$$
G \curvearrowright R = \begin{array}{ccccccccc}
\vdots & & \vdots & & & & \vdots & \\
\oplus & & \oplus & & & & \oplus & \\
R_{2l} & \oplus & R_{2l+1} & \oplus & \cdots & \oplus & R_{3l-1} & \oplus \\
\oplus & & \oplus & & & & \oplus & \\
R_l & \oplus & R_{l+1} & \oplus & \cdots & \oplus & R_{2l-1} & \oplus \\
\oplus & & \oplus & & & & \oplus & \\
R_0 & \oplus & R_1 & \oplus & \cdots & \oplus & R_{l-1} & \oplus \\
\| & & \| & & & & \| & \\
R(0;l) & & R(1;l) & & \cdots & & R(l-1;l) &
\end{array}
$$

For some l, $\dim R(0;l) = \dim R(1;l) = \dim R(2;l) = \cdots$.

$$
\begin{array}{lcccc}
G \curvearrowright & R(0;l), & R(1;l), & \cdots, & R(l-1;l) \\
 & \wr\| & \wr\| & & \wr\| \\
G \curvearrowright & \operatorname{Ind} z(0;l), & \operatorname{Ind} z(1;l), & \cdots, & \operatorname{Ind} z(l-1;l) \\
\cup & \uparrow & \uparrow & & \uparrow \\
H \curvearrowright & z(0;l), & z(1;l), & \cdots, & z(l-1;l), \\
 & & & & (\dim z(i;l) = \dim z(j;l))
\end{array}
$$

We can prove it by $(H, \{\, z(i;l)\,\})$ providing the picture.

In the case when

- $G = S_m$: the symmetric group,
- $R = R_\mu = \bigoplus_i R_\mu^i$: the Springer module,
  where μ is a partition whose multiplicities are divisible by l,

$$\dim R_\mu(0;l) = \dim R_\mu(1;l) = \dim R_\mu(2;l) = \cdots,$$

where $R_\mu(k;l) = \bigoplus_i R_\mu^{il+k}$.

- $H = S_\mu \rtimes C_{\mu,l}$, where
  - S_μ: the Young subgroup,
  - $C_{\mu,l} = \langle a_{\mu,l} \rangle$: some l-th cyclic group.
- $z(i,l)$: the following 1-dimensional representation:

$$
\begin{array}{rccc}
z(i,l): & S_\mu \rtimes C_{\mu,l} & \longrightarrow & \mathbb{C}^\times \\
 & \Psi & & \Psi \\
S_\mu \ni & \sigma & \longmapsto & 1 \\
C_{\mu,l} \ni & a_{\mu,l} & \longmapsto & \zeta_l^i,
\end{array}
$$

where ζ_l is the primitive root of unity.

- $\dim z(i;l) = \dim z(j;l)$
  - Clear by the definition of $z(i;l)$.
- $\operatorname{Ind} z(i;l) \simeq R(i;l)$
  - They showed the coincidence of their characters.
  - To show the fact, they explicitly calculated values $Q_\rho^\mu(\zeta_l^i)$ of Green polynomial at roots of unity.

By their result,

- $Q_\rho^\mu(\zeta_l^i)$ is an nonnegative integer.

So we have the following questions:

- Does the number have a combinatorial presentation?
  (Describe as a number of some combinatorial objects.)
- What do the combinatorial objects mean?

Fix the following:

- l: a positive integer.
  - Let $l = 3$.
- μ: a partition of m whose multiplicities are divisible by l.
  - Let $\mu = (3^3, 2^3) \vdash 15$.
- ρ: a partition of m.
  - Let $\rho = (6, 6, 3) \vdash 15$.

A (ρ, l)-tableau on μ is a filling satisfying:

- weakly increasing for each row.
- the number of boxes where i lies is ρ_i.
- the boxes where i lies form a rectangle of height l.

