

- $\Delta(A_3)$ and its dual complex.

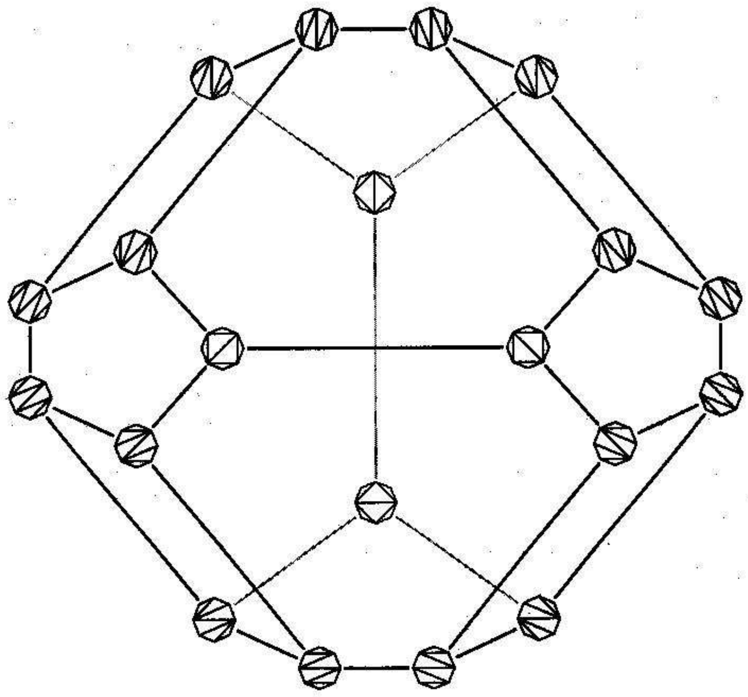




- $\Delta(A_{n-1})$ is the dual complex of the associahedron.

Step 5: Cluster complex $\Delta(\Phi)$

- dual complex of $\Delta(B_3)$



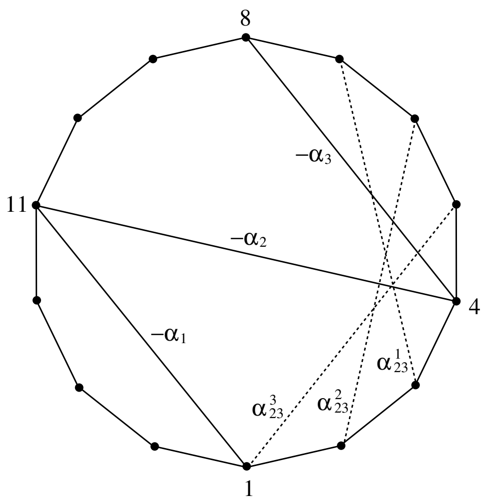
- $\Delta(B_{n-1})$ is the dual complex of the cyclohedron.

Generalized Cluster complex $\Delta(\Phi)$

- Developed by Fomin and Reading (Int. Math. Res. Notices, 2005).

