Cluster algebras

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References

S.Fomin and A.Zelevinsky, Cluster algebras, *Current Developments in Mathematics*, Intern. Press, 2003, 1–34.

S.Fomin and N.Reading, Root systems and generalized associahedra, *IAS/Park City Math. Ser.*, AMS, to appear.

S.Fomin, M.Shapiro, and D.Thurston, Cluster algebras and triangulated surfaces. Part I, *Acta Math.*, to appear.

Cluster Algebras Portal:
`http://www.math.lsa.umich.edu/~fomin/cluster.html`

Introduction

Cluster algebras are a class of commutative rings equipped with a distinguished set of generators grouped into overlapping subsets (clusters) of the same finite cardinality.

Cluster algebras were introduced in [S.F.-A.Z., *JAMS* **15** (2002)] as an algebraic/combinatorial tool for the study of *total positivity* and *dual canonical bases* in semisimple algebraic groups.

In recent years, cluster-algebraic structures have been identified and explored in several mathematical disciplines, including:

- Lie theory and quantum groups;
- Quiver representations;
- Poisson geometry and Teichmüller theory;
- Algebraic and geometric combinatorics.

Plan

This talk will survey the most basic notions and results of the theory of cluster algebras from a combinatorial perspective.

1. Prototypical example

2. Fundamentals of general theory

3. Cluster algebras of finite type

4. Cluster combinatorics

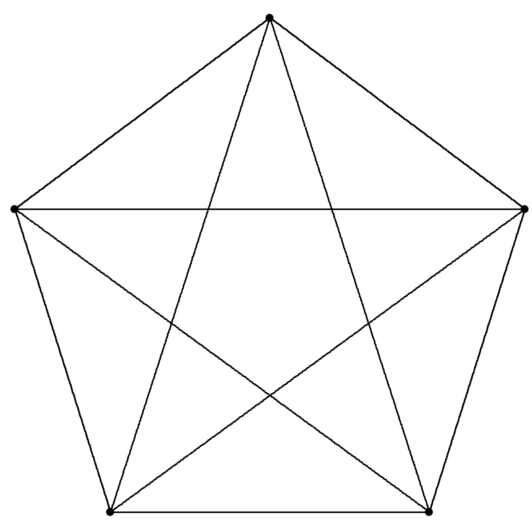
5. Cluster algebras and triangulated surfaces

Unless stated otherwise, results are joint with Andrei Zelevinsky (Northeastern University).

1. Prototypical example

Many cluster algebras arise as coordinate rings of classical algebraic varieties. A case in point is the cluster algebra $\mathcal{A}_n$ defined as follows.

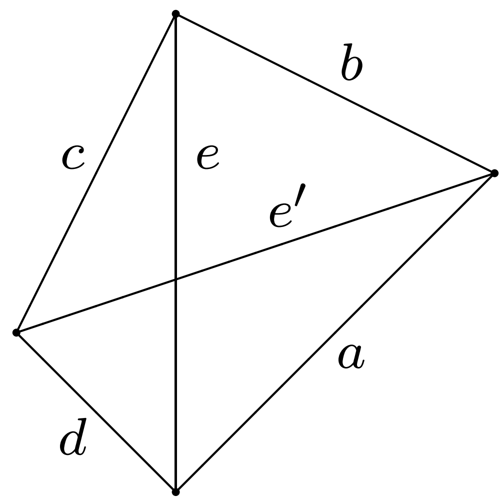
$\mathcal{A}_n$ is a commutative ring generated over $\mathbb{C}$ by $\binom{n+3}{2}$ generators x_a, where a runs over all sides and diagonals of a convex $(n+3)$-gon.



Ptolemy relations

The generators x_a are subject to $\binom{n+3}{4}$ defining relations, called the *Ptolemy relations*:

$$x_e\, x_{e'} = x_a\, x_c + x_b\, x_d\,. \qquad (*)$$



The algebra $\mathcal{A}_n$ is isomorphic to the homogeneous coordinate ring of the Grassmannian $\mathrm{Gr}_{2,n+3}$ of 2-dimensional subspaces in $\mathbb{C}^{n+3}$, with respect to its Plücker embedding.

(Identify the generators x_a with the Plücker coordinates on $\mathrm{Gr}_{2,n+3}$; then $(*)$ become the *Grassmann-Plücker relations*.)

Clusters

Let T be a *triangulation* of our $(n+3)$-gon by n non-crossing diagonals. The *cluster* $\mathbf{x}(T)$ is the n-element set of generators x_a corresponding to the diagonals of T. The *extended cluster* $\tilde{\mathbf{x}}(T)$ is the set of $2n+3$ generators corresponding to the sides and diagonals of T.