Example The following are (ρ, l)-tableaux on μ:

$$\begin{array}{|c|c|c|}\hline 1&1&3\\ \hline 1&1&3\\ \hline 1&1&3\\ \hline 2&2\\ \cline{1-2} 2&2\\ \cline{1-2} 2&2\\ \cline{1-2}\end{array}\ , \qquad \begin{array}{|c|c|c|}\hline 2&2&3\\ \hline 2&2&3\\ \hline 2&2&3\\ \hline 1&1\\ \cline{1-2} 1&1\\ \cline{1-2} 1&1\\ \cline{1-2}\end{array}\ .$$

A marked (ρ, l)-tableau on μ is a diagram obtained

- from a (ρ, l)-tableau on μ
- by putting $*$ on one of left-most boxes where i lies for each i.

Example The following are marked (ρ, l)-tableaux:

$$\begin{array}{|c|c|c|}\hline 1^*&1&3^*\\ \hline 1&1&3\\ \hline 1&1&3\\ \hline 2^*&2\\ \cline{1-2} 2&2\\ \cline{1-2} 2&2\\ \cline{1-2}\end{array}, \begin{array}{|c|c|c|}\hline 1&1&3^*\\ \hline 1^*&1&3\\ \hline 1&1&3\\ \hline 2^*&2\\ \cline{1-2} 2&2\\ \cline{1-2} 2&2\\ \cline{1-2}\end{array}, \begin{array}{|c|c|c|}\hline 1&1&3^*\\ \hline 1&1&3\\ \hline 1^*&1&3\\ \hline 2^*&2\\ \cline{1-2} 2&2\\ \cline{1-2} 2&2\\ \cline{1-2}\end{array}, \begin{array}{|c|c|c|}\hline 1^*&1&3^*\\ \hline 1&1&3\\ \hline 1&1&3\\ \hline 2&2\\ \cline{1-2} 2^*&2\\ \cline{1-2} 2&2\\ \cline{1-2}\end{array}, \begin{array}{|c|c|c|}\hline 1&1&3^*\\ \hline 1^*&1&3\\ \hline 1&1&3\\ \hline 2&2\\ \cline{1-2} 2^*&2\\ \cline{1-2} 2&2\\ \cline{1-2}\end{array}, \begin{array}{|c|c|c|}\hline 1&1&3^*\\ \hline 1&1&3\\ \hline 1^*&1&3\\ \hline 2&2\\ \cline{1-2} 2^*&2\\ \cline{1-2} 2&2\\ \cline{1-2}\end{array}, \ldots$$

Theorem For $\sigma \in S_m$ of cycle type ρ,

$$\sum_{k \in \mathbb{Z}/l\mathbb{Z}} \zeta_l^k \operatorname{Char}\left(R^\mu(k; l)\right)(\sigma) = \left|\{ \text{ marked } (\rho, l)\text{-tableaux on } \mu \}\right|.$$

Remark $Q_\rho^\mu(\zeta_l) = \sum_{k \in \mathbb{Z}/l\mathbb{Z}} \zeta_l^k \operatorname{Char}\left(R^\mu(k; l)\right)(\sigma)$

Lemma For an integer j and $\sigma \in S_m$ of cycle type ρ,

$$\sum_{k \in \mathbb{Z}/l\mathbb{Z}} \zeta_l^k \operatorname{Char}\left(R_\mu(k; l)\right)(\sigma) = \left|\left\{ \{T\}\text{: a tabloid on } \mu \,\middle|\, \sigma\{T\} = \{T\} a_{\mu,l}^{-j} \right\}\right|.$$

Lemma For an integer j and $\sigma \in S_m$ of cycle type ρ,

$$\{ \text{ marked } (\rho, l)\text{-tableaux on } \mu \} \overset{1:1}{\longleftrightarrow} \left\{ \{T\}\text{: a tabloid on } \mu \,\middle|\, \sigma\{T\} = \{T\} a_{\mu,l}^{-j} \right\}.$$