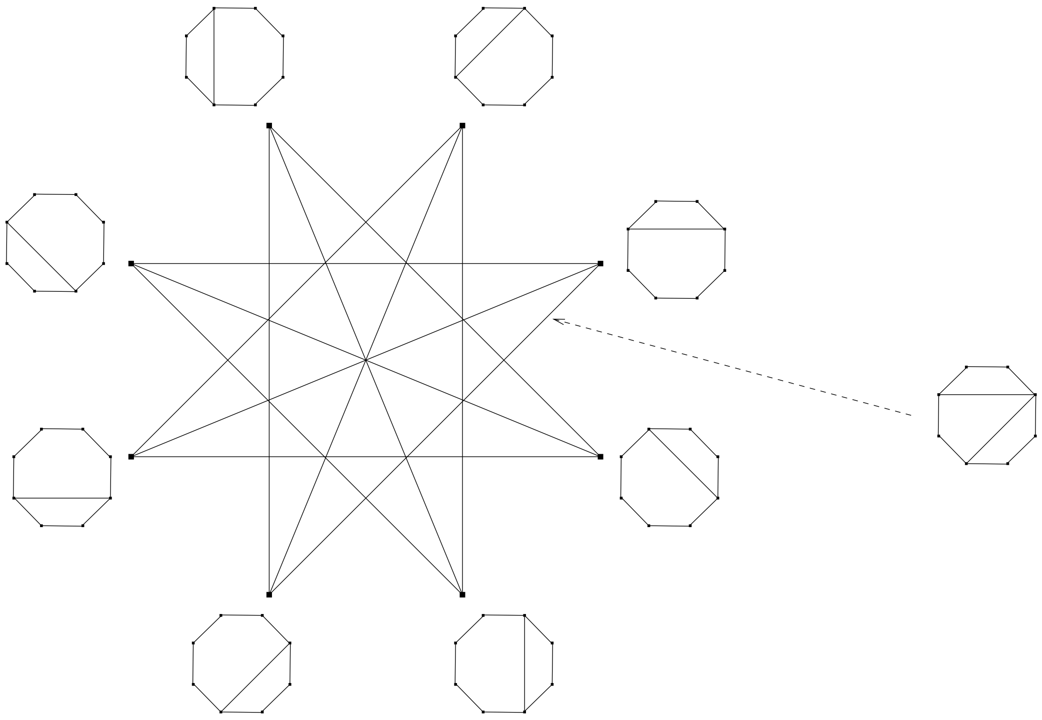
  Root system Φ and $s \rightsquigarrow$ generalized cluster complex $\Delta^s(\Phi)$

- The Steps 1-5 are similar.
- What is s?
  - 1 set of simple negative roots.
  - s sets of positive roots.
- $A_3, s = 3$



Generalized Cluster complex $\Delta^s(\Phi)$

- $\Delta^2(A_2)$



- We will consider all types, for all s, and all k-dim faces.
- Our Goal: k-faces of $\Delta^s(\Phi)$ exhibits CSP.....

$\Delta^s(A_{n-1})$

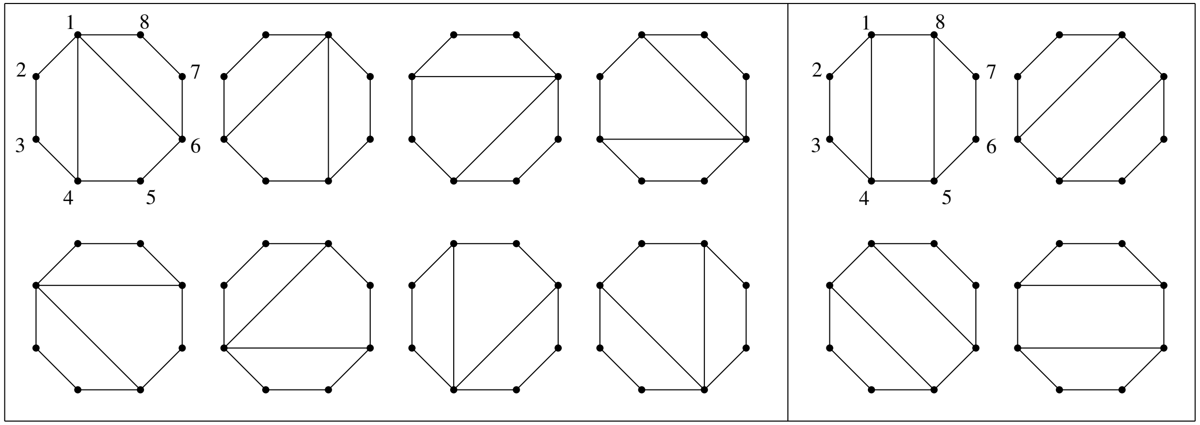
- In type A_{n-1}, combinatorial model realized in dissections.

· $(sn+2)$-gon.

· A-diagnoal $=$ diagonal.

· compatible $=$ noncrossing diagonal.

- Type A, $s=2$, $n=3$, $k=2$. These are 2-faces of $\Delta^2(A_2)$:



$$\frac{1}{k+1}\binom{sn+k+1}{k}\binom{n-1}{k}$$

CSP on $\Delta^s(A_{n-1})$

Theorem. (Eu, Fu, 2007)

· $X :=$ k-faces of $\Delta^s(A_{n-1})$.