Cluster monomials

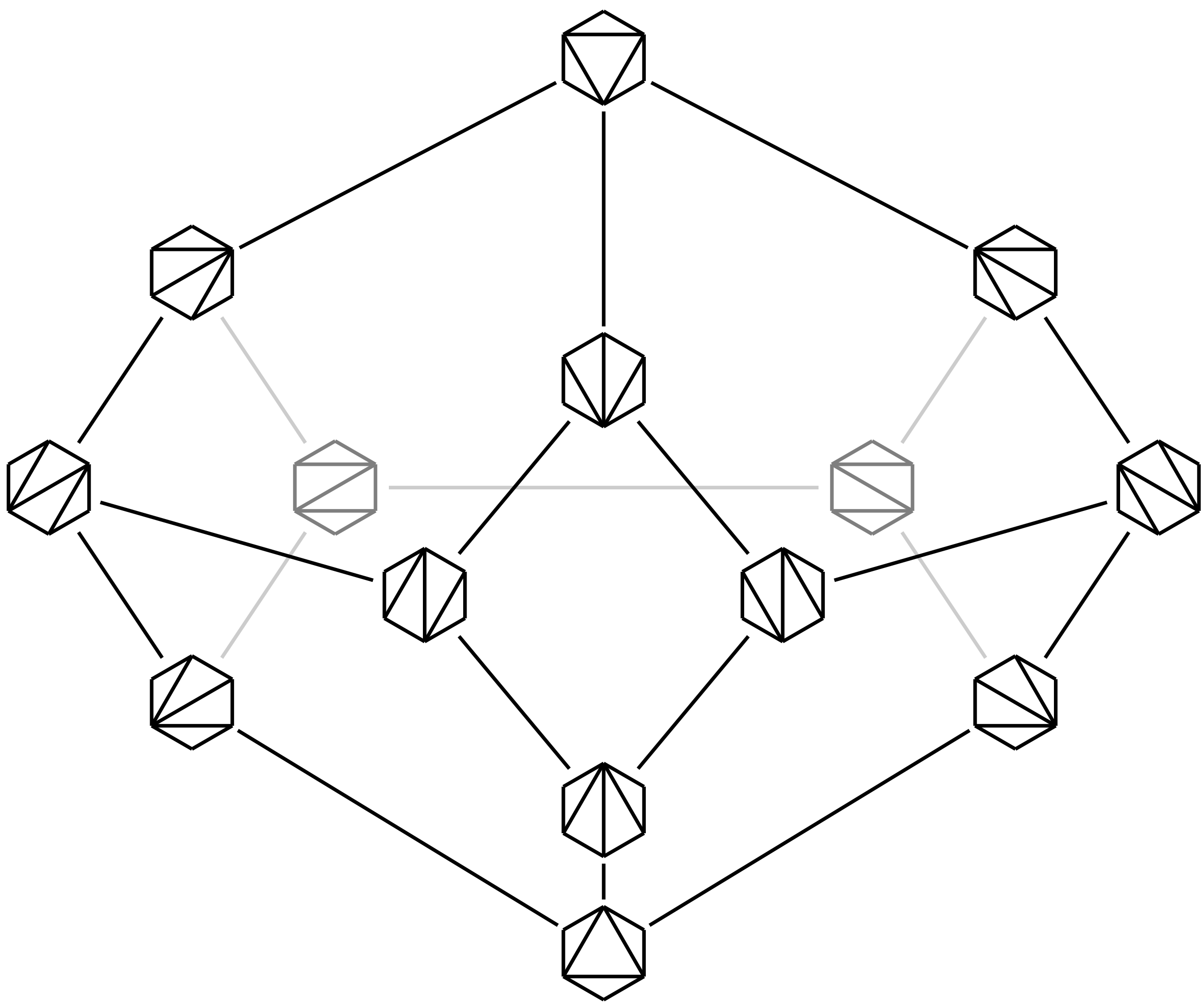
A *cluster monomial* is any monomial in the elements of some extended cluster $\tilde{\mathbf{x}}(T)$.

The following result can be traced back to classical 19$^{\text{th}}$ century literature on invariant theory.

Theorem 1 *Cluster monomials form an additive basis of* $\mathcal{A}_n$ *.*

Flips

Triangulations/clusters are related to each other by *flips*. The graph of flips is the 1-skeleton of the n-dimensional *associahedron*, also known as the *Stasheff polytope*.



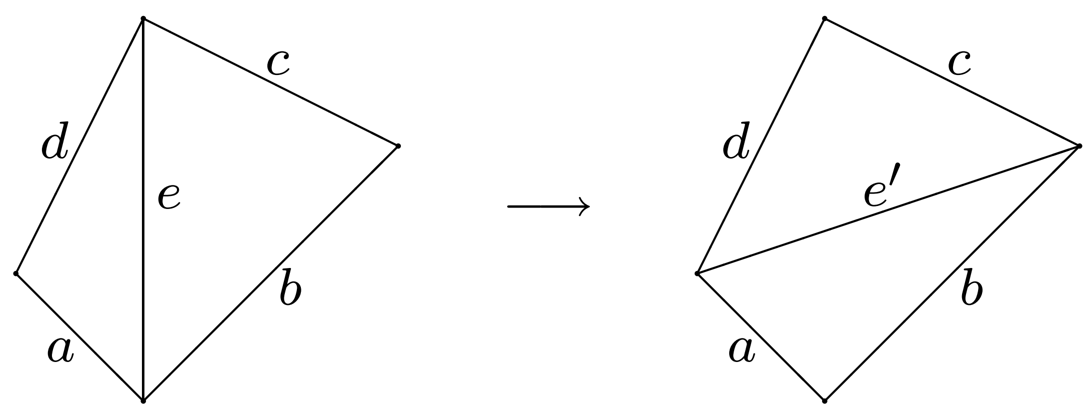
Birational maps. Ambient field

Let us associate to each triangulation T a field of rational functions in $2n+3$ variables:

$$\mathcal{F}(T) = \mathbb{C}(\tilde{\mathbf{x}}(T)).$$

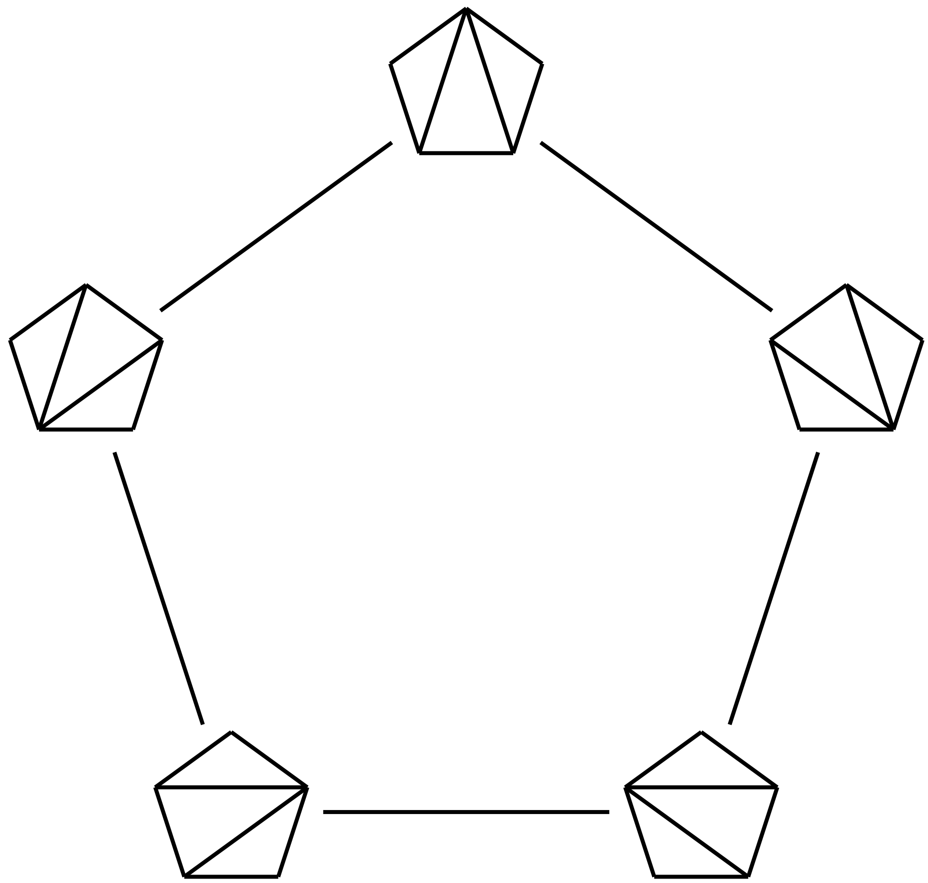
Naturally associated to each flip $T \to T'$ is a birational isomorphism $\mathcal{F}(T) \to \mathcal{F}(T')$ defined by

$$x_{e'} = \frac{x_a\, x_c + x_b\, x_d}{x_e}\,.$$



The diagram of all such isomorphisms commutes. Consequently, all fields $\mathcal{F}(T)$ can be identified with a canonical *ambient field* $\mathcal{F} \supset \mathcal{A}_n$.

Laurent phenomenon. Positivity



Theorem 2 *Each generator* x_a *is a Laurent polynomial in the elements of a given extended cluster* $\tilde{\mathrm{x}}(T)$*. All these Laurent polynomials have positive integer coefficients.*

Towards a general theory

Our next goal is to generalize this example, and in particular Theorems 1 and 2.

This will require axiomatizing

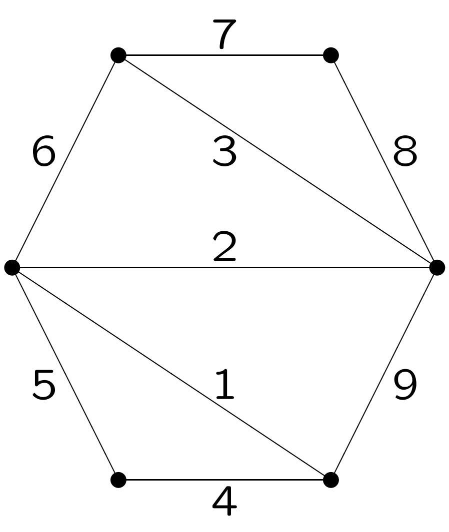
- the combinatorics of clusters and
- the algebra of birational maps between them.