· $X(q) := \dfrac{1}{[k+1]_q} \begin{bmatrix} sn+k+1 \\ k \end{bmatrix}_q \begin{bmatrix} n-1 \\ k \end{bmatrix}_q$.

· $C :=$ cyclic group of order $sn+2$, by rotation.

Then $(X, X(q), C)$ exhibits CSP.

- Note that $X(q)$ is the natural q-analogue.
- When $s = 1$, it is Reiner, Stanton, White's result.

Idea of Proof

- Check the definition of CSP, LHS=RHS.

$$[X(q)]_{q=\omega} = |\{x \in X : c(x) = x\}|.$$

- LHS is easy, once we have a correct $X(q)$.
  - Not always the natural q-analogue.
  - Hence, it not so easy.

- RHS is not easy, we count # elements invariant under d-fold rotation.
  - By using bijective argument.
  - Once we have $X(q)$, it is not so un-easy.

- Take $\Delta^s(A_{n-1})$ as an example.

LHS, A_{n-1}

For LHS,

- Take $X(q) := \frac{1}{[k+1]_q} \begin{bmatrix} sn+k+1 \\ k \end{bmatrix}_q \begin{bmatrix} n-1 \\ k \end{bmatrix}_q$.
- We use the lemma $\begin{bmatrix} m+k-1 \\ k \end{bmatrix}_{q=\omega} = \begin{cases} \binom{\frac{m+k}{d}-1}{\frac{k}{d}} & \text{if } d|k, \\ 0 & \text{otherwise.} \end{cases}$
- The result is

$$[X(q)]_{q=\omega} = \begin{cases} \binom{\frac{sn+k+1}{2}}{\frac{k+1}{2}} \binom{\frac{n-2}{2}}{\frac{k-1}{2}} & \text{if } d = 2,\ k \text{ odd, and } n \text{ even} \\ \binom{\frac{sn+2+k}{d} - 1}{\frac{k}{d}} \binom{\lfloor \frac{n-1}{d} \rfloor}{\frac{k}{d}} & \text{if } d \geq 2 \text{ and } d|k, \\ 0 & \text{otherwise.} \end{cases}$$

RHS, A_{n-1}

For RHS,

• We count number of dissections with $(sn+2)$-gon, k diagonal, and is invariant under d-fold rotation.

• Case 1: If it has a center line $\rightsquigarrow d = 2$, $sn+2$ is even, k odd.

· Simple recurrence.

$$\frac{sn+2}{2} \cdot G\left(s, \frac{n}{2}, \frac{k-1}{2}; 1\right) = \binom{\frac{sn+k+1}{2}}{\frac{k+1}{2}} \binom{\frac{n-2}{2}}{\frac{k-1}{2}},$$

• Case 2: If it has no center line $\rightsquigarrow d \geq 2$, $d|k$

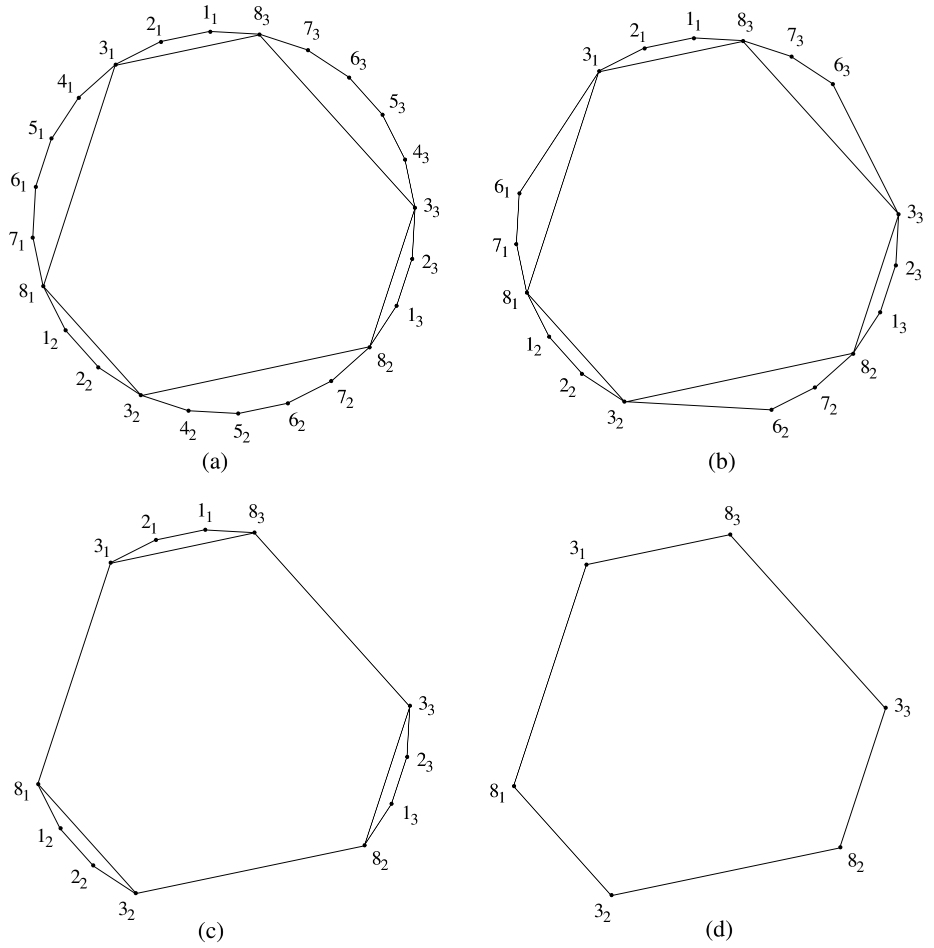
· Biject to $\{(\mu,\nu)|\mu \in A(m,k), \nu \in B(m,k), m = \frac{n-1-r}{d}\}$,

$$A(m,k) = \{(a_1, \ldots, a_{\frac{k}{d}}) : 1 \leq a_1 \leq \cdots \leq a_{\frac{k}{d}} \leq b\},$$

$$B(m,k) = \{(\epsilon_1, \ldots, \epsilon_m) \in \{0,1\}^m : \text{ exactly } \tfrac{k}{d} \text{ entries } \epsilon_j = 1\}.$$

RHS, A_{n-1}, cont.

- For example, this will map to $((3,8),(0,1,1))$



- LHS=RHS, and we are done.

$\Delta^s(B_n)$

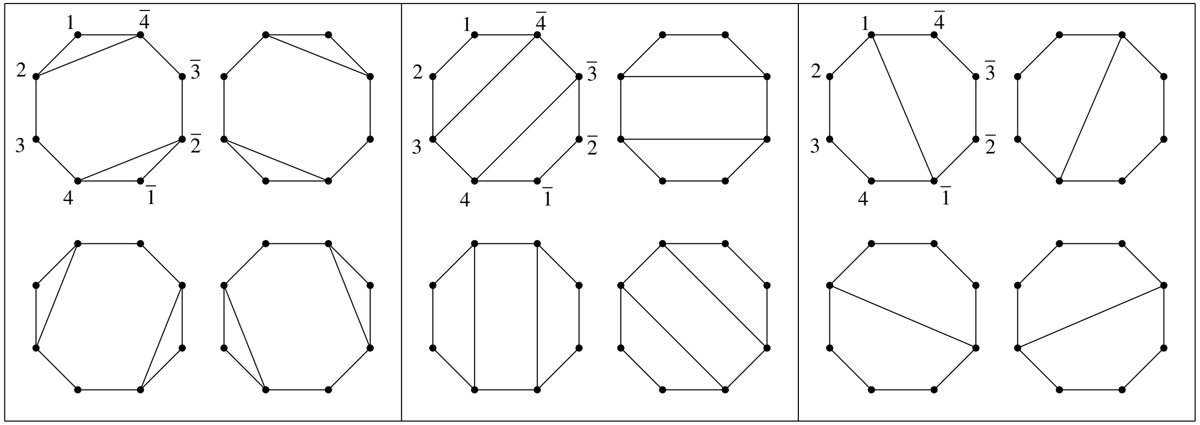
- In type B_n, combinatorial model realized in dissections.