Main idea: The entire structure of clusters and birational exchanges is uniquely determined, in a canonical fashion, by a certain integer matrix $\tilde{B} = \tilde{B}(T)$ which encodes the combinatorics of an arbitrary triangulation T.

Matrices $\tilde{B}(T)$

Label the diagonals of a triangulation T by the numbers $1, \ldots, n$. Label the sides of the $(n+3)$-gon by $n+1, \ldots, 2n+3 = m$.

Let $\tilde{B} = \tilde{B}(T) = (b_{ij})$ be the $m \times n$ integer matrix with rows labeled by $[1, m]$, columns labeled by $[1, n]$, and entries describing *signed adjacencies* between the sides and diagonals.



$$\tilde{B} = \left[\begin{array}{rrr} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \\ \hline -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{array}\right]$$

Ptolemy relations in terms of $\tilde{B}$

For a diagonal of T labeled k, let x_k denote the corresponding generator of $\mathcal{A}_n$. The Ptolemy relation associated with flipping that diagonal can now be written as

$$x_k\, x_k' = \prod_{b_{ik}>0} x_i^{b_{ik}} + \prod_{b_{ik}<0} x_i^{-b_{ik}}\,.$$

Flips in terms of $\tilde{B}$

We say that an $m \times n$ matrix $\tilde{B}' = (b'_{ij})$ is obtained from a matrix $\tilde{B}$ by *matrix mutation* in direction $k \in [1,n]$, and write $\tilde{B}' = \mu_k(\tilde{B})$, if

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k; \\ b_{ij} + \dfrac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} & \text{otherwise.} \end{cases}$$

If T' is obtained from T by a flip replacing a diagonal labeled k, then $\tilde{B}(T') = \mu_k(\tilde{B}(T))$.

2. Fundamentals of general theory

Seeds

Let $0 < n \leq m$. Let $\mathcal{F}$ be a field of rational functions over $\mathbb{C}$ in m independent variables. A *seed* in $\mathcal{F}$ is a pair $(\tilde{\mathbf{x}}, \tilde{B})$, where

- $\tilde{\mathbf{x}} = \{x_1, \ldots, x_m\}$ is a set of algebraically independent generators of $\mathcal{F}$;
- $\tilde{B} = (b_{ij})$ is an $m \times n$ integer matrix of rank n whose $n \times n$ submatrix $B = (b_{i,j})_{i,j \leq n}$ is *skew-symmetrizable*.

We call B the *exchange matrix* of the seed $(\tilde{\mathbf{x}}, \tilde{B})$. The set $\mathbf{x} = \{x_1, \ldots, x_n\} \subset \tilde{\mathbf{x}}$ is the *cluster*.

Each seed is defined up to a relabeling of elements of $\mathbf{x}$ together with the corresponding relabeling of rows and columns of $\tilde{B}$.

A matrix $\tilde{B}(T)$ associated with a triangulation T satisfies these conditions, with $m = 2n + 3$. Hence $(\tilde{\mathbf{x}}(T), \tilde{B}(T))$ is an example of a seed.

Seed mutations

Let $(\tilde{\mathbf{x}}, \tilde{B})$ be a seed as above. Let $1 \leq k \leq n$, so that x_k is an element of the cluster $\mathbf{x} \subset \tilde{\mathbf{x}}$. By analogy with the Ptolemy relations, we set:

$$\tilde{\mathbf{x}}_k = \tilde{\mathbf{x}} - \{x_k\} \cup \{x_k'\},$$

where

$$x_k' = \frac{\prod_{b_{ik}>0} x_i^{b_{ik}} + \prod_{b_{ik}<0} x_i^{-b_{ik}}}{x_k} \in \mathcal{F}.$$