· $(s(2n)+2)$-gon.

· a B-diagonal = a symmetric pair, or a 2-colored antipodal.

· compatible = noncrossing diagonal.

- Type B, $s=1$, $n=3$, $k=1$. These are 1-faces of $\Delta^1(B_3)$:



$$\binom{sn+k}{k}\binom{n}{k}$$

CSP on $\Delta^s(B_n)$

Theorem. (Eu, Fu, 2007)

· $X :=$ k-faces of $\Delta^s(B_n)$.
· $X(q) := \begin{bmatrix} sn+k \\ k \end{bmatrix}_{q^2} \begin{bmatrix} n \\ k \end{bmatrix}_{q^2}$
· $C :=$ cyclic group of order $2sn+2$, by rotation.

Then $(X, X(q), C)$ exhibits CSP.

$\Delta^s(D_n)$

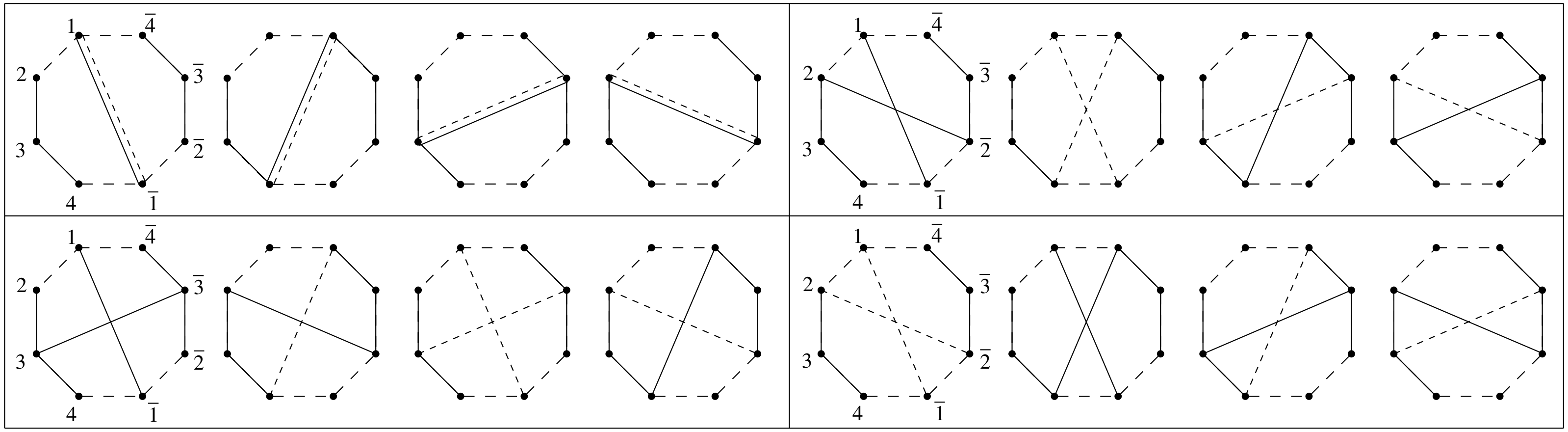
- In type D_n, combinatorial model realized in dissections.

· $(s(2n-2)+2)$-gon.

· a D-diagnoal $=$ a B-diagonal, or a 2-color antipodal.

· compatible $=$ complicated.

- Type D, $s=3$, $n=2$, $k=2$. These are 2-faces of $\Delta^3(D_2)$:



$$\binom{s(n-1)+k}{k}\binom{n}{k} + \binom{s(n-1)+k-1}{k}\binom{n-2}{k-2}$$

CSP on $\Delta^s(D_n)$

Theorem. (Eu, Fu, 2007)

$\cdot$ $X :=$ k-faces of $\Delta^s(D_n)$.

$\cdot$ $X(q) := \begin{bmatrix} s(n-1)+k \\ k \end{bmatrix}_{q^2} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{q^2} + \begin{bmatrix} s(n-1)+k \\ k \end{bmatrix}_{q^2} \begin{bmatrix} n-2 \\ k-1 \end{bmatrix}_{q^2} \cdot q^n$

$$+ \begin{bmatrix} s(n-1)+k \\ k \end{bmatrix}_{q^2} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_{q^2} + \begin{bmatrix} s(n-1)+k-1 \\ k \end{bmatrix}_{q^2} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_{q^2} \cdot q^n.$$

$\cdot$ $C :=$ cyclic group of order $2s(n-1)+2$, by rotation.

Then $(X, X(q), C)$ exhibits CSP.

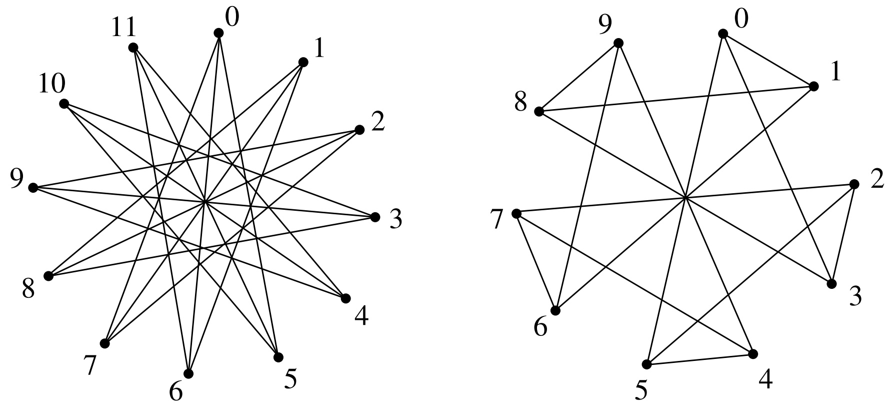
$\Delta^s(I_2(a))$

- In type $I_2(a)$, combinatorial model realized in graphs on the plane.

$\cdot$ $(sa+2)$-vertices.

$\cdot$ $(s+1)$-regular graphs.

- Type $I_2(a)$, these are $\Delta^2(I_2(5))$ and $\Delta^2(I_2(4))$.



(a) (b)

CSP on $\Delta^s(I_2(a))$

Theorem. (Eu, Fu, 2007)

$\cdot$ X := edge set of the graphs $\Delta^s(I_2(a))$.

$\cdot$ $X(q) := \dfrac{[sa+2]_q}{[2]_q} \cdot \dfrac{[sa+a]_q}{[a]_q}$

$\cdot$ C := cyclic group of order $sa+2$, by rotation.

Then $(X, X(q), C)$ exhibits CSP.

Exceptional Types

• For Exceptional types $E_6, E_7, E_8, F_4, H_3, H_4$, we have no combinatorial models. (We will have, see Reading's lecture.)

We still can form the $\Delta^s(\Phi)$,
We still can investigate the orbits under rotation for k-faces,
We still can check if CSP holds.

• These can be done by computer, once we have

· $X := \Delta^s(\Phi)$ OK

· $X(q) :=?????$ NOT OK

· cyclic group $:= \mathbb{Z}_{sh+2}$ OK

• So far we can only check the facets for $s = 1$ in the exceptional types.

Results summary

	A_n	B_n	D_n	$E_6, E_7, E_8, F_4, H_3, H_4$	$I_2(a)$
$\forall s$	•	•	•	$s = 1$, some $s \geq 2$	•
$\forall$ dim	•	•	•	Facets ok, some k	•

Generalized Catalan Numbers

Since all facets when $s = 1$ in all types have CSP,...

- Let

$$\mathrm{Cat}(\Phi) := \prod_{i=1}^{n} \frac{h + e_i + 1}{e_i + 1}$$

be the generalized Catalan numbers. We have proved the following:

Theorem. (Eu, Fu, 2007)

$\cdot\ X :=$ facets of $\Delta(\Phi)$

$\cdot\ X(q) = \mathrm{Cat}(\Phi, q) := \prod_{i=1}^{n} \frac{[h + e_i + 1]_q}{[e_i + 1]_q}$

$\cdot\ C := \mathbb{Z}_{h+2}$

Then $(X, X(q), C)$ exhibits CSP.