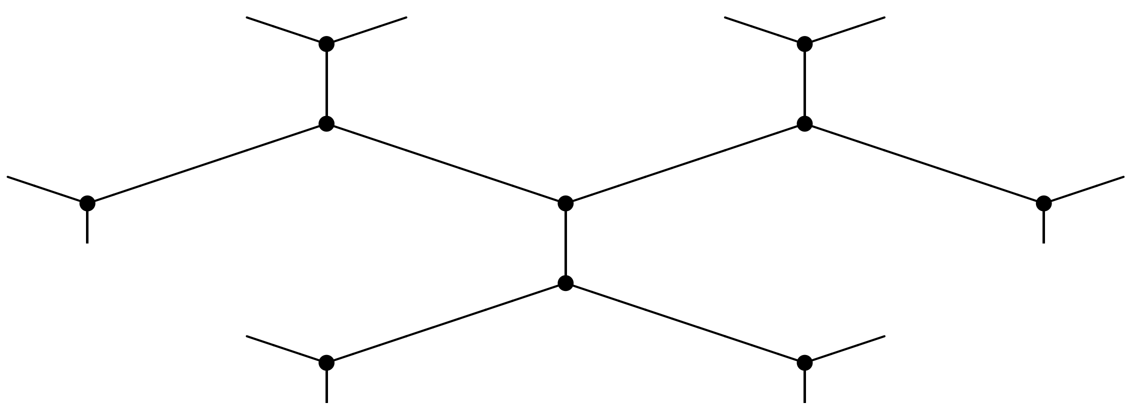
We also set

$$\tilde{B}' = \mu_k(\tilde{B}).$$

It is easy to check that $(\tilde{\mathbf{x}}', \tilde{B}')$ is again a seed. We say that $(\tilde{\mathbf{x}}', \tilde{B}')$ is obtained from $(\tilde{\mathbf{x}}, \tilde{B})$ by a *seed mutation* in direction k. Applying the same mutation to $(\tilde{\mathbf{x}}', \tilde{B}')$ recovers $(\tilde{\mathbf{x}}, \tilde{B})$.

Mutation equivalence

Seeds $(\tilde{\mathbf{x}}, \tilde{B})$ and $(\tilde{\mathbf{x}}', \tilde{B}')$ are called *mutation-equivalent* if $(\tilde{\mathbf{x}}', \tilde{B}')$ can be obtained from $(\tilde{\mathbf{x}}, \tilde{B})$ by a sequence of seed mutations.



Mutation equivalence class

Cluster algebra

Let $\mathcal{S}$ be a *mutation equivalence class* of seeds.

All seeds $(\tilde{\mathbf{x}}, \tilde{B}) \in \mathcal{S}$ share the same set $\mathbf{c} = \tilde{\mathbf{x}} - \mathbf{x}$. Fix a *ground ring* R sandwiched between $\mathbb{Z}[\mathbf{c}]$ and $\mathbb{C}[\mathbf{c}^{\pm 1}]$.

Let $\mathcal{X} = \mathcal{X}(\mathcal{S})$ denote the union of all clusters $\mathbf{x}$ in all the seeds in $\mathcal{S}$. The elements of $\mathcal{X}$ are called *cluster variables*.

The *cluster algebra* $\mathcal{A}(\mathcal{S})$ associated with $\mathcal{S}$ is the R-subalgebra of the ambient field $\mathcal{F}$ generated by all cluster variables: $\mathcal{A}(\mathcal{S}) = R[\mathcal{X}]$.

In our running example, taking $R = \mathbb{C}[\mathbf{c}]$, we recover $\mathcal{A}(\mathcal{S}) = \mathcal{A}_n = \mathbb{C}[\mathrm{Gr}_{2,n+3}]$.

(Strictly speaking, the above definition is that of a cluster algebra of *geometric type*.)

Examples

Theorem 3 [J.Scott, *Proc. LMS* **92** (2006)] *The homogeneous coordinate ring of every Grassmannian* $\mathrm{Gr}_k(\mathbb{C}^r)$ *has a natural cluster algebra structure.*

Conjecturally, this extends to any homogeneous space G/P, and any Schubert variety therein. (Proved for G/P's in $G = SL_m(\mathbb{C})$ by C.Geiss, B.Leclerc, and J.Schröer [`math.RT/0609138`].)

Theorem 4 *The coordinate ring of any affine base space* G/N *is a cluster algebra.*

Laurent phenomenon for cluster algebras

Theorem 5 *Every cluster variable is a Laurent polynomial in the elements of any extended cluster.*

It is conjectured (and in many instances proved) that all such Laurent polynomials have positive coefficients.

Theorem 5 is a special case of the main result in [S.F.-A.Z., *Adv. in Appl. Math.* **28** (2002)], which we used to prove a conjecture of D.Gale and R.Robinson on integrality of *generalized Somos sequences*, and conjectures by J.Propp on *the cube and octahedron recurrences.*

Cluster monomials

A *cluster monomial* is a monomial in cluster variables all of which belong to the same cluster.

Conjecture 6 *The cluster monomials are linearly independent over the ground ring R.*

Many special cases have been proved.

In examples of geometric origin, we expect the cluster monomials to form part of the suitably defined *dual canonical basis* in $\mathcal{A}$.

3. Cluster algebras of finite type

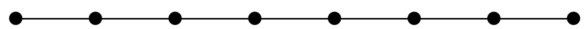
A cluster algebra $\mathcal{A}(\mathcal{S})$ is of *finite type* if the mutation equivalence class $\mathcal{S}$ is finite. Equivalently, there are finitely many cluster variables.

Conjecture 7 *The cluster monomials form an additive basis of a cluster algebra if and only if it is of finite type.*

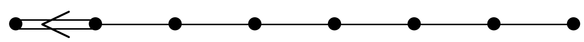
The classification of cluster algebras of finite type turns out to be completely parallel to the classical Cartan-Killing classification of semisimple Lie algebras and finite root systems.