For larger s

Putting s into picture...

- Let

$$\mathsf{Cat}^{(s)}(\Phi) := \prod_{i=1}^{n} \frac{sh + e_i + 1}{e_i + 1}$$

be the generalized[2] Catalan numbers.

Conjecture. (Reiner, Stanton, White)

$\cdot$ X := facets of $\Delta(\Phi)$

$\cdot$ $X(q) = \mathsf{Cat}^{(s)}(\Phi, q) := \prod_{i=1}^{n} \frac{[sh + e_i + 1]_q}{[e_i + 1]_q}$

$\cdot$ $C := \mathbb{Z}_{sh+2}$

Then $(X, X(q), C)$ exhibits CSP.

- Note that we have proved the case A_{n-1}, B_n, D_n, $I_2(a)$.

For larger s, and any k-faces

We want to do more...

- Let

$$\mathrm{Cat}_k^{(s)}(\Phi) := ???$$

be the generalized^2 k-face numbers.

Conjecture. (An ill-posed problem)

$\cdot$ $X :=$ any k-faces of $\Delta(\Phi)$

$\cdot$ $X(q) =???$

$\cdot$ $C := \mathbb{Z}_{sh+2}$

Then $(X, X(q), C)$ exhibits CSP.

- Note that we have proved the case A_{n-1}, B_n, D_n, $I_2(a)$.

- So far we do almost nothing on the exceptional types.

Discussion

	A_n	B_n	D_n	$E_6, E_7, E_8, F_4, H_3, H_4$	$I_2(a)$
s	•	•	•	$s = 1$, (& some $s \geq 2$)	•
dim	•	•	•	mainly facets	•

What's the obstacle(s)?

- For $s > 1$ and facets, we have no systematic method.
  - · s can be very big, too many points.
  - · How can you check on computer, say, $s = 10000$, and E_8?

- For k-faces of exceptional types, it is worse... we even have no $X(q)$.
  - · The q-nify of counting formulae does not work.
  - · Sometimes it is even not polynomial.

Discussion

	A_n	B_n	D_n	$E_6, E_7, E_8, F_4, H_3, H_4$	$I_2(a)$
s	•	•	•	$s = 1$, (& some $s \geq 2$)	•
dim	•	•	•	mainly facets	•

- The $X(q)$ is not easily known, e.g. D_n.
- In Type D_n, we find four q-analogues perfoming CSP.

$$X(q) = \begin{bmatrix} s(n-1)+k \\ k \end{bmatrix}_{q^2} \begin{bmatrix} n-2 \\ k \end{bmatrix}_{q^2} + \begin{bmatrix} s(n-1)+k \\ k \end{bmatrix}_{q^2} \begin{bmatrix} n-2 \\ k-1 \end{bmatrix}_{q^2} \cdot (1+q^n)$$
$$+ \begin{bmatrix} s(n-1)+k \\ k \end{bmatrix}_{q^2} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_{q^2} + \begin{bmatrix} s(n-1)+k-1 \\ k \end{bmatrix}_{q^2} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_{q^2} \cdot q^n$$

also works!

Discussion

	A_n	B_n	D_n	$E_6, E_7, E_8, F_4, H_3, H_4$	$I_2(a)$
s	•	•	•	$s = 1$, (& some $s \geq 2$)	•
dim	•	•	•	mainly facets	•

- Fomin & Reading gave a formula for faces:

$$f_k(\Phi, s) = c(\Phi, k, s) \binom{n}{k} \prod_{L(e_i) \leq k} \frac{sh + e_i + 1}{e_i + 1},$$

where $c(\Phi, k, s)$ and $L(e_i)$ are case-by-case functions.

- The natural q-analogue does not work, even in $s = 1$.
- That is, when $k < n$, we have no $X(q)$ in exceptional types.

Open Questions:

So here comes the open problems:

Open Question: What (and Where) is the genuine $X(q)$?

Conjecture: With this $X(q)$, then $(X, X(q), C)$ has CSP.

Open Question: Give a more conceptual united proof. Maybe invariant theory?

Thanks for your listening.

Welcome any discussions and future collaborations.