Dynkin diagrams of finite crystallographic root systems

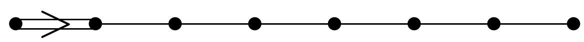
A_n $(n \geq 1)$



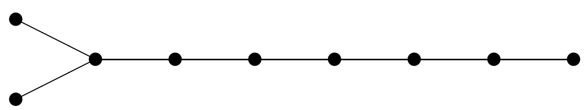
B_n $(n \geq 2)$



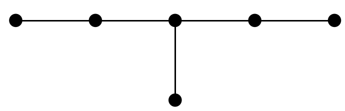
C_n $(n \geq 3)$



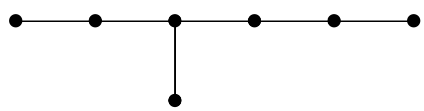
D_n $(n \geq 4)$



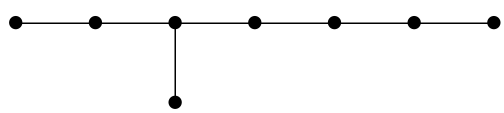
E_6



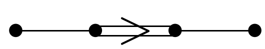
E_7



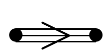
E_8



F_4

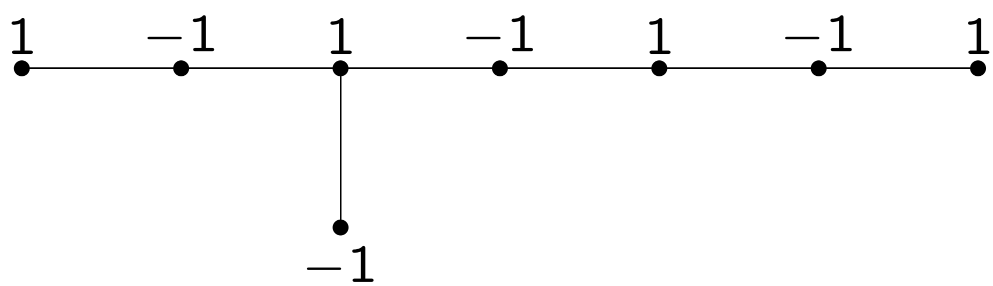


G_2



Bi-partition of a Dynkin diagram

Let $A = (a_{ij})$ be an $n \times n$ Cartan matrix of finite type. Let $\varepsilon : [1, n] \to \{1, -1\}$ be a sign function such that $a_{ij} < 0 \implies \varepsilon(i) = -\varepsilon(j)$.



Let $B(A) = (b_{ij})$ be the skew-symmetrizable matrix defined by

$$b_{ij} = \begin{cases} 0 & \text{if } i = j; \\ \varepsilon(i)\, a_{ij} & \text{if } i \neq j. \end{cases}$$

Finite type classification

Theorem 8 *A cluster algebra $\mathcal{A}$ is of finite type if and only if the exchange matrix at some seed of $\mathcal{A}$ is of the form $B(A)$, where A is a Cartan matrix of finite type.*

The type of the Cartan matrix A in the Cartan-Killing nomenclature is uniquely determined by the cluster algebra $\mathcal{A}$, and is called the *cluster type* of $\mathcal{A}$.

The cluster algebra $\mathcal{A}_n$ of our running example has cluster type A_n.

Cluster types of some coordinate rings

The symmetry exhibited by the cluster type of a cluster algebra is usually not apparent at all from its geometric realization.

$\mathbb{C}[\mathrm{Gr}(2, n+3)]$	A_n
$\mathbb{C}[\mathrm{Gr}(3, 6)]$	D_4
$\mathbb{C}[\mathrm{Gr}(3, 7)]$	E_6
$\mathbb{C}[\mathrm{Gr}(3, 8)]$	E_8
$\mathbb{C}[SL_3/N]$	A_1
$\mathbb{C}[SL_4/N]$	A_3
$\mathbb{C}[SL_5/N]$	D_6
$\mathbb{C}[Sp_4/N]$	B_2
$\mathbb{C}[SL_2]$	A_1
$\mathbb{C}[SL_3]$	D_4

(beyond this table—infinite types)

4. Cluster combinatorics

Cluster complex

The underlying combinatorics of a cluster algebra $\mathcal{A}$ of finite type is encoded by the *cluster complex* $\Delta(\mathcal{A})$, the simplicial complex on the ground set of all cluster variables whose maximal simplices are the clusters.

Theorem 9 [F.Chapoton-S.F.-A.Z.]
The cluster complex of a cluster algebra of finite type is the dual simplicial complex of a simple convex polytope.

This polytope is the *generalized associahedron* of the appropriate Cartan-Killing type. In types A_n and B_n, we recover, respectively, Stasheff's associahedron and Bott-Taubes' *cyclohedron*.

Enumerative results

Let $\mathcal{A}$ be a cluster algebra of finite type. Let Φ be a finite crystallographic root system of the corresponding Cartan-Killing type.

Theorem 10 *The number of cluster variables in $\mathcal{A}$ (=the number of facets of a generalized associahedron) is equal to the number of roots in Φ that are either positive or negative simple.*

Theorem 11 *The number of clusters in $\mathcal{A}$ (=the number of vertices of a generalized associahedron) is equal to*

$$N(\Phi) = \prod_{i=1}^{n} \frac{e_i + h + 1}{e_i + 1},$$

where $e_1, \ldots, e_n$ are the exponents of Φ, and h is the Coxeter number.

Catalan combinatorics of arbitrary type

The numbers $N(\Phi)$ can be viewed as generalizations of the *Catalan numbers* to arbitrary Cartan-Killing type. Besides clusters, they are known to enumerate a variety of combinatorial objects related to the root system Φ:

- ad-*nilpotent ideals* in a Borel subalgebra of a semisimple Lie algebra;
- antichains in the *root poset*;
- regions of the *Shi arrangement* contained in the fundamental chamber;
- orbits of the Weyl group action on the quotient $Q/(h+1)Q$ of the root lattice;
- conjugacy classes of elements x of a semisimple Lie group which satisfy $x^{h+1} = 1$;
- *non-crossing partitions* of the appropriate type.

5. Cluster algebras and triangulated surfaces

Cluster-algebraic structures associated with triangulated surfaces were discovered in:

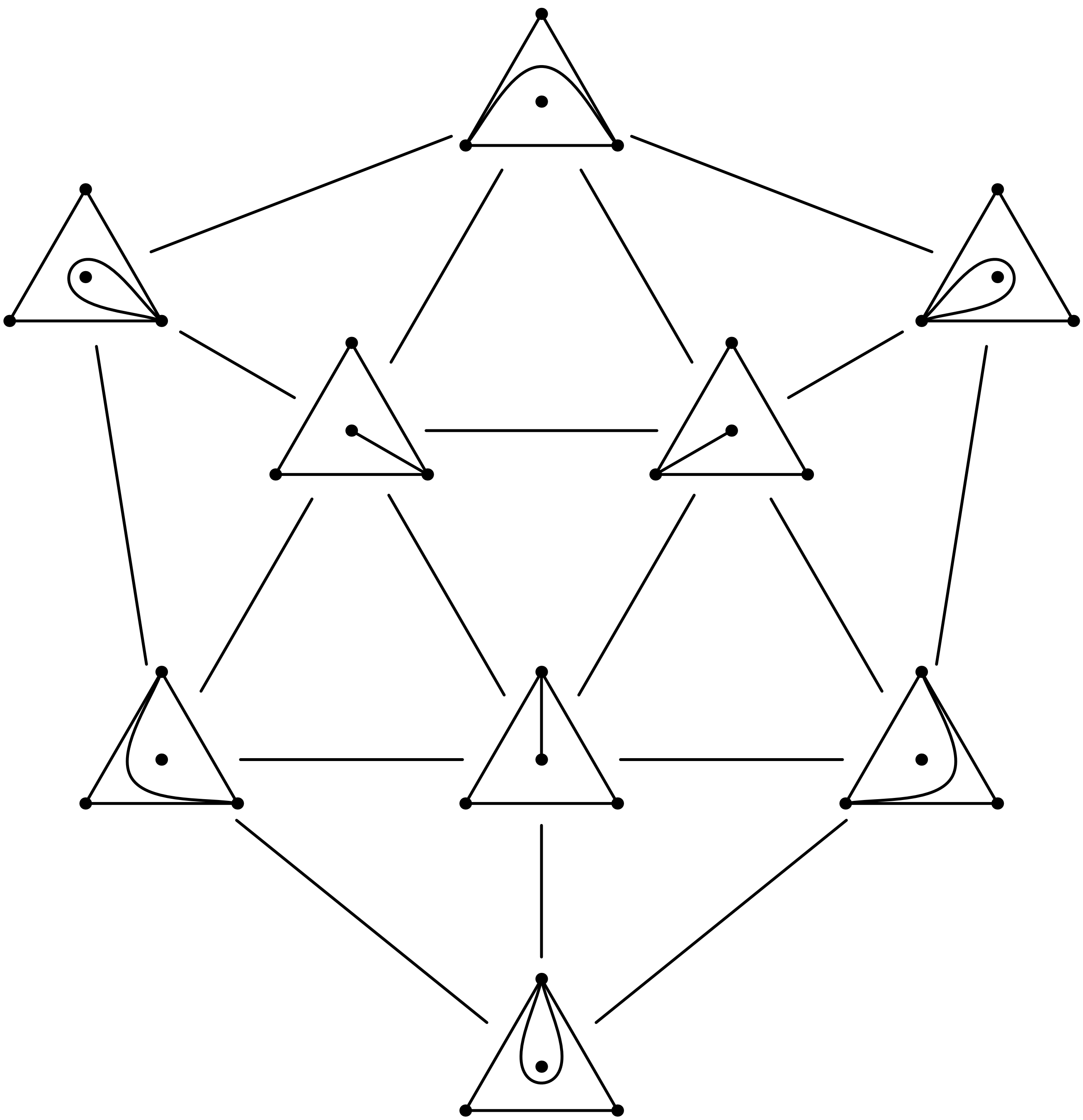
V.V.Fock and A.B.Goncharov,
Publ. Math. IHES **103** (2006),

M.Gekhtman, M.Shapiro, and A.Vainshtein,
Duke Math. J. **127** (2005).

Let $\mathbf{S}$ be a connected oriented surface with boundary. Fix a finite non-empty set $\mathbf{M}$ of *marked points* in the closure of $\mathbf{S}$.

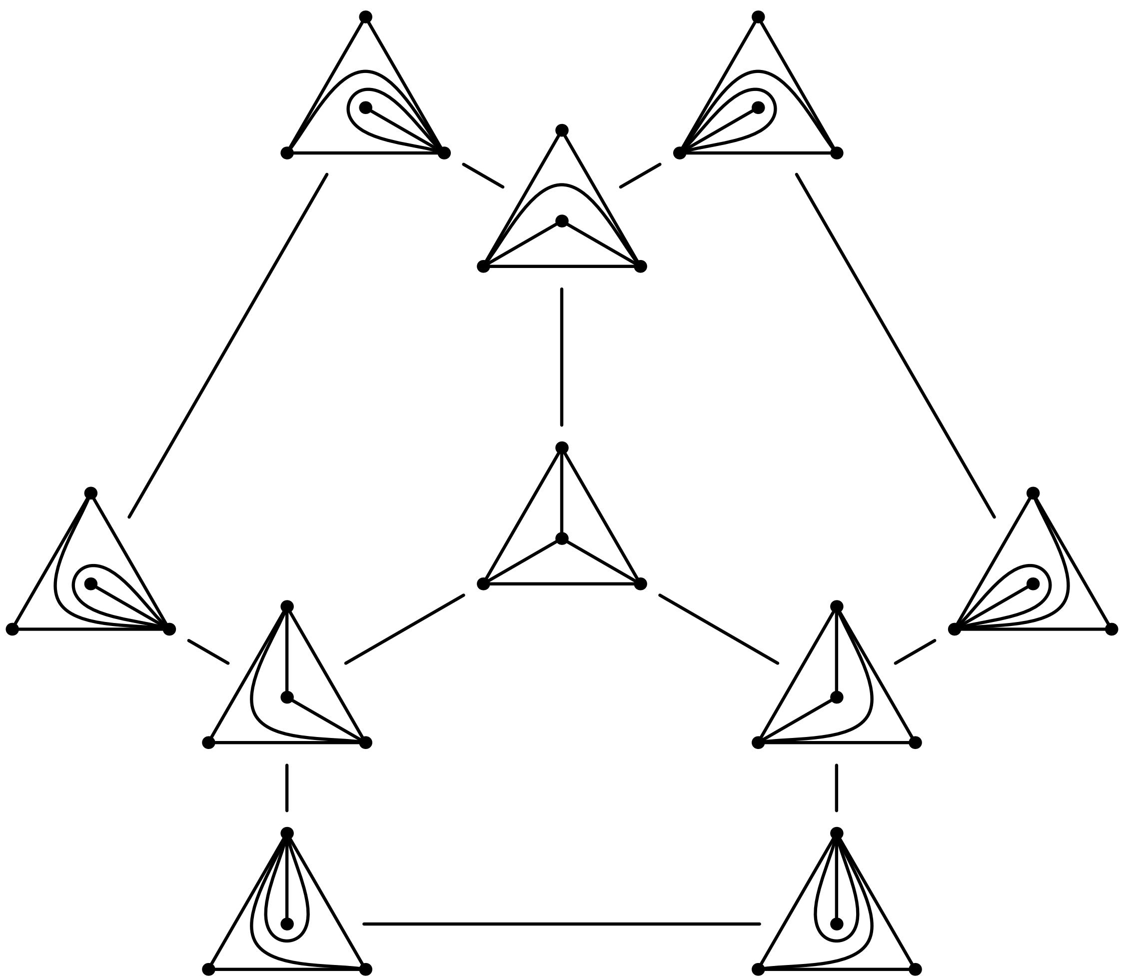
An *arc* in $(\mathbf{S}, \mathbf{M})$ is a non-selfintersecting curve in $\mathbf{S}$ connecting marked points in $\mathbf{M}$ and not passing through $\mathbf{M}$. Each arc is considered up to isotopy rel $\mathbf{M}$, and must not cut out an unpunctured monogon or digon.

Arc complex



Flips on a surface

Maximal collections of compatible arcs form *triangulations* of $(\mathbf{S},\mathbf{M})$. Triangulations are connected by *flips*.



The notion of a signed adjacency matrix $\tilde{B}(T)$ can be generalized to triangulations of surfaces. Under flips, such matrices change according to the general mutation rules, as before.

Cluster variables

Let us associate a formal variable to every arc. Each triangulation gives rise to a cluster. Flips correspond to birational maps, which form a commuting diagram. This leads to a family of cluster-algebraic structures associated with the given bordered surface with marked points.

The cluster variables in a resulting cluster algebra generalize Penner's coordinates on the *decorated Teichmüller space*, also known as *lambda-lengths*.

With M.Shapiro and D.Thurston, we explicitly described the cluster complex associated with an arbitrary bordered surface $(\mathbf{S}, \mathbf{M})$, and determined its homotopy type and growth rate.